THE GENERALIZED RECURRENT WEYL SPACES HAVING A DECOMPOSABLE CURVATURE TENSOR

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Abstract

In this paper, we have studied the generalized recurrent Weyl spaces the curvature tensor of which is decomposed in the form $L_{jkl}^i = v^i \varphi_{jkl}$, and proved some theorems concerning such spaces.

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1. INTRODUCTION

A differentiable manifold W_n of dimension n having a symmetric connection ∇^* and a symmetric conformal metric tensor g_{ij} preserved by ∇^* is called a Weyl space. Accordingly, in local coordinates, there exists a covariant vector field T (complementary vector field) such that the condition

$$\nabla_k^* g_{ij} - 2T_k g_{ij} = 0, \quad (1.1)$$

holds [1], [2], [3].

Writing (1.1) out in full, we have

$$\partial_k g_{ij} - g_{hj} \Gamma_{ik}^h - g_{ih} \Gamma_{jk}^h - 2T_k g_{ij} = 0 \quad (1.2)$$

where Γ_{kl}^i are the connection coefficients of the connection ∇^* are given by

$$\Gamma_{kl}^i = \left\{ \begin{matrix} i \\ kl \end{matrix} \right\} - g^{im} (g_{mk} T_l + g_{ml} T_k - g_{kl} T_m) . \quad (1.3)$$

We denote such a Weyl space as $W_n (\Gamma_{jk}^i, g_{ij}, T_k)$.

An n dimensional manifold GW_n is said to be a generalized Weyl space if it has an asymmetric connection ∇ and a symmetric conformal metric tensor g_{ij} satisfying the compatibility condition given by the equation

$$\nabla_k g_{ij} - 2T_k g_{ij} = 0 \quad (1.4)$$

where T_k denotes a covariant vector field and $\nabla_k g_{ij}$ denotes the usual covariant derivative [4], [5]. Writing (1.4) out in full, we have

$$\partial_k g_{ij} - g_{hj} L_{ik}^h - g_{ih} L_{jk}^h - 2T_k g_{ij} = 0 \quad (1.5)$$

where L_{kl}^i are the connection coefficients of the connection ∇_k are given by

$$L_{kl}^i = \Gamma_{kl}^i + \chi_{kl}^i \quad (1.6)$$

where Γ_{kl}^i and χ_{kl}^i are the symmetric and anti-symmetric parts respectively of L_{kl}^i namely

$$\Gamma_{kl}^i = \frac{1}{2} (L_{kl}^i + L_{lk}^i) \quad (1.7)$$

and

$$\chi_{kl}^i = \frac{1}{2} (L_{kl}^i - L_{lk}^i). \quad (1.8)$$

We denote such a Weyl space $GW_n (L_{jk}^i, g_{ij}, T_k)$.

The Weyl space $W_n (\Gamma_{jk}^i, g_{ij}, T_k)$ is called the associate Weyl space to the generalized Weyl space $GW_n (L_{jk}^i, g_{ij}, T_k)$.

Using the relations (1.2), (1.5) and (1.6) we obtain

$$\chi_{jik} + \chi_{ijk} = 0 \quad (1.9)$$

where the tensor χ_{jik} is defined by

$$\chi_{jik} = g_{hj} \chi_{ik}^h. \quad (1.10)$$

Under a renormalization of the fundamental tensor of the form

$$\check{g}_{ij} = \lambda^2 g_{ij}, \quad (1.11)$$

the complementary vector field T_k is transformed by the law

$$\check{T}_k = T_k + \partial_k \ln \lambda, \quad (1.12)$$

where λ is a scalar function defined on $GW_n (L_{jk}^i, g_{ij}, T_k)$.

A quantity A is called a satellite of weight $\{p\}$ of the tensor g_{ij} , if it admits a transformation of the form

$$\check{A} = \lambda^p A \quad (1.13)$$

under the renormalization (1.11) of the metric tensor g_{ij} of weight $\{p\}$ is defined by [3], [6]

$$\dot{\nabla}_k A = \nabla_k A - p T_k A. \quad (1.14)$$

We note that the prolonged covariant derivative preserves the weight.

The components of the mixed curvature tensor and the Ricci tensor of $GW_n (L_{jk}^i, g_{ij}, T_k)$ are respectively

$$L_{jkl}^i = \frac{\partial}{\partial x^k} L_{jl}^i - \frac{\partial}{\partial x^l} L_{jk}^i + L_{hk}^i L_{jl}^h - L_{hl}^i L_{jk}^h \quad (1.15)$$

$$L_{ij} = L_{ija}^a . \quad (1.16)$$

On the other hand, the scalar curvature of $GW_n (L_{jk}^i, g_{ij}, T_k)$ is defined by

$$L = g^{ij} L_{ij} . \quad (1.17)$$

It is easy to see that the curvature tensor L_{jkl}^i of $GW_n (L_{jk}^i, g_{ij}, T_k)$ can be written as

$$L_{jkl}^i = B_{jkl}^i + \chi_{jkl}^i \quad (1.18)$$

where the tensors B_{jkl}^i and χ_{jkl}^i are defined respectively

$$B_{jkl}^i = \frac{\partial}{\partial x^k} \Gamma_{jl}^i - \frac{\partial}{\partial x^l} \Gamma_{jk}^i + \Gamma_{hk}^i \Gamma_{jl}^h - \Gamma_{hl}^i \Gamma_{jk}^h , \quad (1.19)$$

$$\chi_{jkl}^i = \nabla_k \chi_{jl}^i - \nabla_l \chi_{jk}^i + \chi_{hl}^i \chi_{jk}^h - \chi_{hk}^i \chi_{jl}^h - 2\chi_{jh}^i \chi_{kl}^h . \quad (1.20)$$

The curvature tensor of $GW_n (L_{jk}^i, g_{ij}, T_k)$ satisfies the relations [5]

$$L_{jkl}^h + L_{jlk}^h = 0 \quad (1.21)$$

$$L_{hlk}^j + L_{hkl}^j + L_{klh}^j$$

$$= 2(\nabla_k \chi_{lh}^j + \nabla_l \chi_{hk}^j + \nabla_h \chi_{kl}^j + 2\chi_{lm}^j \chi_{hk}^m + 2\chi_{hm}^j \chi_{kl}^m + 2\chi_{km}^j \chi_{lh}^m) \quad (1.22)$$

$$\dot{\nabla}_m L_{jkl}^i + \dot{\nabla}_k L_{jlm}^i + \dot{\nabla}_l L_{jmk}^i = 2(L_{jpl}^i \chi_{mk}^p + L_{jpk}^i \chi_{lm}^p + L_{jpm}^i \chi_{kl}^p) . \quad (1.23)$$

2. THE GENERALIZED RECURRENT WEYL SPACES HAVING A DECOMPOSABLE CURVATURE TENSOR

If the curvature tensor L_{jkl}^i of $GW_n (L_{jk}^i, g_{ij}, T_k)$ satisfies the condition

$$\dot{\nabla}_m L_{jkl}^i = \psi_m L_{jkl}^i \quad (2.1)$$

where ψ_m is a covariant vector field, then $GW_n (L_{jk}^i, g_{ij}, T_k)$ is called recurrent.

In this section we consider the generalized recurrent Weyl spaces denoted by $RGW_n (L_{jk}^i, g_{ij}, T_k)$ the curvature tensor of which is decomposed in the form

$$L_{jkl}^i = v^i \varphi_{jkl} \quad (2.2)$$

where v^i is a contravariant vector field of weight $\{-1\}$ and φ_{jkl} is a covariant tensor field of weight $\{1\}$.

Using the relations (1.21) and (2.2) we get

$$\varphi_{jkl} + \varphi_{jlk} = 0. \quad (2.3)$$

Taking into account the relations (1.23), (2.1) and (2.2) we obtain

$$\psi_m \varphi_{jkl} + \psi_k \varphi_{jlm} + \psi_l \varphi_{jmk} = 2(\varphi_{jpl} \chi_{mk}^p + \varphi_{jpk} \chi_{lm}^p + \varphi_{jpm} \chi_{kl}^p). \quad (2.4)$$

Multiplying both hand sides of (2.4) by v^m and summing for m we have

$$\varphi_{jkl} = \alpha[\psi_l \phi_{jk} - \psi_k \phi_{jl} + 2(\varphi_{jpl} v^m \chi_{mk}^p + \varphi_{jpk} v^m \chi_{lm}^p + \phi_{jp} \chi_{kl}^p)] \quad (2.5)$$

where α is a scalar function of weight $\{1\}$ which is defined by

$$\alpha = \frac{1}{\mu} \quad (\mu = v^k \psi_k) \quad (2.6)$$

and ϕ_{jk} is a covariant tensor field of weight $\{0\}$ which is defined by

$$\phi_{jk} = v^l \varphi_{jkl}. \quad (2.7)$$

Using the relations (1.16), (2.2) and (2.7) we obtain

$$L_{jk} = \phi_{jk}. \quad (2.8)$$

Using (2.8) in (2.5) we get

$$\varphi_{jkl} = \alpha[\psi_l L_{jk} - \psi_k L_{jl} + 2(\varphi_{jpl} v^m \chi_{mk}^p + \varphi_{jpk} v^m \chi_{lm}^p + L_{jp} \chi_{kl}^p)]. \quad (2.9)$$

Multiplying the relation (2.3) by $v^k v^l$ and summing for k and l and using the relations (2.7) and (2.8) we obtain

$$L_{jk} v^k = 0 \quad (2.10)$$

from which it follows that

$$\det(L_{jk}) = 0. \quad (2.11)$$

Thus we obtain the

THEOREM 2.1: If a space $RGW_n (L_{jk}^i, g_{ij}, T_k)$ has a decomposable curvature tensor in the form $L_{jkl}^i = v^i \varphi_{jkl}$, then we have

$$\varphi_{jkl} = \alpha[\psi_l L_{jk} - \psi_k L_{jl} + 2(\varphi_{jpl} v^m \chi_{mk}^p + \varphi_{jpk} v^m \chi_{lm}^p + L_{jp} \chi_{kl}^p)]$$

and

$$\det(L_{jk}) = 0.$$

If the connection of $RGW_n (L_{jk}^i, g_{ij}, T_k)$ is symmetric that is $\chi_{mk}^p = 0$ then the relations (2.9) reduces to

$$\varphi_{jkl} = \alpha(\psi_l L_{jk} - \psi_k L_{jl}). \quad (2.12)$$

THEOREM 2.2: If a space $RGW_n (L_{jk}^i, g_{ij}, T_k)$ has a decomposable curvature tensor in the form $L_{jkl}^i = v^i \varphi_{jkl}$, then the vector field v^i and the tensor field φ_{jkl} are recurrent.

Proof: By (1.16) and (2.1) we get

$$\dot{\nabla}_m L_{jk} = \psi_m L_{jk}. \quad (2.13)$$

On the other hand, from (2.2) and (2.9) we can write the curvature tensor L_{jkl}^i of RGW_n (L_{jk}^i , g_{ij} , T_k) in the form

$$L_{jkl}^i = \alpha v^i [\psi_l L_{jk} - \psi_k L_{jl} + 2(\varphi_{jpl} v^m \chi_{mk}^p + \varphi_{jpk} v^m \chi_{lm}^p + L_{jp} \chi_{kl}^p)]. \quad (2.14)$$

Taking the prolonged covariant derivative both hand sides of (2.14) with respect to the coordinate u^m and using the relations (2.1) and (2.13) we obtain

$$L_{jkl}^a \dot{\nabla}_m v^i = L_{jkl}^i \dot{\nabla}_m v^a. \quad (2.15)$$

If we use the relation (2.2) then the relation (2.15) reduces to

$$v^a \dot{\nabla}_m v^i = v^i \dot{\nabla}_m v^a \quad (2.16)$$

from which it follows that

$$\dot{\nabla}_m v^i = \lambda_m v^i \quad (2.17)$$

where λ_m is a covariant vector field of weight $\{0\}$. From the relation (2.16) we have the vector field v^i is recurrent.

On the other hand taking the prolonged covariant derivative of (2.2) with respect to the coordinate u^m and using the relations (2.1) and (2.17) we get

$$\dot{\nabla}_m \varphi_{jkl} = (\psi_m - \lambda_m) \varphi_{jkl}. \quad (2.18)$$

From the relation (2.18) we have the tensor field φ_{jkl} is recurrent.

Now we consider the RGW_n (L_{jk}^i , g_{ij} , T_k) spaces having a decomposable curvature tensor in the form

$$L_{jkl}^i = v^i \psi_j \varphi_{kl}, \quad \varphi_{jkl} = \psi_j \varphi_{kl} \quad (2.19)$$

where φ_{kl} is a covariant tensor field of weight $\{1\}$ and concerning these spaces we prove the following theorem:

THEOREM 2.3: The tensor field φ_{jkl} may be decomposed in the form $\varphi_{jkl} = \psi_j \varphi_{kl}$ if and only if the condition

$$\dot{\nabla}_m \psi_j + (\lambda_m - \alpha \dot{\nabla}_m \mu) \psi_j = 0$$

holds true.

Proof: We first prove the necessity of the condition. Suppose that the tensor φ_{jkl} is decomposed in the form (2.19). Under this condition the equation (2.18) transforms into

$$(\psi_m - \lambda_m)\psi_j\varphi_{kl} = (\dot{\nabla}_m\psi_j)\varphi_{kl} + (\dot{\nabla}_m\varphi_{kl})\psi_j . \quad (2.20)$$

Taking the prolonged covariant derivative of the equation $\mu = v^i\psi_i$ with respect to u^m and using (2.17) we obtain

$$v^i(\dot{\nabla}_m\psi_i) = -\mu\lambda_m\dot{\nabla}_m\mu . \quad (2.21)$$

Multiplying both sides of (2.20) by v^j and summing for j and taking (2.21) into account, we get

$$\dot{\nabla}_m\varphi_{kl} = (\psi_m - \alpha\dot{\nabla}_m\mu)\varphi_{kl} . \quad (2.22)$$

Taking the prolonged covariant derivative of both sides of the equation $L^i_{jkl} = v^i\psi_j\varphi_{kl}$ with respect to u^m and using (2.1), (2.17) and (2.22) we obtain

$$\psi_m L^i_{jkl} = \psi_m L^i_{jkl} + (\dot{\nabla}_m\psi_j + \lambda_m\psi_j - \alpha\psi_j\dot{\nabla}_m\mu)\varphi_{kl}v^i \quad (2.23)$$

from which it follows that

$$\dot{\nabla}_m\psi_j + (\lambda_m - \alpha\dot{\nabla}_m\mu)\psi_j = 0 . \quad (2.24)$$

Conversely, the condition (2.24) is sufficient. To see this, take the prolonged covariant derivative of both sides of (2.24) with respect to u^p we get

$$\dot{\nabla}_p\dot{\nabla}_m\psi_j + \dot{\nabla}_p[(\lambda_m - \alpha\dot{\nabla}_m\mu)\psi_j] = 0 . \quad (2.25)$$

Interchanging the indices m and p in (2.25) and subtracting the equation so obtained from (2.25) we obtain

$$\varphi_{jmp} = \alpha[\dot{\nabla}_m(\lambda_p - \alpha\dot{\nabla}_p\mu) - \dot{\nabla}_p(\lambda_m - \alpha\dot{\nabla}_m\mu) + 2\chi_{mp}^a(\alpha\dot{\nabla}_a\mu - \lambda_a)]\psi_j \quad (2.26)$$

where we have used the relations (2.2), (2.24) and the Ricci identity [7].

From (2.26) it follows that the tensor φ_{jmp} may be written in the form

$$\varphi_{jmp} = \psi_j \varphi_{mp}$$

where φ_{mp} is given by

$$\varphi_{mp} = \alpha [\dot{\nabla}_m (\lambda_p - \alpha \dot{\nabla}_p \mu) - \dot{\nabla}_p (\lambda_m - \alpha \dot{\nabla}_m \mu) + 2\chi_{mp}^a (\alpha \dot{\nabla}_a \mu - \lambda_a)] . \quad (2.27)$$

The proof of the theorem is completed.

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