ON STRUCTURE OF KS-SEMIGROUPS

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Abstract

In this paper, we introduce a new class algebras related to BCK-algebras and semigroups, called a KS-semigroup and define an ideal of a KS-semigroups and a strong KS-semigroup, and characterizations of ideals is given. Also we define a congruence relation on a KS-semigroups and a quotient KS-semigroups and prove the isomorphism.

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1. INTRODUCTION

The notion of BCK-algebras was proposed by Y. Imai and K. Iséki in 1966. In the same year, K. Iséki introduced the notion of BCI-algebra which is a generalization of a BCK-algebra. For the general development of BCK/BCI-algebras, the ideal theory plays an important role. We introduce a new class algebras related to BCK-algebras and semigroups, called a KS-semigroup. In this paper, we define an ideal of a KS-semigroups and a strong KS-semigroup, and characterizations of ideals is given. Also we define a congruence relation on a KS-semigroups and a quotient KS-semigroups and prove the isomorphism.

2. PRELIMINARIES

We review some definitions and properties that will be useful in our results.

A *BCI-algebra* is a triple $(X, *, 0)$, where X is a nonempty set, “ $*$ ” is a binary operation on X , $0 \in H$ is an element such that the following four axioms are satisfied for every $x, y, z \in X$:

- (I) $((x * y) * (x * z)) * (z * y) = 0$,
- (II) $(x * (x * y)) * y = 0$,
- (III) $x * x = 0$,
- (IV) $x * y = 0, y * x = 0$ implies $x = y$.

A BCI-algebra X satisfying $0 * x = 0$ for all $x \in X$ is called a *BCK-algebra*. If X is a BCK-algebra, then the relation $x \leq y$ if and only if $x * y = 0$ is a partial order on X , which will be called the *natural ordering* on X .

A BCK-algebra X has the following properties for any $x, y, z \in X$:

- (1) $x * 0 = x$,
- (2) $(x * y) * z = (x * z) * y$,
- (3) $x \leq y$ implies that $x * z \leq y * z$ and $z * y \leq z * x$,
- (4) $(x * z) * (y * z) \leq x * y$,

A nonempty subset I of a BCK/BCI-algebra is called an *ideal* if it satisfies

- (i) $0 \in I$,
- (ii) $x * y \in I$ and $y \in I$ imply $x \in I$ for all $x, y \in X$.

Any ideal I has the property: $y \in I$ and $x \leq y$ imply $x \in I$.

For a BCK-algebra X , the set $X_+ := \{x \in X \mid 0 \leq x\}$ is called the *BCK-part* of X . If $X_+ = \{0\}$, then we say that X is a *p-semisimple* BCI-algebra. Note that a BCI-algebra X is p-semisimple if and only if $0 * (0 * x) = x$ for all $x \in X$.

An KS-semigroup is a non-empty set X with two binary operation “ $*$ ” and “ $\cdot$ ” and constant 0 satisfying the axioms:

- (V) $(X, *, 0)$ is a BCK-algebra,
- (VI) $(X, \cdot)$ is a semigroup,
- (VII) The operation “ $\cdot$ ” is distributive (on both sides) over the operation “ $*$ ”, that is, $x \cdot (y * z) = (x \cdot y) * (x \cdot z)$ and $(x * y) \cdot z = (x \cdot z) * (y \cdot z)$ for all $x, y, z \in X$.

We shall write the multiplication $x \cdot y$ by xy for convenience.

Example 2.1. Let $X = \{0, a, b, c, d\}$. Define $*$ -operation and multiplication “ $\cdot$ ” by the following tables

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	a	0
b	b	b	0	0	0
c	c	c	c	0	0
d	d	d	d	d	0

$\cdot$	0	a	b	c	d
0	0	0	0	0	0
a	0	0	0	0	0
b	0	0	0	0	b
c	0	0	0	b	c
d	0	a	b	c	d

Then, by routine calculations, we can see that X is an *KS-semigroup*.

Lemma 2.2. Let X be an *KS-semigroup*. Then we have

- (i) $0x = x0 = 0$,
- (2) $x \leq y$ implies that $xz \leq yz$ and $zx \leq zy$, for all $x, y, z \in X$.

A non-empty subset A of a semigroup $(X, \cdot)$ is said to be *left* (resp. *right*) *stable* if $xa \in A$ (resp. $ax \in A$) whenever $x \in X$ and $a \in A$. Both left and right stable is *two-sided stable* or simply *stable*.

Definition 2.3. A non-empty subset A of a KS-semigroup X is called a *left*(resp. *right*) *ideal* of X if

- (i) A is a left(resp. right) stable subset of $(X, \cdot)$,
- (ii) for any $x, y \in X, x * y \in A$ and $y \in A$ imply that $x \in A$.

A both of left and right ideal is called a *two-sided ideal* or simply an *ideal*. Note that $\{0\}$ and X are ideals. If A is a left(resp. right) ideal of an KS-semigroup of X , then $0 \in A$. Thus A is an ideal of X .

Example 2.4. Let $X = \{0, a, b, c\}$. Define $*$ -operation and multiplication “ $\cdot$ ” by the following tables

$*$	0	a	b	c
0	0	0	0	0
a	a	0	a	0
b	b	b	0	0
c	c	b	a	0

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
c	0	a	b	c

Then, by routine calculations, we can see that X is an KS-semigroup. If $A = \{0, a\}$, then A is an *ideal* of a KS-semigroup X .

Definition 2.5. An ideal A of an KS-semigroup X is said to be *closed* if $x \in X$ implies $0 * x \in A$.

3. MAIN RESULTS

In what follows, let X denote a KS-semigroup unless otherwise specified.

Definition 3.1. A non-empty subset A of a KS-semigroup X is called a *left*(resp. *right*) *P-ideal* of X if

- (i) A is a left(resp. right) stable subset of $(X, \cdot)$,
- (ii) for any $x, y, z \in X, (x * y) * z \in A$ and $y * z \in A$ imply that $x * z \in A$.

Example 3.2. Let $X = \{0, a, b, c\}$. Define $*$ -operation and multiplication “ $\cdot$ ” by the following tables

$*$	0	a	b	c
0	0	0	0	0
a	a	0	a	a
b	b	b	0	b
c	c	c	c	0

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	a	0	0
b	0	0	b	0
c	0	c	0	0

Then, by routine calculations, we can see that X is an KS-semigroup. Let $A = \{0, a\}$. Then A is a *P-ideal* of a KS-semigroup X .

Example 3.3. Let $X = \{0, a, b, c\}$. Define $*$ -operation and multiplication “ $\cdot$ ” by the following tables

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	a	0	b
c	c	c	c	0

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	0	0	a
b	0	0	0	b
c	0	0	0	c

Then, by routine calculations, we can see that X is an KS-semigroup. Let $A = \{0, a\}$. Then A is a P-ideal of a KS-semigroup X .

Theorem 3.4. *Every P-ideal of a KS-semigroup X is an ideal but the converse is not true.*

Proof. Suppose that A is a P-ideal of a

KS-semigroup. Let $x, y \in X$ be such that $x * y \in A$ and $y \in A$. It follows from (1) that

$$(x * y) * 0 = x * y \in A \text{ and } y * 0 = y \in A.$$

Therefore, by Definition 3.1(ii), we obtain $x = x * 0 \in A$. So, A is an ideal of X . In Example 3.2, $\{0, b\}$ is an ideal but not a P-ideal of X because $(a * c) * b = a * b = 0 \in \{0, b\}$ but $c * b = c \notin \{0, b\}$.

Definition 3.5. A *strong KS-semigroup* is a KS-semigroup X satisfying $x * y = x * xy$ for each $x, y \in X$.

Example 3.6. In Example 2.4, we are easy to prove that $(X, *, \cdot, 0)$ is a strong KS-semigroup.

Lemma 3.7. *Let X be a strong KS-semigroup. Then*

- (i) $xy * y = 0$ for all $x, y \in X$,
- (ii) $x * y = 0$ if and only if $x * xy = 0$ for any $x, y \in X$.

Proof. (1) For any $x, y \in X$, $xy * y = xy * (xy)y = xy * x(yy) = x(y * yy) = x(y * y) = 0$.

(2) It is easy to show from the definition of strong KS-semigroup and the above (1). $\square$

The element 1 is called a *unity* in a KS-semigroup X if $1x = x1 = x$ for all $x \in X$. If X is a strong KS-semigroup with a unity 1, then 1 is the greatest element in X since $x * 1 = x * x1 = x * x = 0$ for all $x \in X$.

Theorem 3.8. *Let X be a strong KS-semigroup with a unity 1 and A any non-empty subset of X . If $y \in A$ and $x \leq y$ imply $x \in A$, then A is an ideal of X .*

Proof. Suppose that $y \in A$ and $x \leq y$ imply $x \in A$. If $x, y \in A$, then $x * y \in A$, since $x * y = x * xy = x(1 * y) \leq x1 = x \in A$. Next, let $s \in X$ and $a \in A$. Then

$$as * a = a(s * 1) = a0 = 0$$

and

$$sa * a = (s * 1)a = 0a = 0,$$

hence $as \leq a$ and $sa \leq a$, that is, $AX \subseteq A$ and $XA \subseteq A$. It follows that A is an ideal of X . $\square$

For any x, y in X , denote $x \wedge y = y * (y * x)$. Obviously, $x \wedge y$ is a lower bound of x and y , and $x \wedge x = x, x \wedge 0 = 0 \wedge x = 0$.

Theorem 3.9. *Let X be a strong KS-semigroup satisfying $x*(x*y) = y*(y*x)$. Then we have*

$$x \wedge y = xy.$$

Proof. For any $x, y \in X$, we have

$$\begin{aligned} x \wedge y = x * (x * y) &= x * (x * xy) = xy * (xy * x) \\ &= xy * x(y * 1) = xy * x0 \\ &= xy * 0 = xy \end{aligned}$$

$\square$

Proposition 3.10. *Let X be a strong KS-semigroup with 1 and satisfying $x * (x * y) = y * (y * x)$. Then the set*

$$\text{ann}(a) = \{x \in X \mid x \wedge a = 0, a \in X\}$$

is an ideal of X .

Proof. Let $y \in \text{ann}(a)$ and $x * y \in \text{ann}(a)$. Then we have $y \wedge a = ya = 0$, and $xa = xa * 0 = xa * ya \in \text{ann}(a)$. Hence we get $x \in \text{ann}(a)$. Also, let $x \in \text{ann}(a)$ and $s \in X$. Then we obtain $x \wedge a = xa = 0$, and so, $sx \wedge a = (sx)a = s(xa) = s0 = 0$. Thus $sx \in \text{ann}(a)$. Similarly, we have $xs \in \text{ann}(a)$. This completes the proof. $\square$

Let X be a strong KS-semigroup satisfying $x * (x * y) = y * (y * x)$. If $s \leq t$ for all $s, t \in X$, then we have $\text{ann}(s) \subseteq \text{ann}(t)$.

Definition 3.11. Let X be a KS-semigroup and let ρ be a binary relation on X . Then

- (1) ρ is said to be *right* (resp. *left*) *compatible* if whenever $(x, y) \in \rho$ then $(x*z, y*z) \in \rho$ (resp. $(z*x, z*y) \in \rho$) and $(xz, yz) \in \rho$ (resp. $(zx, zy) \in \rho$) for all $x, y, z \in X$;
- (2) ρ is said to be *compatible* if $(x, y) \in \rho$ and $(u, v) \in \rho$ imply $(x*u, y*v) \in \rho$ and $(xu, yv) \in \rho$ for all $x, y, u, v \in X$,
- (3) A compatible equivalence relation is called a *congruence relation*.

Using the notion of left (resp. right) compatible relation, we give a characterization of a congruence relation.

Theorem 3.12. *Let X be a KS-semigroup. Then an equivalence relation ρ on X is congruence if and only if it is both left and right compatible.*

Proof. Assume that ρ is a congruence relation on X . Let $x, y \in X$ be such that $(x, y) \in \rho$. Note that $(z, z) \in \rho$ for all $z \in X$ because ρ is reflexive. It follows from the compatibility of ρ that $(x * z, y * z) \in \rho$ and $(xz, yz) \in \rho$. Hence ρ is right compatible. Similarly, ρ is left compatible.

Conversely, suppose that ρ is both left and right compatible. Let $x, y, u, v \in X$ be such that $(x, y) \in \rho$ and $(u, v) \in \rho$. Then $(x * u, y * u) \in \rho$ and $(xu, yu) \in \rho$ by the right compatibility. Using the left compatibility of ρ , we have $(y * u, y * v) \in \rho$ and $(yu, yv) \in \rho$. It follows from the transitivity of ρ that $(x * u, y * v) \in \rho$ and $(xu, yv) \in \rho$. Hence ρ is congruence. $\square$

For a binary relation ρ on a KS-semigroup X , we denote

$$x\rho := \{y \in X \mid (x, y) \in \rho\} \text{ and } X/\rho := \{x\rho \mid x \in X\}.$$

Theorem 3.13. *Let ρ be a congruence relation on a KS-semigroup X . Then X/ρ is a KS-semigroup under the operations*

$$x\rho * y\rho = (x * y)\rho \text{ and } (x\rho)(y\rho) = (xy)\rho$$

for all $x\rho, y\rho \in X/\rho$.

Proof. Since ρ is a congruence relation, the operations are well-defined. Clearly, $(X/\rho, *)$ is a BCK-algebra and $(X/\rho, \cdot)$ is a semigroup. For every $x\rho, y\rho, z\rho \in X/\rho$, we have

$$\begin{aligned} x\rho(y\rho * z\rho) &= x\rho(y * z)\rho = x(y * z)\rho \\ &= (xy * xz)\rho = (xy)\rho * (xz)\rho \\ &= x\rho y\rho * x\rho z\rho, \end{aligned}$$

and

$$\begin{aligned} (x\rho * y\rho)z\rho &= (x * y)\rho z\rho = ((x * y)z)\rho \\ &= (xz * yz)\rho = (xz)\rho * (yz)\rho \\ &= x\rho z\rho * y\rho z\rho. \end{aligned}$$

Thus X/ρ is a KS-semigroup. $\square$

Definition 3.14. Let X and X' be KS-semigroups. A mapping $f : X \rightarrow X'$ is called a *KS-semigroup homomorphism* (briefly, *homomorphism*) if $f(x * y) = f(x) * f(y)$ and $f(xy) = f(x)f(y)$ for all $x, y \in X$.

Let $f : X \rightarrow Y$ be a homomorphism of KS-semigroup. Then the set $\{x \in X \mid f(x) = 0\}$ is called the *kernel* of f , and denote by $\ker f$. Moreover, the set $\{f(x) \in Y \mid x \in X\}$ is called the *image* of f , and denote by $\text{im } f$.

Lemma 3.15. *Let $f : X \rightarrow X'$ be a KS-semigroup homomorphism. Then*

- (1) $f(0) = 0$,
- (2) $x \leq y$ imply $f(x) \leq f(y)$.
- (3) $f(x \wedge y) = f(x) \wedge f(y)$.

Proof. (1). Suppose that x is an element of X . Then

$$f(0) = f(x * x) = f(x) * f(x) = 0$$

(2) Let $x \leq y$. Then we have $x * y = 0$. Thus we have

$$0 = f(x * y) = f(x) * f(y),$$

and so $f(x) \leq f(y)$.

(3) $f(x \wedge y) = f(y * (y * x)) = f(y) * (f(y) * f(x)) = f(x) \wedge f(y)$. $\square$

Proposition 3.16. *Let $f : X \rightarrow X'$ be a KS-semigroup homomorphism and $J = f^{-1}(0) = \{0\}$. Then $f(x) \leq f(y)$ imply $x \leq y$.*

Proof. If $f(x) \leq f(y)$, then we have $f(x) * f(y) = f(x * y) = 0$, and so $x * y$ is an element of J . Hence $x * y = 0$, and so we obtain $x \leq y$. $\square$

Theorem 3.17. *Let ρ be a congruence relation on a KS-semigroup X . Then the mapping $\rho^* : X \rightarrow X/\rho$ defined by $\rho^*(x) = x\rho$ for all $x \in X$ is a KS-semigroup homomorphism.*

Proof. Let $x, y \in X$. Then $\rho^*(x * y) = (x * y)\rho = x\rho * y\rho = \rho^*(x) * \rho^*(y)$, and $\rho^*(xy) = (xy)\rho = (x\rho)(y\rho) = \rho^*(x)\rho^*(y)$. Hence ρ^* is a KS-semigroup homomorphism. $\square$

Theorem 3.18. *Let X and X' be KS-semigroups and let $f : X \rightarrow X'$ be a KS-semigroup homomorphism. Then the set*

$$K_f := \{(x, y) \in X \times X \mid f(x) = f(y)\}$$

is a congruence relation on X and there exists a unique 1-1 KS-semigroup homomorphism $\bar{f} : X/K_f \rightarrow X'$ such that $\bar{f} \circ K_f^ = f$, where $K_f^* : X \rightarrow X/K_f$. That is, the following diagram commute:*

$$\begin{array}{ccc} X & \xrightarrow{K_f^*} & X/K_f \\ f \downarrow & \nearrow \bar{f} & \\ X' & & \end{array}$$

Proof. It is clear that K_f is an equivalence relation on X . Let $x, y, u, v \in X$ be such that $(x, y), (u, v) \in K_f$. Then $f(x) = f(y)$ and $f(u) = f(v)$, which imply that

$$f(x * u) = f(x) * f(u) = f(y) * f(v) = f(y * v)$$

and

$$f(xu) = f(x)f(u) = f(y)f(v) = f(yv).$$

It follows that $(x * u, y * v) \in K_f$ and $(xu, yv) \in K_f$. Hence K_f is a congruence relation on X . Let $\bar{f} : X/K_f \rightarrow X'$ be a map defined by $\bar{f}(xK_f) = f(x)$ for all $x \in X$. It is clear that $\bar{f}$ is well-defined. For any $xK_f, yK_f \in X/K_f$, we have

$$\begin{aligned} \bar{f}(xK_f * yK_f) &= \bar{f}((x * y)K_f) = f(x * y) \\ &= f(x) * f(y) = \bar{f}(xK_f) * \bar{f}(yK_f) \end{aligned}$$

and

$$\begin{aligned}\bar{f}((xK_f)(yK_f)) &= \bar{f}((xy)K_f) = f(xy) \\ &= f(x)f(y) = \bar{f}(xK_f)\bar{f}(yK_f).\end{aligned}$$

If $\bar{f}(xK_f) = \bar{f}(yK_f)$, then $f(x) = f(y)$ and so $(x, y) \in K_f$, that is, $xK_f = yK_f$. Thus $\bar{f}$ is a 1-1 KS-semigroup homomorphism. Now let g be a KS-semigroup homomorphism from X/K_f to X' such that $g \circ K_f^* = f$. Then

$$g(xK_f) = g(K_f^*(x)) = f(x) = \bar{f}(xK_f)$$

for all $xK_f \in X/K_f$. It follows that $g = \bar{f}$ so that $\bar{f}$ is unique. This completes the proof. $\square$

Corollary 3.19. *Let ρ and σ be congruence relations on a KS-semigroup X such that $\rho \subseteq \sigma$. Then the set*

$$\sigma/\rho := \{(x\rho, y\rho) \in X/\rho \times X/\rho \mid (x, y) \in \sigma\}$$

is a congruence relation on X/ρ and there exists a 1-1 and onto KS-semigroup homomorphism from $\frac{X/\rho}{\sigma/\rho}$ to X/σ .

Proof. Let $g : X/\rho \rightarrow X/\sigma$ be a function defined by $g(x\rho) = x\sigma$ for all $x\rho \in X/\rho$. Since $\rho \subseteq \sigma$, it follows that g is a well-defined onto KS-semigroup homomorphism. According to Theorem 3.18, it is sufficient to show that $K_g = \sigma/\rho$. Let $(x\rho, y\rho) \in K_g$. Then $x\sigma = g(x\rho) = g(y\rho) = y\sigma$ and so $(x, y) \in \sigma$. Hence $(x\rho, y\rho) \in \sigma/\rho$, and thus $K_g \subseteq \sigma/\rho$.

Conversely, if $(x\rho, y\rho) \in \sigma/\rho$, then $(x, y) \in \sigma$ and so $x\sigma = y\sigma$. It follows that

$$g(x\rho) = x\sigma = y\sigma = g(y\rho)$$

so that $(x\rho, y\rho) \in K_g$. Hence $K_g = \sigma/\rho$, and the proof is complete. $\square$

Theorem 3.20. *Let I be an ideal of a KS-semigroup X . Then $\rho_I := (I \times I) \cup \Delta_X$ is a congruence relation on X , where $\Delta_X := \{(x, x) \mid x \in X\}$.*

Proof. Clearly, ρ_I is reflexive and symmetric. Noticing that $(x, y) \in \rho_I$ if and only if $x, y \in I$ or $x = y$, we know that if $(x, y) \in \rho_I$ and $(y, z) \in \rho_I$ then $(x, z) \in \rho_I$. Hence ρ_I is an equivalence relation on X . Assume that $(x, y) \in \rho_I$ and $(u, v) \in \rho_I$. Then we have the following four cases: (i) $x, y \in I$ and $u, v \in I$; (ii) $x, y \in I$ and $u = v$; (iii) $x = y$ and $u, v \in I$; and (iv) $x = y$ and $u = v$. In either case, we get $x * u = y * v$ or $(x * u, y * v) \in I \times I$, and $xu = yv$ or $(xu, yv) \in I \times I$. Therefore ρ_I is a congruence relation on X . $\square$

Theorem 3.21. *Let $f : X \rightarrow Y$ be a homomorphism of a KS-semigroup. Then $\ker f$ is an ideal of X .*

Proof. Let $x * y \in \ker f$ and $y \in \ker f$. Then $0 = f(x * y) = f(x) * f(y) = f(x) * 0 = f(x)$. Hence we have $x \in \ker f$. Let $x \in X$ and $a \in \ker f$. Then we

obtain $f(ax) = f(a)f(x) = 0f(a) = 0$ and $f(xa) = f(x)f(a) = f(x)0 = 0$. Hence $ax, xa \in \ker f$. This completes the proof. $\square$

Let A be an ideal of a KS-semigroup X . We define a relation “ $\sim$ ” on X as follows

$$x \underset{A}{\sim} y \text{ if and only if } x * y \in A \text{ and } y * x \in A.$$

Then “ $\sim$ ” is a congruence relation on X .

Let X be a KS-semigroup and denote by A_x the equivalence class containing $x \in X$, and by X/A the set of all equivalence classes of X with respect to “ $\sim$ ”, that is,

$$A_x := \{y \in X \mid x \underset{A}{\sim} y\} \text{ and } X/A := \{A_x \mid x \in X\}.$$

Note that $A_x = A_y$ if and only if $x \underset{A}{\sim} y$, and $A = A_0$.

Theorem 3.22. *If A is an ideal of a KS-semigroup X , then mapping $\psi : X \rightarrow X/A$ given by $\psi(x) = A_x$ is an epimorphism with kernel A .*

Proof. The map $\psi : X \rightarrow X/A$ is clearly surjective and since $\psi(x * y) = A_{x*y} = A_x * A_y = \psi(x) \otimes \psi(y)$ and $\psi(xy) = A_{xy} = A_x \odot A_y = \psi(x) \odot \psi(y)$, ψ is an epimorphism. Now $\ker \psi = \{x \in X \mid \psi(x) = A_x = A_0\} = \{x \in X \mid x \in A\} = A_0$. $\square$

Theorem 3.23. *If A is an ideal of a KS-semigroup X , then $(X/A, \otimes, \odot, A_0)$ is an KS-semigroup under the binary operations*

$$A_x \otimes A_y = A_{x*y} \text{ and } A_x \odot A_y = A_{xy}$$

for all $A_x, A_y \in X/A$.

Proof. We are easy to prove that $(X/A, \otimes, \odot, A_0)$ is a BCK-algebra. First we show that $\odot$ is well-defined. Let $A_x = A_u$ and $A_y = A_v$. Then we have $xy * xv = x(y * v) \in A$ and $xv * xy = x(v * y) \in A$. Thus $xy \underset{A}{\sim} xv$. On the other hand, $xv * uv = (x * u)v \in A$ and $uv * xv = (u * x)v \in A$. Hence, $xv \underset{A}{\sim} uv$, and so $A_{xy} = A_{uv}$. Therefore, $(X/A, \odot)$ is a semigroup. Moreover, for any $A_x, A_y, A_z \in X/A$, we obtain $A_x \odot (A_y \otimes A_z) = A_x \odot A_{y*z} = A_{x(y*z)} = A_{xy*xz} = A_{xy} \otimes A_{xz} = (A_x \odot A_y) \otimes (A_x \odot A_z)$. Similarly, we get $(A_x \otimes A_y) \odot A_z = (A_x \odot A_z) \otimes (A_y \odot A_z)$. Therefore, X/A is a KS-semigroup. $\square$

Theorem 3.24. *Let $f : X \rightarrow Y$ be a homomorphism of KS-semigroups. Then $\ker f$ is an ideal of X .*

Proof. Let $x * y \in \ker f$ and $y \in \ker f$. Then $0 = f(x * y) = f(x) * f(y) = f(x) * 0 = f(x)$. Hence we have $x \in \ker f$. Let $x \in X$ and $a \in \ker f$. Then by Lemma 2.2, we obtain $f(ax) = f(a)f(x) = 0f(a) = 0$ and $f(xa) = f(x)f(a) = f(x)0 = 0$. Hence $xa, ax \in \ker f$. $\square$

Theorem 3.25. *Let $f : X \rightarrow Y$ be a homomorphism of a KS-semigroup. Then for any ideal of X , $A/(\ker f \cap A) = f(A)$.*

Proof. Clearly, $\ker f \cap A$ is an ideal of A . Let $B = \ker f \cap A$. We define a mapping $\psi : A/B \rightarrow Y$ with $\psi(B_x) = f(x)$ where $x \in A$. Then for any $B_x, B_y \in A/B$, we have

$$\begin{aligned} B_x = B_y &\Leftrightarrow x * y \in B, y * x \in B, \\ &\Leftrightarrow f(x * y) = 0, f(y * x) = 0, \\ &\Leftrightarrow f(x) * f(y) = 0, f(y) * f(x) = 0, \\ &\Leftrightarrow f(x) = f(y), \\ &\Leftrightarrow \psi(B_x) = \psi(B_y). \end{aligned}$$

Hence, ψ is well-defined and one to one. Moreover, for all $B_x, B_y \in A/B$, we get

$$\psi(B_x \otimes B_y) = \psi(B_{x*y}) = f(x * y) = f(x) * f(y) = \psi(B_x) * \psi(B_y),$$

and

$$\psi(B_x \odot B_y) = \psi(B_{xy}) = f(xy) = f(x)f(y) = \psi(B_x)\psi(B_y).$$

So ψ is a homomorphism of a KS-semigroup. Thus we obtain $\text{im } \psi = \{\psi(B_x) \mid x \in A\} = \{f(x) \mid x \in A\} = f(A)$. Therefore $A/(\ker f \cap A) = f(A)$. $\square$

Corollary 3.26. (First Isomorphism Theorem) *If $f : X \rightarrow Y$ is a surjective homomorphism of a KS-semigroup, then $X/\ker f$ is isomorphic to Y .*

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