ON STRUCUTRE OF KS-SEMIGROUPS

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Abstract

In this paper, we introduce a new class algebras related to BCK-algebras and semigroups, called a KS-semigroup and define an ideal of a KS-semigroups and a strong KS-semigroup, and characterizations of ideals is given. Also we define a congruence relation on a KS-semigroups and a quotient KS-semigroups and prove the isomorphism.

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Keywords: KS-semigroup, strong KS-semigroup, P-ideal, quotient KS-semigroup, homomorphism

1. Introduction

The notion of BCK-algebras was proposed by Y. Imai and K. Iséki in 1966. In the same year, K. Iséki introduced the notion of BCI-algebra which is a generalization of a BCK-algebra. For the general development of BCK/BCI-algebras, the ideal theory plays an important role. We introduce a new class algebras related to BCK-algebras and semigroups, called a KS-semigroup. In this paper, we define an ideal of a KS-semigroups and a strong KS-semigroup, and characterizations of ideals is given. Also we define a congruence relation on a KS-semigroups and a quotient KS-semigroups and prove the isomorphism.

2. Preliminaries

We review some definitions and properties that will be useful in our results. 
A BCI-algebra is a triple \((X, \ast, 0)\), where \(X\) is a nonempty set, \(\ast\) is a binary operation on \(X\), \(0 \in H\) is an element such that the following four axioms are satisfied for every \(x, y, z \in X\):
\begin{align*}
(\text{I}) \quad (x \ast y) \ast (x \ast z) \ast (z \ast y) &= 0, \\
(\text{II}) \quad (x \ast (x \ast y)) \ast y &= 0, \\
(\text{III}) \quad x \ast x &= 0, \\
(\text{IV}) \quad x \ast y = 0, y \ast x = 0 \text{ implies } x = y.
\end{align*}

A BCI-algebra \( X \) satisfying \( 0 \ast x = 0 \) for all \( x \in X \) is called a \textit{BCK-algebra}. If \( X \) is a BCK-algebra, then the relation \( x \leq y \) if and only if \( x \ast y = 0 \) is a partial order on \( X \), which will be called the \textit{natural ordering} on \( X \).

A BCK-algebra \( X \) has the following properties for any \( x, y, z \in X \):

\begin{enumerate}
\item \( x \ast 0 = x \),
\item \( (x \ast y) \ast z = (x \ast z) \ast y \),
\item \( x \leq y \) implies that \( x \ast z \leq y \ast z \) and \( z \ast y \leq z \ast x \),
\item \( (x \ast z) \ast (y \ast z) \leq x \ast y \),
\end{enumerate}

A nonempty subset \( I \) of a BCK/BCI-algebra is called an \textit{ideal} if it satisfies

\begin{enumerate}
\item \( 0 \in X \),
\item \( x \ast y \in X \) and \( y \in X \) imply \( x \in X \) for all \( x, y \in X \).
\end{enumerate}

Any ideal \( I \) has the property: \( y \in I \) and \( x \leq y \) imply \( x \in I \).

For a BCK-algebra \( X \), the set \( X_+ := \{ x \in X \mid 0 \leq x \} \) is called the \textit{BCK-part} of \( X \). If \( X_+ = \{0\} \), then we say that \( X \) is a \textit{p-semisimple} BCI-algebra. Note that a BCI-algebra \( X \) is p-semisimple if and only if \( 0 \ast (0 \ast x) = x \) for all \( x \in X \).

An KS-semigroup is a non-empty set \( X \) with two binary operation “\( \ast \)” and “\( \cdot \)” and constant 0 satisfying the axioms:

\begin{enumerate}
\item \( (X, \ast, 0) \) is a BCK-algebra,
\item \( (X, \cdot) \) is a semigroup,
\item The operation “\( \cdot \)” is distributive (on both sides) over the operation “\( \ast \)” ,
\end{enumerate}

that is, \( x \cdot (y \ast z) = (x \cdot y) \ast (x \cdot z) \) and \( (x \ast y) \cdot z = (x \cdot z) \ast (y \cdot z) \) for all \( x, y, z \in X \).

We shall write the multiplication \( x \cdot y \) by \( xy \) for convenience.

\textbf{Example 2.1.} Let \( X = \{0, a, b, c, d\} \). Define \( \ast \)-operation and multiplication “\( \cdot \)” by the following tables

\begin{table}[h]
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\begin{tabular}{c|cccc}
\hline
\( \ast \) & 0 & a & b & c & d \\
\hline
0 & 0 & 0 & 0 & 0 & 0 \\
a & a & 0 & a & a & 0 \\
b & b & b & 0 & 0 & 0 \\
c & c & c & c & 0 & 0 \\
d & d & d & d & d & d \\
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and

\begin{table}[h]
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\begin{tabular}{c|cccc}
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\( \cdot \) & 0 & a & b & c & d \\
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0 & 0 & 0 & 0 & 0 & 0 \\
a & a & 0 & a & a & 0 \\
b & b & 0 & 0 & 0 & b \\
c & c & 0 & 0 & b & c \\
d & d & 0 & a & b & c \\
\hline
\end{tabular}
\end{table}

Then, by routine calculations, we can see that \( X \) is an KS-semigroup.

\textbf{Lemma 2.2.} Let \( X \) be an KS-semigroup. Then we have

\begin{enumerate}
\item \( 0x = x0 = 0 \),
\item \( x \leq y \) implies that \( xz \leq yz \) and \( zx \leq zy \), for all \( x, y, z \in X \).
\end{enumerate}
A non-empty subset $A$ of a semigroup $(X, \cdot)$ is said to be left (resp. right) stable if $xa \in A$ (resp. $ax \in A$) whenever $x \in X$ and $a \in A$. Both left and right stable is two-sided stable or simply stable.

**Definition 2.3.** A non-empty subset $A$ of a KS-semigroup $X$ is called a left (resp. right) ideal of $X$ if

(i) $A$ is a left (resp. right) stable subset of $(X, \cdot)$,

(ii) for any $x, y \in X, x \cdot y \in A$ and $y \in A$ imply that $x \in A$.

A both of left and right ideal is called a two-sided ideal or simply an ideal.

Note that $\{0\}$ and $X$ are ideals. If $A$ is a left (resp. right) ideal of an KS-semigroup of $X$, then $0 \in A$. Thus $A$ is an ideal of $X$.

**Example 2.4.** Let $X = \{0, a, b, c\}$. Define $\ast$-operation and multiplication “.$\cdot$.“ by the following tables

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Then, by routine calculations, we can see that $X$ is an KS-semigroup. If $A = \{0, a\}$, then $A$ is an ideal of a KS-semigroup $X$.

**Definition 2.5.** An ideal $A$ of an KS-semigroup $X$ is said to be closed if $x \in X$ implies $0 \ast x \in A$.

3. **Main results**

In what follows, let $X$ denote a KS-semigroup unless otherwise specified.

**Definition 3.1.** A non-empty subset $A$ of a KS-semigroup $X$ is called a left (resp. right) $P$-ideal of $X$ if

(i) $A$ is a left (resp. right) stable subset of $(X, \cdot)$,

(ii) for any $x, y, z \in X, (x \ast y) \ast z \in A$ and $y \ast z \in A$ imply that $x \ast z \in A$.

**Example 3.2.** Let $X = \{0, a, b, c\}$. Define $\ast$-operation and multiplication “.$\cdot$.“ by the following tables

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Then, by routine calculations, we can see that $X$ is an KS-semigroup. Let $A = \{0, a\}$. Then $A$ is a $P$-ideal of a KS-semigroup $X$.

**Example 3.3.** Let $X = \{0, a, b, c\}$. Define $\ast$-operation and multiplication “.$\cdot$.“ by the following tables
Then, by routine calculations, we can see that $X$ is an KS-semigroup. Let $A = \{0, a\}$. Then $A$ is a P-ideal of a KS-semigroup $X$.

**Theorem 3.4.** Every P-ideal of a KS-semigroup $X$ is an ideal but the converse is not true.

**Proof.** Suppose that $A$ is a P-ideal of a KS-semigroup. Let $x, y \in X$ be such that $x \ast y \in A$ and $y \in A$. It follows from (1) that

$$(x \ast y) \ast 0 = x \ast y \in A \text{ and } y \ast 0 = y \in A.$$ 

Therefore, by Definition 3.1(ii), we obtain $x = x \ast 0 \in A$. So, $A$ is an ideal of $X$. In Example 3.2, $\{0, b\}$ is an ideal but not a P-ideal of $X$ because $(a \ast c) \ast b = a \ast b = 0 \notin \{0, b\}$ but $c \ast b = c \notin \{0, b\}$.

**Definition 3.5.** A strong KS-semigroup is a KS-semigroup $X$ satisfying $x \ast y = x \ast xy$ for each $x, y \in X$.

**Example 3.6.** In Example 2.4, we are easy to prove that $(X, \ast, \cdot, 0)$ is a strong KS-semigroup.

**Lemma 3.7.** Let $X$ be a strong KS-semigroup. Then

(i) $xy \ast y = 0$ for all $x, y \in X$,  
(ii) $x \ast y = 0$ if and only if $x \ast xy = 0$ for any $x, y \in X$.

**Proof.** (1) For any $x, y \in X$, $xy \ast y = xy \ast (xy) = xy \ast x(yy) = x(y \ast y) = 0$.

(2) It is easy to show from the definition of strong KS-semigroup and the above (1). $\square$

The element 1 is called a unity in a KS-semigroup $X$ if $1x = x1 = x$ for all $x \in X$. If $X$ is a strong KS-semigroup with a unity 1, then 1 is the greatest element in $X$ since $x \ast 1 = x \ast x1 = x \ast x = 0$ for all $x \in X$.

**Theorem 3.8.** Let $X$ be a strong KS-semigroup with a unity 1 and $A$ any non-empty subset of $X$. If $y \in A$ and $x \leq y$ imply $x \in A$, then $A$ is an ideal of $X$.

**Proof.** Suppose that $y \in A$ and $x \leq y$ imply $x \in A$. If $x, y \in A$, then $x \ast y \in A$, since $x \ast y = x \ast xy = x(1 \ast y) \leq x1 = x \in A$. Next, let $s \in X$ and $a \in A$. Then

$$as \ast a = a(s \ast 1) = a0 = 0$$
and

\[ sa * a = (s * 1)a = 0a = 0, \]

hence \( as \leq a \) and \( sa \leq a \), that is, \( AX \subseteq A \) and \(XA \subseteq A \). It follows that \( A \) is an ideal of \( X \).

For any \( x, y \) in \( X \), denote \( x \land y = y * (y * x) \). Obviously, \( x \land y \) is a lower bound of \( x \) and \( y \), and \( x \land x = x, x \land 0 = 0 \land x = 0 \).

**Theorem 3.9.** Let \( X \) be a strong KS-semigroup satisfying \( x * (x * y) = y * (y * x) \). Then we have

\[ x \land y = xy. \]

**Proof.** For any \( x, y \in X \), we have

\[
x \land y = x * (x * y) = x * (x * xy) = xy * (xy * x) \\
= xy * x(y * 1) = xy * 0 \\
= xy * 0 = xy
\]

\[ \square \]

**Proposition 3.10.** Let \( X \) be a strong KS-semigroup with \( 1 \) and satisfying \( x * (x * y) = y * (y * x) \). Then the set

\[ \text{ann}(a) = \{ x \in X \mid x \land a = 0, a \in X \} \]

is an ideal of \( X \).

**Proof.** Let \( y \in \text{ann}(a) \) and \( x * y \in \text{ann}(a) \). Then we have \( y \land a = ya = 0 \), and \( xa = xa * 0 = xa * ya \in \text{ann}(a) \). Hence we get \( x \in \text{ann}(a) \). Also, let \( x \in \text{ann}(a) \) and \( s \in X \). Then we obtain \( x \land a = xa = 0 \), and so, \( sx \land a = (sx)a = s(xa) = s0 = 0 \). Thus \( sx \in \text{ann}(a) \). Similarly, we have \( xs \in \text{ann}(a) \). This completes the proof.

Let \( X \) be a strong KS-semigroup satisfying \( x * (x * y) = y * (y * x) \). If \( s \leq t \) for all \( s, t \in X \), then we have \( \text{ann}(s) \subseteq \text{ann}(t) \).

**Definition 3.11.** Let \( X \) be a KS-semigroup and let \( \rho \) be a binary relation on \( X \). Then

1. \( \rho \) is said to be right (resp. left) compatible if whenever \( (x, y) \in \rho \) then \( (x * z, y * z) \in \rho \) (resp. \( (z * x, z * y) \in \rho \)) and \( (x z, y z) \in \rho \) (resp. \( (z x, z y) \in \rho \)) for all \( x, y, z \in X \);
2. \( \rho \) is said to be compatible if \( (x, y) \in \rho \) and \( (u, v) \in \rho \) imply \( (x * u, y * v) \in \rho \) and \( (x u, y v) \in \rho \) for all \( x, y, u, v \in X \);
3. A compatible equivalence relation is called a congruence relation.

Using the notion of left (resp. right) compatible relation, we give a characterization of a congruence relation.

**Theorem 3.12.** Let \( X \) be a KS-semigroup. Then an equivalence relation \( \rho \) on \( X \) is congruence if and only if it is both left and right compatible.
Proof. Assume that \( \rho \) is a congruence relation on \( X \). Let \( x, y \in X \) be such that \((x, y) \in \rho\). Note that \((z, z) \in \rho\) for all \( z \in X \) because \( \rho \) is reflexive. It follows from the compatibility of \( \rho \) that \((x \ast z, y \ast z) \in \rho\) and \((xz, yz) \in \rho\). Hence \( \rho \) is right compatible. Similarly, \( \rho \) is left compatible.

Conversely, suppose that \( \rho \) is both left and right compatible. Let \( x, y, u, v \in X \) be such that \((x, y) \in \rho\) and \((u, v) \in \rho\). Then \((x \ast u, y \ast u) \in \rho\) and \((xu, yu) \in \rho\) by the right compatibility. Using the left compatibility of \( \rho \), we have \((y \ast u, y \ast v) \in \rho\) and \((yu, yv) \in \rho\). It follows from the transitivity of \( \rho \) that \((x \ast u, y \ast v) \in \rho\) and \((xu, yv) \in \rho\). Hence \( \rho \) is congruence.

For a binary relation \( \rho \) on a KS-semigroup \( X \), we denote
\[
x \rho := \{y \in X \mid (x, y) \in \rho\} \quad \text{and} \quad X/\rho := \{x \rho \mid x \in X\}.
\]

**Theorem 3.13.** Let \( \rho \) be a congruence relation on a KS-semigroup \( X \). Then \( X/\rho \) is a KS-semigroup under the operations
\[
x \rho \ast y \rho = (x \ast y) \rho \quad \text{and} \quad (x \rho)(y \rho) = (xy) \rho
\]
for all \( x \rho, y \rho \in X/\rho \).

**Proof.** Since \( \rho \) is a congruence relation, the operations are well-defined. Clearly, \((X/\rho, \ast)\) is a BCK-algebra and \((X/\rho, \cdot)\) is a semigroup. For every \( x \rho, y \rho, z \rho \in X/\rho \), we have
\[
x \rho(y \rho \ast z \rho) = x \rho(y \ast z) \rho = x(y \ast z) \rho
\]
\[
= (xy \ast xz) \rho = (xy) \rho \ast (xz) \rho
\]
\[
= x \rho y \rho \ast x \rho z \rho,
\]
and
\[
(x \rho \ast y \rho) z \rho = (x \ast y) \rho z \rho = ((x \ast y)z) \rho
\]
\[
= (xz \ast yz) \rho = (xz) \rho \ast (yz) \rho
\]
\[
= x \rho z \rho \ast y \rho z \rho.
\]
Thus \( X/\rho \) is a KS-semigroup. \( \square \)

**Definition 3.14.** Let \( X \) and \( X' \) be KS-semigroups. A mapping \( f : X \rightarrow X' \) is called a **KS-semigroup homomorphism** (briefly, **homomorphism**) if \( f(x \ast y) = f(x) \ast f(y) \) and \( f(xy) = f(x)f(y) \) for all \( x, y \in X \).

Let \( f : X \rightarrow Y \) be a homomorphism of KS-semigroup. Then the set \( \{x \in X \mid f(x) = 0\} \) is called the **kernel** of \( f \), and denote by \( \ker f \). Moreover, the set \( \{f(x) \in Y \mid x \in X\} \) is called the **image** of \( f \), and denote by \( \text{im} f \).

**Lemma 3.15.** Let \( f : X \rightarrow X' \) be a KS-semigroup homomorphism. Then
\[
(1) \quad f(0) = 0,
\]
\[
(2) \quad x \leq y \text{ imply } f(x) \leq f(y).
\]
\[
(3) \quad f(x \wedge y) = f(x) \wedge f(y).
\]

**Proof.** (1) Suppose that \( x \) is an element of \( X \). Then
\[
f(0) = f(x \ast x) = f(x) \ast f(x) = 0
\]
(2) Let $x \leq y$. Then we have $x \ast y = 0$. Thus we have
$$0 = f(x \ast y) = f(x) \ast f(y),$$
and so $f(x) \leq f(y)$.

(3) $f(x \wedge y) = f(y \ast (y \ast x)) = f(y) \ast (f(y) \ast f(x)) = f(x) \wedge f(y)$. \hfill \Box

**Proposition 3.16.** Let $f : X \to X'$ be a KS-semigroup homomorphism and $J = f^{-1}(0) = \{0\}$. Then $f(x) \leq f(y)$ imply $x \leq y$.

**Proof.** If $f(x) \leq f(y)$, then we have $f(x) \ast f(y) = f(x \ast y) = 0$, and so $x \ast y$ is an element of $J$. Hence $x \ast y = 0$, and so we obtain $x \leq y$. \hfill \Box

**Theorem 3.17.** Let $\rho$ be a congruence relation on a KS-semigroup $X$. Then the mapping $\rho^* : X \to X/\rho$ defined by $\rho^*(x) = x\rho$ for all $x \in X$ is a KS-semigroup homomorphism.

**Proof.** Let $x, y \in X$. Then $\rho^*(x \ast y) = (x \ast y)\rho = x\rho \ast y\rho = \rho^*(x) \ast \rho^*(y)$, and $\rho^*(xy) = (xy)\rho = (x\rho)(y\rho) = \rho^*(x) \ast \rho^*(y)$. Hence $\rho^*$ is a KS-semigroup homomorphism. \hfill \Box

**Theorem 3.18.** Let $X$ and $X'$ be KS-semigroups and let $f : X \to X'$ be a KS-semigroup homomorphism. Then the set

$$K_f := \{(x, y) \in X \times X \mid f(x) = f(y)\}$$

is a congruence relation on $X$ and there exists a unique 1-1 KS-semigroup homomorphism $\bar{f} : X/K_f \to X'$ such that $\bar{f} \circ K^*_f = f$, where $K^*_f : X \to X/K_f$.

**Proof.** It is clear that $K_f$ is an equivalence relation on $X$. Let $x, y, u, v \in X$ be such that $(x, y), (u, v) \in K_f$. Then $f(x) = f(y)$ and $f(u) = f(v)$, which imply that
$$f(x \ast u) = f(x) \ast f(u) = f(y) \ast f(v) = f(y \ast v)$$
and
$$f(xu) = f(x)f(u) = f(y)f(v) = f(yv).$$

It follows that $(x \ast u, y \ast v) \in K_f$ and $(xu, yv) \in K_f$. Hence $K_f$ is a congruence relation on $X$. Let $\bar{f} : X/K_f \to X'$ be a map defined by $\bar{f}(xK_f) = f(x)$ for all $x \in X$. It is clear that $\bar{f}$ is well-defined. For any $xK_f, yK_f \in X/K_f$, we have
\[
\bar{f}(xK_f \ast yK_f) = \bar{f}((x \ast y)K_f) = f(x \ast y) = f(x) \ast f(y) = f(xK_f) \ast f(yK_f)
\]
and

\[ \tilde{f}(xK_f(yK_f)) = \tilde{f}(xK_f) = f(xy) = f(x)f(y) = \tilde{f}(xK_f)\tilde{f}(yK_f). \]

If \( \tilde{f}(xK_f) = \tilde{f}(yK_f) \), then \( f(x) = f(y) \) and so \( (x, y) \in K_f \), that is, \( xK_f = yK_f \). Thus \( \tilde{f} \) is a 1-1 KS-semigroup homomorphism. Now let \( g \) be a KS-semigroup homomorphism from \( X/K_f \) to \( X' \) such that \( g \circ K_f = \tilde{f} \). Then

\[ g(xK_f) = g(K_f^*(x)) = f(x) = \tilde{f}(xK_f) \]

for all \( xK_f \in X/K_f \). It follows that \( g = \tilde{f} \) so that \( \tilde{f} \) is unique. This completes the proof.

\[ \square \]

**Corollary 3.19.** Let \( \rho \) and \( \sigma \) be congruence relations on a KS-semigroup \( X \) such that \( \rho \subseteq \sigma \). Then the set

\[ \sigma/\rho := \{(x\rho, y\rho) \in X/\rho \times X/\rho | (x, y) \in \sigma \} \]

is a congruence relation on \( X/\rho \) and there exists a 1-1 and onto KS-semigroup homomorphism from \( X/\rho \) to \( X/\sigma \).

**Proof.** Let \( g : X/\rho \rightarrow X/\sigma \) be a function defined by \( g(x\rho) = x\sigma \) for all \( x\rho \in X/\rho \). Since \( \rho \subseteq \sigma \), it follows that \( g \) is a well-defined onto KS-semigroup homomorphism. According to Theorem 3.18, it is sufficient to show that \( K_g = \sigma/\rho \). Let \( (x\rho, y\rho) \in K_g \). Then \( x\sigma = g(x\rho) = g(y\rho) = y\sigma \) and so \( (x, y) \in \sigma \). Hence \( (x\rho, y\rho) \in \sigma/\rho \), and thus \( K_g \subseteq \sigma/\rho \).

Conversely, if \( (x\rho, y\rho) \in \sigma/\rho \), then \( (x, y) \in \sigma \) and so \( x\sigma = y\sigma \). It follows that

\[ g(x\rho) = x\sigma = y\sigma = g(y\rho) \]

so that \( (x\rho, y\rho) \in K_g \). Hence \( K_g = \sigma/\rho \), and the proof is complete.

\[ \square \]

**Theorem 3.20.** Let \( I \) be an ideal of a KS-semigroup \( X \). Then \( \rho_I := (I \times I) \cup \Delta_X \) is a congruence relation on \( X \), where \( \Delta_X := \{(x, x) \mid x \in X\} \).

**Proof.** Clearly, \( \rho_I \) is reflexive and symmetric. Noticing that \( (x, y) \in \rho_I \) if and only if \( x, y \in I \) or \( x = y \), we know that if \( (x, y) \in \rho_I \) and \( (y, z) \in \rho_I \) then \( (x, z) \in \rho_I \). Hence \( \rho_I \) is an equivalence relation on \( X \). Assume that \( (x, y) \in \rho_I \) and \( (u, v) \in \rho_I \). Then we have the following four cases: (i) \( x, y \in I \) and \( u, v \in I \); (ii) \( x, y \in I \) and \( u = v \); (iii) \( x = y \) and \( u, v \in I \); and (iv) \( x = y \) and \( u = v \). In either case, we get \( x * u = y * v \) or \( (x * u, y * v) \in I \times I \), and \( xu = yv \) or \( (xu, yv) \in I \times I \). Therefore \( \rho_I \) is a congruence relation on \( X \).

\[ \square \]

**Theorem 3.21.** Let \( f : X \rightarrow Y \) be a homomorphism of a KS-semigroup. Then \( \ker f \) is an ideal of \( X \).

**Proof.** Let \( xy \in \ker f \) and \( y \in \ker f \). Then \( 0 = f(xy) = f(x) * f(y) = f(x) * 0 = f(x) \). Hence we have \( x \in \ker f \). Let \( x \in X \) and \( a \in \ker f \). Then we
obtain \( f(ax) = f(a)f(x) = 0 \) and \( f(xa) = f(x)f(a) = f(x)0 = 0 \). Hence \( ax, xa \in \ker f \). This completes the proof.

Let \( A \) be an ideal of a KS-semigroup \( X \). We define a relation \( \sim \) on \( X \) as follows

\[
x \sim_A y \text{ if and only if } x \ast y \in A \text{ and } y \ast x \in A.
\]

Then \( \sim \) is a congruence relation on \( X \).

Let \( X \) be a KS-semigroup and denote by \( A_x \) the equivalence class containing \( x \in X \), and by \( X/A \) the set of all equivalence classes of \( X \) with respect to \( \sim \), that is,

\[
A_x := \{ y \in X | x \sim_A y \} \quad \text{and} \quad X/A := \{ A_x | x \in X \}.
\]

Note that \( A_x = A_y \) if and only if \( x \sim_A y \), and \( A = A_0 \).

**Theorem 3.22.** If \( A \) is an ideal of a KS-semigroup \( X \), then mapping \( \psi : X \to X/A \) given by \( \psi(x) = A_x \) is an epimorphism with kernel \( A \).

**Proof.** The map \( \psi : X \to X/A \) is clearly surjective and since \( \psi(x \ast y) = A_{x \ast y} = A_x \ast A_y = \psi(x) \otimes \psi(y) \) and \( \psi(xy) = A_{xy} = A_x \circ A_y = \psi(x) \circ \psi(y) \), \( \psi \) is an epimorphism. Now \( \ker \psi = \{ x \in X | \psi(x) = A_x = A_0 \} = \{ x \in X | x \in A \} = A_0 \).

**Theorem 3.23.** If \( A \) is an ideal of a KS-semigroup \( X \), then \( (X/A, \otimes, \circ, A_0) \) is an KS-semigroup under the binary operations

\[
A_x \otimes A_y = A_{x \ast y} \quad \text{and} \quad A_x \circ A_y = A_{xy}
\]

for all \( A_x, A_y \in X/A \).

**Proof.** We are easy to prove that \( (X/A, \otimes, \circ, A_0) \) is a BCK-algebra. First we show that \( \circ \) is well-defined. Let \( A_x = A_y \) and \( A_y = A_z \). Then we have \( xy \ast xv = x(y \ast v) \in A \) and \( xv \ast xy = x(v \ast y) \in A \). Thus \( xy \sim_A xv \). On the other hand, \( xv \ast uw = (x \ast u)v \in A \) and \( uv \ast xv = (u \ast x)v \in A \). Hence, \( xv \sim_A uv \), and so \( A_{xy} = A_{uw} \). Therefore, \( (X/A, \circ) \) is a semigroup. Moreover, for any \( A_x, A_y, A_z \in X/A \), we obtain \( A_x \circ (A_y \circ A_z) = A_x \circ A_{yz} = A_{x(yz)} = A_{xy} \otimes A_{xz} = (A_x \circ A_y) \otimes (A_x \circ A_z) \). Similarly, we get \( (A_x \circ A_y) \circ A_z = (A_x \circ A_z) \otimes (A_y \circ A_z) \). Therefore, \( X/A \) is a KS-semigroup.

**Theorem 3.24.** Let \( f : X \to Y \) be a homomorphism of KS-semigroups. Then \( \ker f \) is an ideal of \( X \).

**Proof.** Let \( x \ast y \in \ker f \) and \( y \in \ker f \). Then \( 0 = f(x \ast y) = f(x) \ast f(y) = f(x) \ast 0 = f(x) \). Hence we have \( x \in \ker f \). Let \( x \in X \) and \( a \in \ker f \). Then by Lemma 2.2, we obtain \( f(ax) = f(a)f(x) = 0f(a) = 0 \) and \( f(xa) = f(x)f(a) = f(x)0 = 0 \). Hence \( xa, ax \in \ker f \).

**Theorem 3.25.** Let \( f : X \to Y \) be a homomorphism of a KS-semigroup. Then for any ideal of \( X \), \( A/(\ker f \cap A) = f(A) \).
Proof. Clearly, $\ker f \cap A$ is an ideal of $A$. Let $B = \ker f \cap A$. We define a mapping $\psi : A/B \to Y$ with $\psi(B_x) = f(x)$ where $x \in A$. Then for any $B_x, B_y \in A/B$, we have
\[
B_x = B_y \iff x * y \in B, y * x \in B,
\]
\[
\iff f(x * y) = 0, f(y * x) = 0,
\]
\[
\iff f(x) * f(y) = 0, f(y) * f(x) = 0,
\]
\[
\iff f(x) = f(y),
\]
\[
\iff \psi(B_x) = \psi(B_y).
\]
Hence, $\psi$ is well-defined and one to one. Moreover, for all $B_x, B_y \in A/B$, we get
\[
\psi(B_x \otimes B_y) = \psi(B_{xy}) = f(x * y) = f(x) * f(y) = \psi(B_x) * \psi(B_y),
\]
and
\[
\psi(B_x \odot B_y) = \psi(B_{xy}) = f(xy) = f(x)f(y) = \psi(B_x)\psi(B_y).
\]
So $\psi$ is a homomorphism of a KS-semigroup. Thus we obtain $\text{im } \psi = \{\psi(B_x) \mid x \in A\} = \{f(x) \mid x \in A\} = f(A)$. Therefore $A/(\ker f \cap A) = f(A)$. \qed

Corollary 3.26. (First Isomorphism Theorem) If $f : X \to Y$ is a surjective homomorphism of a KS-semigroup, then $X/\ker f$ is isomorphic to $Y$.

References


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