Certain Shifts on Banach Spaces of Formal Power Series¹

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Abstract

In this paper, we represent certain shifts on a Banach space as M_z , the operator of multiplication by the independent variable z , on some space of sequences. Also we will give necessary and sufficient conditions for the unitarily equivalence, similarity and quasisimilarity of such operators.

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1 Introduction

A bounded linear operator A on a Banach space $\mathcal{Y}$ is called a *shift* if

- (i) A is an injective,
- (ii) The range of A has codimension 1;
- (iii) $\bigcap_{i=1}^{\infty} A^i \mathcal{Y} = \{0\}$.

R. Crownover was the first to give a basis free definition of a shift on a Banach space. He has obtained some results concerning spectrum, and the commutant of such shifts [2]. In this connection, one can see [1], [3] and [5].

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Let $p > 0$ and $\{\beta(n)\}$ denote a sequence of positive numbers such that $\beta(0) = 1$. We consider the space of all sequences $f = \{\hat{f}(n)\}$ such that

$$\|f\|_p^p = \sum_{n=0}^{\infty} |\hat{f}(n)|^p \beta(n)^p < \infty.$$

We shall use the formal notation $f(z) = \sum_{n=0}^{\infty} \hat{f}(n)z^n$ whether or not the series converges for any complex values of z . Throughout this article, we consider the space $H^p(\beta)$ to be defined by

$$H^p(\beta) = \{f : f(z) = \sum_{n=0}^{\infty} \hat{f}(n)z^n, \|f\|_p < \infty\}.$$

One of our goal in this article, is to study the representation of certain shifts on a Banach space as the operator M_z on $H^p(\beta)$.

In all that follows we consider $1 \leq p < \infty$. Define the σ -finite measure μ on the nonnegative integers, N_0 , by $\mu(K) = \sum_{n \in K} \beta(n)^p, K \subseteq N_0$. Since $H^p(\beta) \cong l^p(\mu)$, we conclude that $H^p(\beta)$ is, indeed, a Banach space. For some more information on the space $H^p(\beta)$ see [4] and [7].

It is natural to ask when the operators $(M_z, H^p(\beta))$ and $(M_z, H^p(\alpha))$ are unitarily equivalent, similar or quasisimilar. We would like to give necessary and sufficient conditions for unitarily equivalence, similarity and quasisimilarity of the operators $(M_z, H^p(\beta))$ and $(M_z, H^p(\alpha)), 1 \leq p < \infty$. Furthermore, we show that if A_1 and A_2 are two injective unilateral weighted shifts on separable Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, and if there exist dense range operators $S : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ and $T : \mathcal{H}_2 \rightarrow \mathcal{H}_1$ such that $A_2 S = S A_1$ and $T A_2 = A_1 T$, then A_1 is similar to A_2 and vice-versa.

2 Main Results

It is worthwhile remarking that if A be a shift on a Banach space $\mathcal{Y}$ then A is a closed range operator. This is a consequence of an easy automatic continuity result: If a finite codimensional linear subspace $\mathcal{X}$ of a Banach space $\mathcal{Y}$ is the range of a continuous linear mapping N on $\mathcal{X}$, then it must be closed [9, Lemma 3.3]. It is known [2] that there is an element $y_0 \in \mathcal{Y}$ with $\|y_0\| = 1$ so that $\mathcal{Y} = \langle y_0 \rangle \oplus A\mathcal{Y}$ where $\langle y_0 \rangle$ is a one-dimensional subspace of $\mathcal{Y}$ containing y_0 .

Suppose $y \in \mathcal{Y}$. Then there exist a scalar $\hat{y}(0)$ and a vector $y_1 \in \mathcal{Y}$ such that $y = \hat{y}(0)y_0 + Ay_1$. Similarly, there exist a scalar $\hat{y}(1)$ and a vector $y_2 \in \mathcal{Y}$

such that $y_1 = \hat{y}(1)y_0 + Ay_2$. So $y = \hat{y}(0)y_0 + \hat{y}(1)Ay_0 + A^2y_2$. By induction, we obtain unique sequences $\{\hat{y}(n)\}_{n=0}^\infty$ and $\{y_n\}_{n=0}^\infty$ of scalars and vectors, respectively, such that for every n ,

$$y = \hat{y}(0)y_0 + \hat{y}(1)Ay_0 + \cdots + \hat{y}(n)A^n y_0 + A^{n+1}y_{n+1}. \quad (1)$$

Equation (1) is called the Taylor formula for y , and the coefficients $\hat{y}(n)$ are called the Taylor coefficients of y .

Theorem 2.1. Let A be a shift on the Banach space $\mathcal{Y} = \langle y_0 \rangle \oplus A\mathcal{Y}$. Suppose that there exists $1 \leq p < \infty$ such that for every $y \in \mathcal{Y}$

(a) $\|\sum_{i=0}^n \hat{y}(i)A^i y_0\|^p = \sum_{i=0}^n \|\hat{y}(i)A^i y_0\|^p$ for all nonnegative integers n , where A^0 is the identity operator, and

(b) $\sum_{n=0}^\infty \|\hat{y}(n)A^n y_0\|^p < \infty$

Then A is unitarily equivalent to M_z on $H^p(\beta)$ for a suitable choice of β . Moreover, such p is unique.

Proof. Let $\beta(n) = \|A^n y_0\|$ and the linear map B from $\mathcal{Y}$ into $H^p(\beta)$ be defined by $By = \sum_{n=0}^\infty \hat{y}(n)z^n$. We first show that B is an isometry. Let $y \in \mathcal{Y}$. The condition (b) implies the convergence of the sequence $\{A^{n+1}y_{n+1}\}$, appearing in (1). Define the linear map $S : \mathcal{Y} \times \langle y_0 \rangle \rightarrow \mathcal{Y}$ by $S(y, \alpha y_0) = \alpha y_0 + Ay$. Obviously, S is bijective and continuous. Now, by the inverse mapping theorem, we conclude that S^{-1} is also continuous, and so

$$\|Ay\| = \|S(y, 0)\| \geq \frac{\|(y, 0)\|}{\|S^{-1}\|} = c\|y\|,$$

where $c = \frac{1}{\|S^{-1}\|}$. Thus, A is bounded below. This fact implies that $\text{ran } A^n, n \geq 1$, is closed. Suppose that the sequence $\{A^{n+1}y_{n+1}\}_n$ converges to $y' \in \mathcal{Y}$. Then y' is in $\bigcap_{n=1}^\infty A^n \mathcal{Y}$, and consequently, condition (iii) says that y' must be the zero vector. Therefore,

$$\begin{aligned} \|y\|^p &= \lim_{n \rightarrow \infty} \left\| \sum_{i=0}^n \hat{y}(i)A^i y_0 \right\|^p \\ &\stackrel{(a)}{=} \lim_{n \rightarrow \infty} \sum_{i=0}^n \|\hat{y}(i)A^i y_0\|^p \\ &= \|By\|^p, \end{aligned}$$

which means that B is an isometry.

Now, suppose that $f \in H^p(\beta)$, and $\varepsilon > 0$ is given. Then there exists $N > 0$ such that $\sum_{i=N}^\infty |\hat{f}(i)|^p \|A^i y_0\|^p < \varepsilon^p$. It results from this fact and condition (a)

that the sequence $\{y_n\}_{n=0}^\infty$ defined by $y_n = \sum_{i=0}^n \hat{f}(i)A^i y_0$ is Cauchy and so is convergent. Suppose its limit is y . It is easily checked that $By = f$, and so B is surjective.

In the next step, we will first show that M_z is bounded, and then $BA = M_z B$. Put $M = \sup_n \frac{\beta(n+1)}{\beta(n)}$. It is clear that $M \leq \|A\| < \infty$. Therefore,

$$\begin{aligned} \|zf\|_p^p &= \sum_{n=0}^{\infty} |(zf)(n)|^p \beta(n)^p \\ &= \sum_{n=1}^{\infty} |\hat{f}(n-1)|^p \beta(n)^p \\ &= \sum_{n=0}^{\infty} |\hat{f}(n)|^p \beta(n+1)^p \\ &\leq M^p \sum_{n=0}^{\infty} |\hat{f}(n)|^p \beta(n)^p \\ &= M^p \|f\|_p^p. \end{aligned}$$

Thus M_z is bounded. Finally, if $y \in \mathcal{Y}$ then

$$\begin{aligned} BAy &= B(A(\sum_{i=0}^{\infty} \hat{y}(i)A^i y_0)) \\ &= B(\sum_{i=0}^{\infty} \hat{y}(i)A^{i+1} y_0) \\ &= \sum_{i=1}^{\infty} \hat{y}(i)z^{i+1} \\ &= M_z(\sum_{i=0}^{\infty} \hat{y}(i)z^i) \\ &= (M_z B)(y); \end{aligned}$$

which is the desired conclusion.

To complete the proof, it remains to show that p is unique. Suppose $1 \leq q < \infty$ and (a) and (b) hold for q besides p . Thus, the operator M_z on $H^p(\beta)$ is unitarily equivalent to the operator $\tilde{M}_z$ on $H^q(\beta)$ defined by $\tilde{M}_z(f) = zf$; consequently, there exists a unitary operator $U : H^p(\beta) \rightarrow H^q(\beta)$ such that $UM_z = \tilde{M}_z U$. Suppose g is in $H^p(\beta)$ so that $Ug = 1$. Then

$$1 = Ug = U(\sum_{n=0}^{\infty} \hat{g}(n)z^n) = \sum_{n=0}^{\infty} \hat{g}(n)z^n U1.$$

It is clear from the definition of the norm that the mapping $f \mapsto \hat{f}(0)$ is bounded, thus $1 = \hat{g}(0)(U1)^\wedge(0)$. Furthermore, U is an isometry, and so $\sum_{n=0}^{\infty} |\hat{g}(n)|^p \beta(n)^p = 1$. Since $\beta(0) = 1$, we observe that $|\hat{g}(0)| \leq 1$. Hence $|(U1)^\wedge(0)| \geq 1$. On the other hand, $\|U1\| = \|1\|$, that is, $\sum_{n=0}^{\infty} |(U1)^\wedge(n)|^q \beta(n)^q = 1$, and so $|(U1)^\wedge(0)| \leq 1$; consequently, $|(U1)^\wedge(0)| = 1$. Moreover, since $1 = \|U1\|^q = \sum_{n=0}^{\infty} |(U1)^\wedge(n)|^q \beta(n)^q$, we have $(U1)^\wedge(n) = 0, n \geq 1$. Putting $U1 = c$, for some constant c , we see that

$$2^{1/p} = \|1 + \frac{z}{\beta(1)}\| = \|U(1 + \frac{z}{\beta(1)})\| = \|c + \frac{z}{\beta(1)}c\| = 2^{1/q},$$

which results that $p = q$. $\diamond$

Example 2.2. Let $p = 2$ and suppose that A is a shift isometry on $H^p(\beta)$. Then $H^p(\beta) = \bigoplus_{i=0}^{\infty} \langle A^i y_0 \rangle$ and the uniqueness of the Taylor coefficients implies that A is unitary equivalent to M_z on $H^p(\beta)$.

Recall that if A_1 and A_2 are operators on Banach spaces X_1 and X_2 , respectively, then A_1 and A_2 are *unitarily equivalent*, if there is an isomorphism (i.e., surjective linear isometry) $U : X_1 \rightarrow X_2$ such that $UA_1U^{-1} = A_2$.

Theorem 2.3. Two operators $(M_z, H^p(\beta))$ and $(M_z, H^p(\alpha))$ are unitarily equivalent if and only if there exists a constant c such that $\beta(n) = |c|\alpha(n)$, for every nonnegative integer n .

Proof. First, suppose that $U : H^p(\beta) \rightarrow H^p(\alpha)$ is an isomorphism such that $UM_z = M_zU$. Then for every $k > 0$,

$$\begin{aligned} \|U1 + z^k\|^p - \|U1 - z^k\|^p &= \|U1 + z^k Ug\|^p - \|U1 - z^k Ug\|^p \quad (\text{where } Ug = 1) \\ &= \|U(1 + z^k g)\|^p - \|U(1 - z^k g)\|^p \\ &= \|1 + z^k g\|^p - \|1 - z^k g\|^p = 0. \end{aligned}$$

On the other hand,

$$\|U1 \pm z^k\|^p = \sum_{n \neq k} |(U1)^\wedge(n)|^p \beta(n)^p + |(U1)^\wedge(k) \pm 1|^p \beta(k)^p.$$

So $|(U1)^\wedge(k) + 1|^p \beta(k)^p - |(U1)^\wedge(k) - 1|^p \beta(k)^p = 0$. Since $\beta(k) > 0$ it follows that

$$|(U1)^\wedge(k) + 1| = |(U1)^\wedge(k) - 1|. \quad (3)$$

The same argument shows that

$$|(U1)^\wedge(k) + i| = |(U1)^\wedge(k) - i|. \quad (4)$$

Formula (3) combined with (4) shows that $(U1)^\wedge(k) = 0$, for every $k > 0$. Therefore, $U1 = c$ for some constant c . This, in turn, implies that $Uz^n = cz^n$ for all $n \geq 0$. Hence

$$\beta(n) = \|z^n\|_{H^p(\beta)} = \|Uz^n\|_{H^p(\alpha)} = |c|\alpha(n) \quad \forall n \geq 0.$$

For the converse, define $U : H^p(\beta) \rightarrow H^p(\alpha)$ by $Uf = cf$. It can be easily verified that U is an isomorphism and $UM_z = M_zU$. $\diamond$

Definition 2.4. If A_1 and A_2 are operators on Banach spaces X_1 and X_2 , respectively, then A_1 is *similar* to A_2 if there is an invertible operator $S : X_1 \rightarrow X_2$ such that $SA_1 = A_2S$. The operators A_1 and A_2 are *quasisimilar* if there are operators $S_{12} : X_2 \rightarrow X_1$ and $S_{21} : X_1 \rightarrow X_2$ satisfying the following conditions for $i, j = 1, 2$:

(a) $\ker S_{ij} = \{0\}$ and $\text{ran } S_{ij}$ is dense;

(b) $S_{ij}A_j = A_iS_{ij}$,

(take S_{11} and S_{22} to be the identity operators). Operators satisfying (a), are said to be quasi-affinities.

Lemma 2.5. Let $S : H^p(\beta) \rightarrow H^p(\alpha)$ be an operator such that $M_zS = SM_z$. If $\text{ran } S$ is dense then there exists a constant c such that

$$0 < c \leq \frac{\beta(n)}{\alpha(n)} \quad \text{for all } n \geq 0.$$

Proof. First, note that if $(S1)^\wedge(0) = 0$, then

$$(Sz^n)^\wedge(0) = (z^nS1)^\wedge(0) = 0 \quad \text{for all } n \geq 0.$$

Since the mapping $f \mapsto \hat{f}(0)$ is bounded we conclude that $(Sf)^\wedge(0) = 0$ for every f in $H^p(\beta)$. Furthermore $\text{ran } S$ is dense in $H^p(\alpha)$, and so there would exist a sequence $\{f_n\}$ in $H^p(\beta)$ such that Sf_n converges to 1. So $(Sf_n)^\wedge(0) \rightarrow 1$, contradicting the fact that $(Sf_n)^\wedge(0) = 0$ for every nonnegative integer n . Thus, we see that $(S1)^\wedge(0) \neq 0$. Now

$$\begin{aligned} \alpha(n)|(S1)^\wedge(0)| &= \alpha(n)|(z^nS1)^\wedge(n)| \\ &= \alpha(n)|(Sz^n)^\wedge(n)| \\ &\leq \|Sz^n\| \leq \|S\| \cdot \|z^n\| = \|S\|\beta(n) \quad \text{for all } n \geq 0. \end{aligned}$$

Hence if we put $c = \frac{|(S1)(0)|}{\|S\|}$ then $0 < c \leq \frac{\beta(n)}{\alpha(n)}$ for every $n \geq 0$. $\diamond$

We have already seen that when the operators $(M_z, H^p(\beta))$ and $(M_z, H^p(\alpha))$ are unitarily equivalent. Now, we pose the question of similarity of these operators.

Theorem 2.6. The operator $(M_z, H^p(\beta))$ is similar to $(M_z, H^p(\alpha))$ if and only if there exist positive constants C_1 and C_2 such that $0 < C_1 \leq \frac{\beta(n)}{\alpha(n)} \leq C_2$ for every nonnegative integer n .

Proof. Suppose $(M_z, H^p(\beta))$ is similar to $(M_z, H^p(\alpha))$. Then there is an invertible operator $S : H^p(\beta) \rightarrow H^p(\alpha)$ such that $SM_z = M_zS$; or equivalently $M_zS^{-1} = S^{-1}M_z$. By Lemma 3.3, there exist constants C_1 and C_2 such that

$$0 < C_1 \leq \frac{\beta(n)}{\alpha(n)} \text{ and } 0 < \frac{1}{C_2} \leq \frac{\alpha(n)}{\beta(n)},$$

for all $n \geq 0$. For the converse, suppose that there exist positive constants C_1 and C_2 so that $0 < C_1 \leq \frac{\beta(n)}{\alpha(n)} \leq C_2$ for every $n \geq 0$. It follows that $H^p(\alpha)$ and $H^p(\beta)$ coincide as sets with equivalent norms. Take for S the identity operator. $\diamond$

Corollary 2.7. Suppose $S : H^p(\beta) \rightarrow H^p(\alpha)$ and $T : H^p(\alpha) \rightarrow H^p(\beta)$ be two dense range operators such that $M_zS = SM_z$ and $TM_z = M_zT$. Then $(M_z, H^p(\beta))$ is similar to $(M_z, H^p(\alpha))$.

Proof. By Lemma 2.5, there exist constants c and d such that $0 < c < \frac{\beta(n)}{\alpha(n)}$ and $0 < d < \frac{\alpha(n)}{\beta(n)}$ for every $n \geq 0$. Put $C_1 = c$ and $C_2 = \frac{1}{d}$, and apply Theorem 2.6. $\diamond$

Another immediate consequence is the following.

Corollary 2.8. The operators $(M_z, H^p(\beta))$ and $(M_z, H^p(\alpha))$ are similar if and only if they are quasisimilar.

We recall that a *unilateral weighted shift operator* A on a separable Hilbert space $\mathcal{H}$ is an operator that maps each vector in some orthonormal basis $\{e_n\}_{n=0}^{+\infty}$ into a scalar multiple of the next vector; i.e., $Ae_n = w_n e_{n+1}$. The operator A is *injective* if and only if $w_n \neq 0$ for every n . It is known that every injective unilateral weighted shift can be represented as M_z on $H^2(\beta)$ for a suitable choice of β ; see [8]. In this view, Corollary 2.7 provides the following.

Corollary 2.9. Let A_1 and A_2 be two injective unilateral weighted shift on the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$. If there exist dense range operators $S : \mathcal{H}_1 \rightarrow \mathcal{H}_2$

and $T : \mathcal{H}_2 \rightarrow \mathcal{H}_1$ such that $A_2S = SA_1$ and $TA_2 = A_1T$, then A_1 is similar to A_2 .

It is known that if two injective unilateral weighted shifts are quasisimilar then they are similar. We can easily get this result as a conclusion of Corollary 2.9. Also it follows from [6, Theorem 4.6.] that if A_1 and A_2 are two injective unilateral weighted shifts on a Hilbert space $\mathcal{H}$ and if S and T are two bounded surjective operators on $\mathcal{H}$ such that $A_2S = SA_1$ and $TA_2 = A_1T$ then $\sigma(A_1) = \sigma(A_2)$. Since similarity preserves the spectrum and the various parts of the spectrum, we also get the following from Corollary 2.9.

Corollary 2.10. Let A_1 and A_2 be two injective unilateral weighted shifts on Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$. Suppose there exist operators $S : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ and $T : \mathcal{H}_2 \rightarrow \mathcal{H}_1$ which have dense range, and satisfy $A_2S = SA_1$ and $TA_2 = A_1T$. Then $\sigma(A_1) = \sigma(A_2)$.

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