International Mathematical Forum, 1, 2006, no. 4, 191-197

Separability of graded fields

L. EL FADIL

Department of Mathematics, Faculty of Sciences Dhar-Mahraz
Fes-Morocco, P.O. Box 1796, Fes-Atlas
Fes-Morocco

Abstract

In this paper, we study separability of graded field extensions. In particular, without restriction to torsion free of the grade group, we show that some separability results of graded field extensions given in [7] hold. On the other hand, the separability notions defined in [1] do not cover many situations, for example group algebras. We show that these notions are simple cases of separability of commutative ring extensions.

1 Introduction

Let $(\Gamma, +)$ be an abelian group and R a graded commutative ring with respect to Γ, i.e., $R = \oplus_{\sigma\in\Gamma} R_\sigma$ such that R_σ is a R_0-submodule of R and $R_\sigma R_\tau \subset R_{\sigma+\tau}$ for every $(\sigma, \tau) \in \Gamma^2$. Set $\Gamma_R = \{\sigma \in \Gamma \mid R_\sigma \neq 0\}$ and $R^h = \cup_{\sigma\in\Gamma} R_\sigma$ the set of homogeneous elements of R. For every nonzero homogeneous element $x \in R_\sigma$, we write $deg(x) = \sigma$ and we call it the degree of x. The graded ring R is called a graded field if every nonzero homogeneous element in R is invertible. In that case, R_0 is a field, R_σ is a R_0-vector space of dimension 1 for every $\sigma \in \Gamma$ and Γ_R is a subgroup of Γ, called the grade group of R.

In [7], Hwang and Wadsworth investigated the separability of a graded field extension S/R with grade groups Γ and $\triangle$ respectively, where $\triangle$ is a torsion free abelian group. In that case, S and R are domains. These results do not cover many situations, for example group algebras. In this paper, we investigate the separability of a graded field extension S/R, where S and R are not necessarily domains. In particular, we state separability results given in [7].

Throughout the paper, S/R is a graded field extension with grade groups $\Gamma_R = \Gamma$ and $\Gamma_S = \triangle$, i.e., $R_\sigma \subset S_\sigma$, for every $\sigma \in \Gamma$. For a subgroup Λ such that $\Gamma \subseteq \Lambda \subseteq \triangle$, we define $S(\Lambda) := \oplus_{x\in\Lambda} S_x$. In particular, $S(\Gamma)$ is a graded field with grade group Γ, and we have $R \subseteq S(\Gamma) \subseteq S$. In the third section, we split the extension S/R into two graded ring extensions $R \subset S(\Gamma)$ and $S(\Gamma) \subset S$. In the fourth section, we investigate separability of graded field extensions. In particular, we state the separability results given in [7]. In the fifth section, we characterize the separability of homogeneous element via its minimal polynomial. In particular, we show that the separability notions defined in [1], which do not cover many situations, are simple cases of separability of commutative ring extensions.

2 Preliminary Notes

For a subgroup Λ such that $\Gamma \subseteq \Lambda \subseteq \triangle$, we define $S(\Lambda) := \oplus_{x\in\Lambda} S_x$. In particular, $S(\Gamma)$ is a graded field with grade group Γ and we have $R \subseteq S(\Gamma) \subseteq S$. In this way we split the extension S/R into two graded ring extensions $R \subset S(\Gamma)$ and $S(\Gamma) \subset S$. The first one is made of graded fields which are graded over the same group Γ, while in the second one $S(\Gamma)$ and S have the same homogeneous components of degree 0, namely S_0.

Lemma 2.1 *S is a crossed product of $S(\Gamma)$ by the group $\triangle/\Gamma$. In particular, S is a free $S(\Gamma)$-module.*

Proof. We can define a new grading on S over the group $\triangle/\Gamma$ by taking
(1) $S_\sigma := \oplus_{x\in\sigma} S_x$, for every $\sigma \in \triangle/\Gamma$. Obviously S is a $\triangle/\Gamma$-graded ring whose homogeneous component of degree 0 is $S(\Gamma)$. Since S is a graded field with grade group $\triangle$, then its nonzero homogeneous elements are invertible. Thus, for every $\sigma \in \triangle/\Gamma$, S_σ contains an invertible element (fix $x \in \sigma$ and a nonzero $a_\sigma \in S_x \subset S_\sigma$). So, S is a crossed product of $S(\Gamma)$ by the group $\triangle/\Gamma$.

Lemma 2.2 *The multiplication of S induces an isomorphism of graded rings*

$$S_0 \otimes_{R_0} R \simeq S(\Gamma)$$

Proof. $S_0 \otimes_{R_0} R$ is a graded ring by $S_0 \otimes_{R_0} R = \oplus_{x\in\Gamma}(S_0 \otimes_{R_0} R_x)$. The map $m : S_0 \otimes_{R_0} R \longrightarrow S(\Gamma)$, defined by $m(s \otimes r) = sr$, is an homomorphism of Γ-graded rings. We have already remarked that any graded field is a crossed product. So, from [**?**, Corollary I.3.9], it follows that m is an isomorphism if and only if the restriction of m to $(S_0 \otimes_{R_0} R)_0$ is an isomorphism. But $(S_0 \otimes_{R_0} R)_0 = S_0 \otimes_{R_0} R_0 \simeq S_0 = S(\Gamma)_0$, where the middle isomorphism is the canonical isomorphism, that is equal to the restriction of m to $(S_0 \otimes_{R_0} R)_0$.

The following Remark generalizes [2, Theorem 3, p. 29].

Remark 2.3 *Let S/R be a graded field extension. Then S is a free R-module and $[S : R] = [S_0 : R_0][\triangle : \Gamma]$.*
In particular, S is a free R-module of finite rank if and only if S_0 is a R_0-vector space of finite dimension.

It results from Lemma 2.1 and Lemma 2.2.

3 Separability of graded fields

In this section, we investigate separability of a graded field extension S/R, where S and R are not necessarily domains. In particular, we state separability results given in [7].

Let S be a free R-algebra of finite rank. Every $x \in S$ induces an R-homomorphism l_x of S defined by $l_x(s) = xs$ for every $s \in S$. Define $T_{S/R}(x) = tr(l_x)$ the trace of l_x. For a free R-submodule M of S, $T_{S/R}$ induces a bilinear form of M by $T_{M/R}(x, y) = T_{S/R}(xy)$ for every $(x, y) \in M^2$. The determinant of the bilinear form $T_{M/R}$ with respect to a R-basis $(e_1, ..., e_n)$ of M is denoted by $D(e_1, ..., e_n)$ and called the discriminant of $(e_1, ..., e_n)$. The discriminant ideal of the R-module M is the ideal of R generated by $D(e_1, ..., e_n)$, where $(e_1, ..., e_n)$ is a R-basis of M. The discriminant plays a key role to investigate the separability of commutative algebras.

Lemma 3.1 *If S/R is separable, then S is a free R-module of finite rank.*

Proof. From Remark 2.3, S is a free R-module. Then by [8, Proposition III.3.2], it follows that S is a finitely generated R-module.

In the sequel of the paper, we assume that S is a finitely generated R-module. Then $\triangle/\Gamma$ is a finite group and S_0 is a R_0-vector space of finite dimension.

Theorem 3.2 *The extension S/R is separable if and only if S_0/R_0 is separable and $[\triangle : \Gamma]$ invertible in R_0.*

Proof. We use the discriminant computation. Recall that from [8, Theorem III 4.7, p. 89], S/R is separable if and only if the map $T_{S/R}$ induces a nonsingular bilinear form of S, i.e., $D_R(S) = R$.
We have already remarked that S is a is a free $S(\Gamma)$ generated by $(u_{\sigma_1}, ..., u_{\sigma_n})$, where $\triangle/\Gamma = \{\bar{\sigma_1}, ..., \bar{\sigma_n}\}$ and for every i, u_{σ_i} is an homogeneous element of S of degree σ_i. Let M be the R-submodule of S, generated by $(u_{\sigma_1}, ..., u_{\sigma_n})$. Then $S \simeq S(\Gamma) \otimes_R M$ as R-modules. From [4, Proposition 3, p. 95], $D_R(S) = (D_R(S(\Gamma)))^n (D_R(M))^f$, where $f = [S_0 : R_0]$. Consequently, $D_R(S) = R$ if and only if $D_R(M) = R$ and $D_R(S(\Gamma)) = R$. Since $S(\Gamma) \simeq S_0 \otimes_{R_0} R$, from [4, Proposition 1, p. 95],
$D_R(S(\Gamma)) = D_{R_0}(S_0)R$. As R_0 is a field, $D_R(S(\Gamma)) = S(\Gamma)$ if and only if $D_{R_0}(S_0) = R_0$, i.e., S_0/R_0 is separable. On the other hand, for every (i, j), there exists c_{σ_i,σ_j} an invertible element of $S(\Gamma)$ such that $u_{\sigma_i} u_{\sigma_j} = c_{\sigma_i,\sigma_j} u_{\sigma_i+\sigma_j}$. Hence $T_{S/S(\Gamma)}(u_{\sigma_i} u_{\sigma_j}) = nc_{\sigma_i,\sigma_j} \delta_0^{\sigma_i+\sigma_j}$, where $\delta_\tau^\sigma = 1$ if $\tau = \sigma$ and $\delta_\tau^\sigma = 0$ elsewhere. Consequently, the determinant of the bilinear form $T_{M/R}$, with respect to the basis $(u_{\sigma_1} ..., u_{\sigma_n})$, is $D(u_{\sigma_1} ..., u_{\sigma_n}) = sn^n$, where s is an invertible element of R. So, $D_R(M) = R$ if and only if n is invertible in R_0.

Corollary 3.3 *1) The extension $S/S(\Gamma)$ is separable if and only if $[\triangle : \Gamma]$ is invertible in R_0.*
2) $S(\Gamma)/R$ is separable if and only if S_0/R_0 is separable.
3) The extension S/R is separable if and only if $S(\Gamma)/R$ and $S/S(\Gamma)$ are separable.

Indeed, for 1) $S_0 = S(\Gamma)_0$. For 2) $\Gamma_{S(\Gamma)} = \Gamma_R$.

Proposition 3.4 *Let S/R be a graded field extension. Then S/R is separable if and only if for every graded subfield extension S'/R of S/R, S/S' and S'/R are separable.*
In particular, we state 3) of Corollary 3.3.

Proof. Let S'/R be a graded subfield extension of S/R. Then its grade group Λ is a subgroup of $\triangle$ and its homogeneous component of degree 0, S'_0 is a subfield of S_0. Let n, m and r be the cardinal orders of $\triangle/\Gamma$, $\triangle/\Lambda$ and Λ/Γ respectively. Then $n = mr$. If S/R is separable, then S_0/R_0 is separable and n is invertible in R_0. Hence m and r are invertible in R_0, S_0/S'_0 and S'_0/R_0 are separable. Consequently, S/S' and S'/R are separable.
Conversely, if S/S' and S'/R are separable, then m and r are invertible in R_0, S_0/S'_0 and S'_0/R_0 are separable. Therefore S_0/R_0 is separable and n is invertible in R_0, i.e., S/R is separable.

Proposition 3.5 *Assume that R is a domain with quotients field K. Then S/R is separable if and only if $K \otimes_R S/K$ is separable.*

Proof. In the proof of Theorem 3.2, we have shown that $D_R(S) = D_R(M)^f D_R(S(\Gamma))^n$ and $D_{S(\Gamma)}(S) = n^n S(\Gamma)$, where $f = [S_0 : R_0]$ and M is the R-submodule of S, generated by $(u_{\sigma_1}, ..., u_{\sigma_n})$. From [4, Proposition 1, p. 95], $D_{S(\Gamma)}(S) = D_R(M)S(\Gamma)$. On the other hand, from Lemma 2.2, $S(\Gamma) \simeq S_0 \otimes_{R_0} R$. So, $D_R(S(\Gamma)) = D_{R_0}(S_0)R$. Therefore $D_R(S) = n^{nf}(D_{R_0}(S_0))^n R$. Consequently, $D_R(S)$ is generated by an homogeneous element of degree 0. Hence S/R is separable if and only if $D_R(S) \neq 0$, i.e., $D_K(K \otimes S) \neq 0$.

Remark 3.6 . *This Proposition generalizes [7, Theorem 3.11(a), p. 833]. Indeed, S is not necessarily a domain.*

Examples 3.7 *1) let L/k be a finite field extension of characteristic p. Denote $R = k[X, X^{-1}]$ and $S = k[X^{\frac{1}{n}}, X^{\frac{-1}{n}}]$. Then S/R is separable if and only if L/k is separable and p does not divide n.*
2) Let G be an abelian group, H a subgroup of G such that G/H is finite and let L/k be a finite field extension of characteristic p. Then $L[G]/k[H]$ is separable if and only if p does not divide $[G : H]$ and L/k is separable.
3) If G is a finite abelian group, then $L[G]/k$ is separable if and only if G is p-torsion free and L/k is separable. In particular, $k[G]/k$ is separable if and only if G is p-torsion free. We find also, the Maschke's Theorem, i.e., if G is a p-torsion free finite abelian group, then $k[G]$ is a semi-simple algebra.

4 Separability and separable polynomials

Let S/R be a graded field extension with grade groups Γ and $\triangle$ respectively, such that $\triangle$ is a torsion free abelian group. Then R is an integrally closed domain (see [7, Corollary 1.3, p. 824]). Hence every homogeneous element x of S, which is integral over R, has its minimal polynomial over R, which is homogenezable (see [7, Proposition 2.2, p. 825]). We extend this result to any graded field extension.

Let $r \in \triangle$. Recall that for a polynomial $P = \sum_{i=0}^{n} a_i X^i \in R[X]$ such that $a_n \neq 0$, P is called a r-homogenezable polynomial if every for (i, j) such that $a_i \neq 0$ and $a_j \neq 0$, a_i and a_j are homogeneous and $\mathrm{d}eg(a_i) + ir = \mathrm{d}eg(a_j) + jr$. Let $\lambda = deg(a_n) + nr$. λ is called the grade of the homogeneous polynomial P, i.e., P is an homogeneous element of degree λ in the graded ring $R[X]^{(r)}$, where $\mathrm{d}eg(X) = r$.

Proposition 4.1 *Let S/R be a graded field extension and $\alpha \in S_\sigma$ an homogeneous element of degree $\sigma \in \triangle$, which is integral over R. Then α has its minimal polynomial over R, which is $\sigma-$homogenezable , i.e., the ideal $I(\alpha) = \{P \in R[X] \,|\, P(\alpha) = 0\}$ of $R[X]$ is a principal ideal which is generated by a monic homogenezable polynomial.*

Proof. Since α is integral over R, there exists a nonzero polynomial of minimal degree in $R[X]$, which annihilate α. Let $P(X) = a_n X^n + ... + a_0 \in R[X]$ be a such polynomial, where $a_n \neq 0$. Set $a_n = s_{1n} + ... + s_{rn}$ the decomposition of a_n of homogeneous elements of R, where $s_{1n} \neq 0$. Let $Q(X)$ be the $deg(s_{1n}) + ndeg(\alpha)$-homogeneous component of $P(X)$ in $R[X]^{(\sigma)}$. Let $Q(X) = s_n X^n + ... + s_1 X + s_0$. Then $s_n \alpha^n + ... + s_0 = 0$ and $deg(s_n) + ndeg(\alpha) = deg(s_{n-1}) + (n-1)deg(\alpha) = ... = deg(s_0)$. Since s_n is a nonzero homogeneous element of R, s_n is invertible in R. So, $f_\alpha(X) = \frac{Q(X)}{s_n} \in R[X]$ is a monic homogenezable polynomial of minimal degree, which annihilate α. Therefore, f_α is the minimal polynomial of α over R, and then $I(\alpha) = f_\alpha R[X]$.

Proposition 4.2 *Let S/R be a graded field extension and $s \in S_\sigma$ a nonzero homogeneous element which is integral over R. Then $R[s]$ is a graded field with grade group $\Gamma < \sigma >$.*

Proof. Since s is integral over R, $R[s]$ is a finitely generated R-module. Let $f_s(X) = X^n + ... + a_0$ be its minimal polynomial. Since s is invertible, a_0 is a nonzero homogeneous element of R. So, a_0 is invertible in R. Hence $s^{-1} = -a_0^{-1}(s^{n-1} + a_{n-1}s^{n-2} + ... + a_1) \in R[s]$. Consequently, $R[s] = \sum_{g \in \Gamma<\sigma>} R[s]_g$, where $R[s]_g = \sum_{\tau + n\sigma = g} R_\tau s^n$. Therefore every homogeneous component of $R[s]$ contains an invertible element. Hence it suffices to show that $R[s]_0$ is a field.

First, $R[s]_0$ is a subring of the domain S_0. Moreover $R[s]_0$ is a R_0-vector space of finite dimension. Then $R[s]_0$ is a field. Finally, $R[s]$ is a graded field with grade group $\Gamma < \sigma >$.

Let $f \in R[X]$ be a monic polynomial. Recall that f is called separable if the R-algebra $S_f = R[X]/(f)$ is separable, where (f) is the principal ideal of $R[X]$ generated by f.

Theorem 4.3 *Let $f \in R[X]$ be a monic $\sigma-$homogenezable polynomial. Then f is separable if and only if f is square free.*

Proof. Set $f(X) = X^n + a_{n-1}X^{n-1} + ... + a_0$. Since $f(X)$ is a σ-homogeneous polynomial, $S_f = \sum_{g\in\Gamma<\sigma>} S_g$, where $S_g = \sum_{\tau+k\sigma=g} R_\tau \bar{X}^k$ is a graded ring with respect to the semigroup $\Gamma < \sigma >$ such that $deg(\bar{X}) = \sigma$. On the other hand, for every i such that $a_i \neq 0$, $deg(a_i)+i\sigma = n\sigma$. So, for every i such that $a_i \neq 0$, $deg(a_i) + (i-1)\sigma = (n-1)\sigma$. Hence $f'(X)$ is a σ-homogeneous polynomial. Consequently, $f'(\bar{X})$ is a homogeneous element of S_f, and then $det(f'(\bar{X}))$ is a homogeneous element of R, where $det(f'(\bar{X}))$ is the determinant of the endomorphism $l_{f'(\bar{X})}$, of S_f, defined by the multiplication by $f'(\bar{X})$. Therefore, S_f/R is separable if and only if $f'(\bar{X}) \neq 0$, i.e., f is square free.

Remark 4.4 *1) If $\triangle$ is a torsion free abelian group, then R and S are domains with quotients fields K and KS, respectively. Hence for every homogeneous element α of S, its minimal polynomial f_α over R is an homogenezable polynomial, which is irreducible over K. Consequently, α is separable over R if and only if it is a simple root of its minimal polynomial f_α.*
3) Let R be a graded field, which is a domain of characteristic p. Let $\alpha \in S$ be an homogeneous, which is integral over R. Then $R[\alpha]$ is separable over R if and only if $f_\alpha \notin R[X^p]$.

Theorem 4.5 *Let S/R be a graded field extension. Then S/R is separable if and only if for every homogeneous element $a \in S$, $R[a]/R$ is separable.*

Proof. Assume that for every homogeneous element $a \in S$, $R[a]/R$ is separable. Set $\triangle/\Gamma =< \sigma_1 > \times ... \times < \sigma_n >$. For every $1 \leq i \leq n$, pick a nonzero element $a_i \in S_{\sigma_i}$. Then the multiplication of S induces an isomorphism of $S(\Gamma)$-algebras $m' : S(\Gamma)[a_1] \otimes ... \otimes S(\Gamma)[a_n] \longrightarrow S$. By assumption, $R[a_i]/R$ is separable for every $1 \leq i \leq n$. Then $S(\Gamma)[a_i]/S(\Gamma)$ is separable for every $1 \leq i \leq n$. We conclude that S/R is separable by using [8, Proposition III.1.7].
Conversely, assume that S/R is separable. Then $[\triangle : \Gamma]$ is invertible in R_0. Let $a \in S$ be a nonzero homogeneous element of degree $x \in \triangle$. From Proposition 4.2, we have that $R[a]$ is a graded subfield of S. So, from Proposition 3.4, $R[a]/R$ is separable.

Corollary 4.6 *Let S/R be a graded field extension. Then S/R is separable if and only if for every homogeneous element $a \in S$, its minimal polynomial over R is square free.*
In particular, if S is a domain, then S/R is separable if and only if for every homogeneous element is a simple root of its minimal polynomial over R.

Proof. It suffices to show that for every homogeneous element $a \in S$, $R[a]/R$ is separable if and only if its minimal polynomial over R is square free.

Remark 4.7 *The separability notions defined in [1] do not cover many situations, for example group algebras. The previous Corollary shows that these notions are simple cases of separability of commutative rings.*

References

[1] M. Boulagouaz, An introduction to the galois theory for graded fields,*Lect. Notes in Pure and Appl. math., Marcel Dekker,* **208**(1999), 21-31.

[2] M. Boulagouaz, The graded and tame extensions, *Lect. Notes in Pure and Appl. math., Marcel Dekker,* **153**(1994), 27-40.

[3] F. DeMeyer E. Ingraham, Separable Algebras Over Commutative Rings, *Lect. Notes in Maths,* **181** Springer-Verlag, Berlin, New York, 1971.

[4] L. El Fadil and M. Charkani, Generalization of the discriminant and applications, *Arabian Journal for Science and Engineering,* **1A** (29)(2004), 93-98.

[5] L. El Fadil,Discriminant and separability, Preprint **547** (2003), *CRM, E-08193 Bellaterra Espana.*

[6] C. Greither, Cyclic Galois extensions of commutative rings, *Lecture Notes in Maths,* **1534**, Springer-Verlag Berlin Heidelberg, New York, 1992.

[7] Y. S. Hwang and A. R. Wadsworth, Algebraic extensions of graded and valued fields, *Comm. Algebra,* **27**(2)(1999), 821-840. math., Marcel Dekker, (153)(1994), 27-40.

[8] M. A. Knus and M. Ojanguren, Théorie de la descente et algèbres d'Azummaya, *Lecture notes in maths,* **389**, Springer-Verlag Berlin, Heidelberg, New York, 1974.

Received: April 12, 2005