CHARACTERIZATION OF THE EXISTENCE OF A WEAK QUALITATIVE PROBABILITY

Gianni BOSI

Dipartimento di Matematica Applicata
“Bruno de Finetti”, Università di Trieste,
Piazzale Europa 1,
34127 Trieste, Italy

Abstract

Necessary and sufficient conditions are presented for the existence of a weak qualitative probability for a binary relation on an algebra of subsets of an arbitrary set.

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1 Introduction

The problem concerning the representability of a total preorder (i.e., a reflexive, transitive and total binary relation) on an algebra of subsets of a given nonempty set by means of a probability measure was already solved by several authors (see e.g. Chateauneuf [4], Clark [5], and Fine [7]). The existence of a weak probability representation was characterized by Cohen [6], who followed a dual approach. Nakamura [10] was concerned with the representability of a binary relation with intransitive indifference by means of a pair of probability measures (see also Fishburn [9]).

On the other hand, the notion of a *lower scale* in a topological preordered space proved to be useful in order to characterize the existence of a continuous or at least semicontinuous utility representation in case that the preorder is not complete (see e.g. Alcantud and Rodríguez-Palmero [1] and Alcantud et al. [2]). Further, when we consider a preorder on a group, it is possible to characterize the existence of an additive utility function by imposing suitable conditions on a lower scale (see e.g. Bosi et al. [3]).

In this paper, we use the concept of a lower (decreasing) scale in order to obtain a characterization of the existence of a probability measure which is

also a weak utility for an acyclic binary relation on an algebra of subsets of a nonempty set. To the best of our knowledge, this approach appears as novel in the literature so far. As an application of our main result, we present a sufficient condition for the existence of a probability measure which is a strong utility for a linear order on an algebra of subsets of a nonempty set under the assumption of order separability.

2 Notation and preliminaries

Throughout this paper we shall denote by $\prec$ a binary relation on an algebra $\mathcal{A}_\Omega$ of subsets of an arbitrary set Ω . As usual, if for two events $A, B \in \mathcal{A}_\Omega$ it happens that $A \prec B$, then this means that “ B is more probable than A ”. The pair $(\mathcal{A}_\Omega, \prec)$ will be referred to as a *related algebra*.

Definition 2.1 A set function $p : \mathcal{A}_\Omega \rightarrow [0, 1]$ on a related algebra $(\mathcal{A}_\Omega, \prec)$ is said to be a *qualitative probability* on $(\Omega, \mathcal{A}_\Omega, \prec)$ if p is both a *probability* on $(\Omega, \mathcal{A}_\Omega)$ (i.e., $p(\emptyset) = 0$, $p(\Omega) = 1$ and $p(A \cup B) = p(A) + p(B)$ for all sets $A, B \in \mathcal{A}_\Omega$ such that $A \cap B = \emptyset$) and a *weak utility* on $(\mathcal{A}_\Omega, \prec)$ (i.e., for all sets $A, B \in \mathcal{A}_\Omega$, $p(A) < p(B)$ whenever $A \prec B$).

If in addition for all sets $A, B \in \mathcal{A}_\Omega$ we have that $p(A) < p(B)$ is equivalent to $A \prec B$ (i.e., p is a *strong utility* on $(\mathcal{A}_\Omega, \prec)$), then p will be referred to as a *strong qualitative probability*. $\square$

Remark 2.2 If there exists a weak utility p on $(\mathcal{A}_\Omega, \prec)$, then $\prec$ is *acyclic* (i.e., $A_1 \prec A_2 \prec \dots \prec A_n$ implies $A_1 \neq A_n$ for all $A_1, \dots, A_n \in \mathcal{A}_\Omega$). $\square$

Given an acyclic binary relation $\prec$ on $\mathcal{A}_\Omega$, we shall denote by $\ll$ the *transitive closure* of $\prec$ (i.e., $A \ll B$ if and only if there exist $A_1, \dots, A_n \in \mathcal{A}_\Omega$ such that $A = A_1 \prec A_2 \prec \dots \prec A_n = B$). The transitive closure of any acyclic binary relation is a *partial order* (i.e., it is irreflexive and transitive). Obviously, a partial order $\prec$ is in particular acyclic and in this case we have that $\prec = \ll$.

Remark 2.3 If there exists a strong utility p on $(\mathcal{A}_\Omega, \prec)$, then $\prec$ is a *weak order* (i.e., $\prec$ is asymmetric and negatively transitive).

Definition 2.4 A binary relation $\prec$ on $\mathcal{A}_\Omega$ is said to be *additive* if for all sets $A, B, C \in \mathcal{A}_\Omega$ such that $(A \cup B) \cap C = \emptyset$ we have that $A \prec B$ if and only if $A \cup C \prec B \cup C$. $\square$

If there exists a weak qualitative probability p on a *linearly ordered* algebra $(\mathcal{A}_\Omega, \prec)$ (i.e., the binary relation $\prec$ is irreflexive, transitive and complete), then p is also a strong qualitative probability and $\prec$ is necessarily additive.

Definition 2.5 If $(\mathcal{A}_\Omega, \prec)$ is a related algebra, then a subset $\mathcal{G}$ of $\mathcal{A}_\Omega$ is said to be a *lower set* if the following condition holds:

$$A \in \mathcal{G} \text{ and } B \prec A \text{ imply } B \in \mathcal{G} \text{ for all sets } A, B \in \mathcal{A}_\Omega.$$

□

Definition 2.6 If $\prec$ is an acyclic binary relation on an algebra $\mathcal{A}_\Omega$ of subsets of Ω , then a countable family $\mathcal{G} = \{G_r : r \in \mathbb{Q} \cap]0, 1]\}$ of lower subsets of $\mathcal{A}_\Omega$ is said to be a *countable lower scale* in $(\Omega, \mathcal{A}_\Omega, \prec)$ if it satisfies the following conditions:

- (1) $G_1 = \mathcal{A}_\Omega$ and $G_r \subset G_s$ for all $r, s \in \mathbb{Q} \cap]0, 1]$ such that $r < s$;
- (2) $\emptyset \in G_r$ for all $r \in \mathbb{Q} \cap]0, 1]$ and $\Omega \notin G_r$ for all $r \in \mathbb{Q} \cap]0, 1[$.

Further, a countable lower scale $\mathcal{G} = \{G_r : r \in \mathbb{Q} \cap]0, 1]\}$ in $(\Omega, \mathcal{A}_\Omega, \prec)$ is said to be *separating* if the following condition holds:

- (3) for all $A, B \in \mathcal{A}_\Omega$ such that $A \prec\prec B$ there exist $r, s \in \mathbb{Q} \cap]0, 1[$ such that $r < s$, $A \in G_r$ and $B \notin G_s$.

□

Definition 2.7 If $\prec$ is an acyclic binary relation on an algebra $\mathcal{A}_\Omega$ of subsets of Ω , then a countable lower scale $\mathcal{G} = \{G_r : r \in \mathbb{Q} \cap]0, 1]\}$ in $(\Omega, \mathcal{A}_\Omega, \prec)$ is said to be *additive* if for all $A, B \in \mathcal{A}_\Omega$ such that $A \cap B = \emptyset$, and for all $r, s \in \mathbb{Q} \cap]0, 1[$ the following conditions hold:

- (4) $A \cup B \in G_{r+s}$ whenever $A \in G_r$, $B \in G_s$, $r + s < 1$;
- (5) $(r + s = 1)$ or $(r + s < 1, A \cup B \notin G_{r+s})$ whenever $A \notin G_r$, $B \notin G_s$.

□

Definition 2.8 If $\prec$ is an acyclic binary relation on an algebra $\mathcal{A}_\Omega$ of subsets of Ω , then a countable lower scale $\mathcal{G} = \{G_r : r \in \mathbb{Q} \cap]0, 1]\}$ in $(\Omega, \mathcal{A}_\Omega, \prec)$ is said to be *monotonic* if the following condition holds:

- (6) for all $A, B \in \mathcal{A}_\Omega$ and $r \in \mathbb{Q} \cap]0, 1]$, $B \in G_r$ whenever $B \subset A$ and $A \in G_r$.

□

Definition 2.9 A binary relation $\prec$ on an algebra $\mathcal{A}_\Omega$ of subsets of Ω is said to be *order separable* if there exists a countable subset $\mathcal{D}$ of $\mathcal{A}_\Omega$ such that for all $A, B \in \mathcal{A}_\Omega$ with $A \prec B$ there exists $D \in \mathcal{D}$ such that $A \prec D \prec B$. $\mathcal{D}$ is said to be an *order dense subset* of the related algebra $(\mathcal{A}_\Omega, \prec)$. □

3 Existence of a weak qualitative probability

The following theorem provides a characterization of the existence of a weak qualitative probability for a binary relation on an algebra of subsets of a given set.

Theorem 3.1 *Let $\prec$ be a binary relation on an algebra $\mathcal{A}_\Omega$ of subsets of an arbitrary set Ω . Then the following conditions are equivalent:*

- (i) *There exists a weak qualitative probability p on $(\Omega, \mathcal{A}_\Omega, \prec)$;*
- (ii) *$\prec$ is acyclic and there exists a countable lower scale $\mathcal{G} = \{G_r : r \in \mathbb{Q} \cap]0, 1]\}$ in $(\Omega, \mathcal{A}_\Omega, \prec)$ which is additive, monotonic and separating.*

Proof. Necessity. Assume that there exists a weak qualitative probability p on $(\Omega, \mathcal{A}_\Omega, \prec)$. Then we have just to define $G_r = p^{-1}([0, r])$ for all $r \in \mathbb{Q} \cap]0, 1]$ and $G_1 = \mathcal{A}_\Omega$ in order to easily check that the family $\mathcal{G} = \{G_r : r \in \mathbb{Q} \cap]0, 1]\}$ is a countable lower scale in $(\Omega, \mathcal{A}_\Omega, \prec)$ which is additive, monotonic and separating. Indeed, it is clear that $\mathcal{G}$ is actually a countable lower scale in $(\Omega, \mathcal{A}_\Omega, \prec)$. If A and B are any two elements of $\mathcal{A}_\Omega$ such that $A \prec B$, then $p(A) < p(B)$ since p is in particular a weak utility on the related set $(\mathcal{A}_\Omega, \prec)$, and therefore if we consider any two rational numbers r, s such that $p(A) < r < s < p(B)$ we have that $A \in G_r, B \notin G_s$. Hence, the countable lower scale $\mathcal{G}$ is also separating according to condition (3). Now consider any two sets $A, B \in \mathcal{A}_\Omega$ such that $A \cap B = \emptyset$. If $A \in G_r, B \in G_s$ for some $r, s \in \mathbb{Q} \cap]0, 1]$ such that $r + s < 1$, then we have that $p(A \cup B) = p(A) + p(B) < r + s$, and therefore $A \cup B \in G_{r+s}$. If $A \notin G_r, B \notin G_s$ for some $r, s \in \mathbb{Q} \cap]0, 1]$ then $r + s > 1$ is contradictory since this would imply that $p(A) + p(B) = p(A \cup B) \geq r + s > 1$. So assume that $r + s < 1$. Then we have that $p(A \cup B) = p(A) + p(B) \geq r + s$, which implies that $A \cup B \notin G_{r+s}$. Hence, the above defined countable lower scale $\mathcal{G}$ is additive according to Definition 2.7. Finally, if for some $A, B \in \mathcal{A}_\Omega$ such that $B \subset A$, and $r \in \mathbb{Q} \cap]0, 1]$ we have that $A \in G_r$, then $p(B) \leq p(A) < r$ obviously implies $B \in G_r$. Hence, $\mathcal{G}$ is monotonic according to Definition 2.8. Sufficiency. Assume that there exists a countable lower scale $\mathcal{G} = \{G_r : r \in \mathbb{Q} \cap]0, 1]\}$ in $(\Omega, \mathcal{A}_\Omega, \prec)$ which is additive, monotonic and separating. Define the function $p : \mathcal{A}_\Omega \rightarrow [0, 1]$ by

$$p(A) = \inf\{r \in \mathbb{Q} \cap]0, 1] : A \in G_r\} \quad (A \in \mathcal{A}_\Omega).$$

We claim that p is a weak qualitative probability on $(\Omega, \mathcal{A}_\Omega, \prec)$. Let us first show that p is a weak utility on $(\mathcal{A}_\Omega, \prec)$. Since $\mathcal{G}$ is separating, we have that for every $A, B \in \mathcal{A}_\Omega$ such that $A \prec B$ there exist $r, s \in \mathbb{Q} \cap]0, 1]$ such that $r < s, A \in G_r, B \notin G_s$, and therefore from the definition of p we have that

$p(A) \leq r < s \leq p(B)$, which obviously implies that $p(A) < p(B)$.

Let us now prove that p is a probability measure on $(\Omega, \mathcal{A}_\Omega)$. By condition (2) in the definition of a countable lower scale (see Definition 2.6), it is clear that $p(\emptyset) = 0$ and $p(\Omega) = 1$. Further, we have that $0 \leq p(A) \leq 1$ directly from the definition of p . In order to prove that p is additive, assume that there exist $A, B \in \mathcal{A}_\Omega$ such that $A \cap B = \emptyset$ and $p(A \cup B) \neq p(A) + p(B)$. If $p(A) + p(B) < p(A \cup B)$, then both $p(A)$ and $p(B)$ must be strictly less than 1. From the definition of p , there exist $r, s \in \mathbb{Q} \cap]0, 1]$ such that $p(A) < r$, $p(B) < s$, $A \in G_r$, $B \in G_s$ and $p(A) + p(B) < r + s < p(A \cup B)$. But here we have a contradiction since by condition (4) in the definition of an additive lower scale (see Definition 2.7), we should have that $A \cup B \in G_{r+s}$, which implies $p(A \cup B) \leq r + s$ from the definition of p . If $p(A \cup B) < p(A) + p(B)$, then first assume that $p(A)$ and $p(B)$ are both strictly positive. By condition (5) we may assume that $p(A \cup B) < 1$ because otherwise there would exist $r, s \in \mathbb{Q} \cap]0, 1[$ such that $r < p(A)$, $s < p(B)$, $r + s > 1$, and therefore it should be $A \notin G_r$, $B \notin G_s$, $r + s > 1$. So assume that $p(A \cup B) < 1$. Then we may consider $r, s \in \mathbb{Q} \cap]0, 1[$ such that $r < p(A)$, $s < p(B)$, $p(A \cup B) < r + s < 1$. Hence we have that $A \notin G_r$, $B \notin G_s$, and therefore $p(A \cup B) < r + s$ is contradictory by condition (5). Finally, if either $p(A)$ or $p(B)$ is equal to 0, then $p(A \cup B) < p(A) + p(B)$ is contradictory by axiom (6). So the proof is complete. $\square$

Remark 3.2 Obviously, from considerations above, we have that Theorem 3.1 applies in case that $\prec$ is a partial order. Further, the previous theorem substantially provides a characterization of the existence of a weak qualitative probability p on a triplet $(\Omega, \mathcal{A}_\Omega, \preceq)$, where $\preceq$ is a preorder (i.e., a reflexive and transitive binary relation) on $\mathcal{A}_\Omega$ and p is now a *utility* on the preordered set $(\mathcal{A}_\Omega, \preceq)$ (i.e., for all sets $A, B \in \mathcal{A}_\Omega$, $p(A) \leq p(B)$ whenever $A \preceq B$ and $p(A) < p(B)$ whenever $A \prec B$). Here, $\prec$ is the *strict part* of $\preceq$ (i.e., for all sets $A, B \in \mathcal{A}_\Omega$, $A \prec B$ if and only if $(A \preceq B)$ and not $(B \preceq A)$). We only have to replace the term “lower scale” by the term “*decreasing scale*” where a decreasing scale is a family $\mathcal{G} = \{G_r : r \in \mathbb{Q} \cap]0, 1]\}$ which consists of decreasing sets (i.e., subsets $\mathcal{G}$ of $\mathcal{A}_\Omega$ such that $A \in \mathcal{G}$ and $B \preceq A$ imply $B \in \mathcal{G}$ for all sets $A, B \in \mathcal{A}_\Omega$).

As an application of the previous characterization we present a sufficient condition for the existence of a strong qualitative probability on an algebra endowed with an order separable linear order.

Corollary 3.3 *Let $\prec$ be a linear order on an algebra $\mathcal{A}_\Omega$ of subsets of an arbitrary set Ω . Then there exists a strong qualitative probability p on $(\Omega, \mathcal{A}_\Omega, \prec)$ provided that the following conditions hold:*

- (a) $\emptyset \prec A \prec \Omega$ for every set $A \in \mathcal{A}_\Omega$ such that $A \neq \emptyset$ and $A \neq \Omega$;
- (b) $\prec$ is additive;
- (c) $\prec$ is order separable and there exists a countable order dense subset $\mathcal{D} = \{D_r : r \in \mathbb{Q} \cap]0, 1[\}$ of $(\mathcal{A}_\Omega, \prec)$ satisfying the following conditions:
 - (c1) $r < s$ is equivalent to $D_r \prec D_s$ for all $r, s \in \mathbb{Q} \cap]0, 1[$;
 - (c2) $A \prec D_r$ and $B \prec D_s$ imply $A \cup B \prec D_{r+s}$ for all $A, B \in \mathcal{A}_\Omega$ such that $A \cap B = \emptyset$ and for all $r, s \in \mathbb{Q} \cap]0, 1[$ such that $r + s < 1$;
 - (c3) $\text{not}(A \prec D_r)$ and $\text{not}(B \prec D_s)$ imply $(r + s = 1)$ or $(r + s < 1, \text{not}(A \cup B \prec D_{r+s}))$ for all $A, B \in \mathcal{A}_\Omega$ such that $A \cap B = \emptyset$ and for all $r, s \in \mathbb{Q} \cap]0, 1[$.

Proof. Assume that conditions (a), (b) and (c) hold, and then define, for every $r \in \mathbb{Q} \cap]0, 1[$, $G_r = \{A \in \mathcal{A}_\Omega : A \prec D_r\}$. Further, let $G_1 = \mathcal{A}_\Omega$ and define $\mathcal{G} = \{G_r : r \in \mathbb{Q} \cap]0, 1[\}$. By Theorem 3.1 it suffices to show that under our assumptions $\mathcal{G}$ is an additive, monotonic and separating countable lower scale in $(\Omega, \mathcal{A}_\Omega, \prec)$. It is immediate to check that $\mathcal{G}$ is actually a lower scale by conditions (a) and (c1). By conditions (c2) and (c3) it is immediate to verify that $\mathcal{G}$ is additive. In order to show that $\mathcal{G}$ is also separating, consider any two sets $A, B \in \mathcal{A}_\Omega$ such that $A \prec B$. Since $\prec$ is order separable and condition (c1) holds, there exist two rational numbers $r < s$ ($r, s \in \mathbb{Q} \cap]0, 1[$) such that $A \prec D_r \prec D_s \prec B$, or equivalently $A \in G_r$, $B \notin G_s$. It remains to show that $\mathcal{G}$ is monotonic. But under our assumptions we have that the following property holds:

$$(*) \quad A \prec B \text{ for all } A, B \in \mathcal{A}_\Omega \text{ such that } A \subsetneq B.$$

Indeed, consider any two sets $A, B \in \mathcal{A}_\Omega$ such that $A \subsetneq B$. Then, since $\prec$ is a complete binary relation, we have that either $A \prec B$ or $B \prec A$. But $B \prec A$ is contradictory since in this case additivity of $\prec$ would imply $B \cup (\Omega \setminus B) = \Omega \prec A \cup (\Omega \setminus B)$. Hence, the above property (*) implies that the countable lower scale $\mathcal{G}$ is also monotonic. This consideration completes the proof. $\square$

References

- [1] J.C.R. Alcantud and C. Rodríguez-Palmero, Characterization of the existence of semicontinuous weak utilities, *Journal of Mathematical Economics*, **32** (1999), 503–509.

- [2] J.C.R. Alcantud, G. Bosi, M.J. Campión, J.C. Candeal, E. Induráin, C. Rodríguez-Palmero and M.E. Zuanon, Continuous utility functions through scales, (2005), submitted.
- [3] G. Bosi, M.J. Campión, J.C. Candeal, E. Induráin and M.E. Zuanon, Numerical isotones of preordered semigroups through the concept of a scale, *Mathematica Pannonica*, **16** (2005), 65-77.
- [4] A. Chateauneuf, On the existence of a probability measure compatible with a total preorder on a Boolean algebra, *Journal of Mathematical Economics*, **14** (1985), 43-52.
- [5] S.A. Clark, The measurement of qualitative probability, *Journal of Mathematical Psychology*, **44** (2000), 464-479.
- [6] M.A. Cohen, Necessary and sufficient conditions for existence and uniqueness of weak probability structures, *Journal of Mathematical Psychology*, **35** (1991), 242-259.
- [7] T. Fine, A note on the existence of quantitative probability, *Annals of Mathematical Statistics*, **42** (1971), 1182-1186.
- [8] P.C. Fishburn, Linear extensions of additive partial orders, *Order*, **14** (1997/98), 153-169.
- [9] P.C. Fishburn, Interval Orders and Interval Graphs, Wiley, New York, 1985.
- [10] Y. Nakamura, Comparative probability with additive threshold, *IPPS Economic Working paper*, 02-02 (2002), University of Tsukuba

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