Coverings of curves and non-existence of certain models of them in the quadric surface

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Abstract. Let $C \subset S := \mathbb{P}^1 \times \mathbb{P}^1$ be an integral curve and $\nu : X \to C$ its normalization. Set $L := \nu^*(\mathcal{O}_S(1,0))$. Here we give numerical conditions which imply that the pair $(X,L)$ is not the pull-back of a pair $(Y,R)$ with $Y$ a smooth curve of genus $q > 0$ and $R \in \text{Pic}(Y)$.

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1. Introduction

Let $X$ be a smooth and connected projective curve. There are many high degree pencils $f_i : X \to \mathbb{P}^1$, $i = 1, 2$, such that the induced map $(f_1, f_2) : X \to S := \mathbb{P}^1 \times \mathbb{P}^1$ is birational onto its image. However, very seldom there are such morphisms for which the image $(f_1, f_2)(X)$ has “very few” singularities. Here we prove the following quantitative versions of this observation.

Theorem 1. Fix integer $u, v, a$ such that $v \geq u \geq 4$ and $0 \leq a \leq (u+1)(v+1)-3a-4$. Fix a general $A \subset S := \mathbb{P}^1 \times \mathbb{P}^1$ such that there $\sharp(A) = a$. Let $\Gamma_A$ denote the set of all integral and nodal curves $C \subset |\mathcal{O}_S(u,v)|$ such that $\text{Sing}(C) = A$. The set $\Gamma_A$ is a non-empty open subset of an $(uv+u+v-3a)$-dimensional projective space. Fix a general $C \in \Gamma_A$ and let $\nu : X \to C$ be the normalization. Then $X$ is not a multiple covering $f : X \to Y$ of a smooth curve of genus $q > 0$ (deg$(f) > 1$) such that there is $R \in \text{Pic}(Y)$ with $\nu^*(\mathcal{O}_C(1,0)) \cong f^*(R)$ and $h^0(X, \nu^*(\mathcal{O}_C(1,0))) = h^0(Y,R)$.

Theorem 2. Fix integers $m, k, q, v, a$ such that $k \geq 2$, $m \geq 2$, $q > 0$, $v \geq mk$ and $0 \leq a \leq \min\{mk-1, v-2 - [(q+1)/m]\}$. Let $C \in |\mathcal{O}_S(mk,v)|$, $S := \mathbb{P}^1 \times \mathbb{P}^1$, be any integral curve such that the conductor of the normalization $\nu : X \to C$ has length $a$. Then $X$ is not a degree $k$ multiple covering $f : X \to Y$ of a smooth curve of genus $q$ such that there is $R \in \text{Pic}^m(Y)$ with $\nu^*(\mathcal{O}_C(1,0)) \cong f^*(R)$ and $h^0(X, \nu^*(\mathcal{O}_C(1,0))) = h^0(Y,R)$.

We work over an algebraically closed field $k$ with $\text{char}(k) = 0$.

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Example 1. Set $S := \mathbb{P}^1 \times \mathbb{P}^1$. Let $\pi_1 : S \to \mathbb{P}^1$ and $\pi_2 : S \to \mathbb{P}^1$ denote the two projections. We have $\text{Pic}(S) \cong \mathbb{Z}^\oplus 2$ with, as generators, the class $\mathcal{O}_S(1,0)$ of a fiber of $\pi_1$ and the class $\mathcal{O}_S(0,1)$ of a fiber of $\pi_2$. For all $P \in S$ let $2P$ denote the first infinitesimal neighborhood of $P$ in $S$, i.e. the closed subscheme of $S$ with $\mathcal{I}_{P,S^2}$ as its ideal sheaf. Hence $(2P)_{\text{red}} \{P\}$, and length$(2P) = 3$ and length$(3P) = 6$. Fix any integral curve $C \subset |\mathcal{O}_S(u,v)|$ and set $A(C) := \text{Sing}(C)$. Let $\nu : X \to C$ be the normalization map and $B(C) \subset S$ the conductor of $X$ in $S$. Hence $p_a(X) = uv - u - v + 1 - \text{length}(B(C))$. We recall that $B(C) = A(C)$ if $C$ has only ordinary ordinary points and ordinary cusps as its singularities. Since $\omega_S \cong \mathcal{O}_S(-2,-2)$, we have $h^0(S, \omega_S) = h^1(S, \omega_S) = 0$. Hence the classical adjunction theory developed for plane curves works for curves in $S$ and gives $H^0(X, \omega_X) \cong H^0(S, \mathcal{I}_{B(C)}(u-2, v-2))$.

Set $L := \nu^*(\mathcal{O}_C(1,0))$. Hence $L$ is a degree $u$ spanned line bundle on $X$. Fix an integer $k$ such that $1 \leq k \leq v - 3$. By adjunction theory we have $h^0(X, L^\otimes k) = k + 1 + h^1(S, \mathcal{O}_S(u-2, v-2 - k))$.

Proof of Theorem 1. The non-emptiness of $\Gamma_A$ and its dimension easily follow from [1], Cor. 4.6, because $\omega_S^* \cong \mathbb{P}^1$. Set $Z := \bigcup_{P \in \mathbb{A}} \{2P\}$. By [1], Cor. 4.6, we have $h^1(S, \mathcal{I}_Z(u,v)) = 0$. Since $P$ is general in $S$ and $h^0(S, \mathcal{I}_Z(u,v)) \neq 0$, we get $h^1(S, \mathcal{I}_{Z \cup \{P\}}(u,v)) = 0$. Fix a general $T \subset |\mathcal{O}_S(1,0)|$ and a general $P \in T$. Hence $T \cap A = \emptyset$. For any quasi-projective scheme $\Theta$ and any $Q \in \Theta_{\text{reg}}$ set $\{2Q, \Theta\}$ denote the first infinitesimal neighborhood of $Q$ in $\Theta$. Hence $\{2P, T\}$ is the degree two closed subscheme of $T$ with $P$ as support. Set $Z' := Z \cup \{2P, T\}$. Take another general $P' \in T$ and set $Z' := Z' \cup \{2P', T\}$.

First Claim: $h^1(S, \mathcal{I}_{Z'}(u,v)) = 0$.

Proof of the First Claim: Set $W := \bigcup_{Q \in \pi_1(A)} \{2Q, \mathbb{P}^1\}$. Assume $h^1(S, \mathcal{I}_{Z'}(u,v)) \neq 0$. Since $h^0(S, \mathcal{I}_Z(u,v)) \geq 2$ and $P$ is general in $S$, we get that the differential of the rational map induced on $S$ by linear system $|\mathcal{I}_Z(u,v)|$ vanishes identically in the direction of the $\pi_1$-fibring. Since $\text{char}(\mathbb{K}) = 0$, this implies that this rational map factors through $\pi_1$. Hence $h^0(S, \mathcal{I}_Z(u,v)) = h^0(\mathbb{P}^1, \mathcal{I}_W(v)) = \max\{0, v + 1 - 2(\pi_1(A))\}$. Since $\mathcal{I}(\pi_1(A)) = 0$, $h^0(S, \mathcal{I}_Z(u,v)) = (u + 1)(v + 1) - 3a$ and $u \geq 2$, we get a contradiction and hence prove the First Claim.

Second Claim: $h^1(S, \mathcal{I}_{Z''}(u,v)) = 0$.

Proof of the Second Claim: Here we use the assumption $(u + 1)(v + 1) - 3a \geq 4$ to get $h^0(S, \mathcal{I}_{Z'}(u,v)) = 0$. Then we may easily adapt the proof of the First Claim to get the Second Claim.

By the Second Claim we get that for a general $C \in |\mathcal{I}_Z(u,v)|$ the morphism $\pi_1 \circ \nu$ has at least one ordinary ramification point which is the only ramification point in its fiber of $\pi_1 \circ \nu$.

Proof of Theorem 2. Set $u := mk$. Assume the existence of a degree $k$ multiple covering $f : X \to Y$ of a smooth curve of genus $q$ such that there is $R \in \text{Pic}(Y)$ with $L := \nu^*(\mathcal{O}_C(1,0)) \cong f^*(R)$ and $h^0(X, \nu^*(\mathcal{O}_C(1,0))) = h^0(Y, R)$.

Fix any integer $t$ such that $1 \leq t \leq v - 2$. By the last part of Example 1 we have $h^0(X, L^\otimes t) = (t + 1) + h^1(S, \mathcal{I}_Z(u,v - 2 - t))$. Notice that $h^0(X, L^\otimes t) \geq h^0(Y, R^\otimes t) \geq tm + 1 - m$ and that $tm + 1 - q \geq t - 2$ if $m(t - 1) \geq q + 1$. Hence $h^1(S, \mathcal{I}_Z(u-2, v-2-t)) \neq 0$ if $m(t-1) \geq q+1$. We have $h^1(S, \mathcal{I}_Z(u-2, v-2-t)) = 0$, if $a \leq \min\{u - 1, v - 1 - t\}$. 

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References


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