Coverings of curves and non-existence of certain models of them in the quadric surface

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Abstract. Let $C \subset S := \mathbf{P}^1 \times \mathbf{P}^1$ be an integral curve and $\nu : X \rightarrow C$ its normalization. Set $L := \nu^*(\mathcal{O}_S(1, 0))$. Here we give numerical conditions which imply that the pair (X, L) is not the pull-back of a pair (Y, R) with Y a smooth curve of genus $q > 0$ and $R \in \text{Pic}(Y)$.

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1. INTRODUCTION

Let X be a smooth and connected projective curve. There are many high degree pencils $f_i : X \rightarrow \mathbf{P}^1$, $i = 1, 2$, such that the induced map $(f_1, f_2) : X \rightarrow S := \mathbf{P}^1 \times \mathbf{P}^1$ is birational onto its image. However, very seldom there are such morphisms for which the image $(f_1, f_2)(X)$ has “very few” singularities. Here we prove the following quantitative versions of this observation.

Theorem 1. *Fix integer u, v, a such that $v \geq u \geq 4$ and $0 \leq a \leq (u+1)(v+1) - 3a - 4$. Fix a general $A \subset S := \mathbf{P}^1 \times \mathbf{P}^1$ such that there $\sharp(A) = a$. Let Γ_A denote the set of all integral and nodal curves $C \subset |\mathcal{O}_S(u, v)|$ such that $\text{Sing}(C) = A$. The set Γ_A is a non-empty open subset of an $(uv + u + v - 3a)$ -dimensional projective space. Fix a general $C \in \Gamma_A$ and let $\nu : X \rightarrow C$ be the normalization. Then X is not a multiple covering $f : X \rightarrow Y$ of a smooth curve of genus > 0 ($\deg(f) > 1$) such that there is $R \in \text{Pic}(Y)$ with $\nu^*(\mathcal{O}_C(1, 0)) \cong f^*(R)$ and $h^0(X, \nu^*(\mathcal{O}_C(1, 0))) = h^0(Y, R)$.*

Theorem 2. *Fix integers m, k, q, v, a such that $k \geq 2$, $m \geq 2$, $q > 0$, $v \geq mk$ and $0 \leq a \leq \min\{mk - 1, v - 2 - \lceil (q + 1)/m \rceil\}$. Let $C \in |\mathcal{O}_S(mk, v)|$, $S := \mathbf{P}^1 \times \mathbf{P}^1$, be any integral curve such that the conductor of the normalization $\nu : X \rightarrow C$ has length a . Then X is not a degree k multiple covering $f : X \rightarrow Y$ of a smooth curve of genus q such that there is $R \in \text{Pic}^m(Y)$ with $\nu^*(\mathcal{O}_C(1, 0)) \cong f^*(R)$ and $h^0(X, \nu^*(\mathcal{O}_C(1, 0))) = h^0(Y, R)$.*

We work over an algebraically closed field $\mathbb{K}$ with $\text{char}(\mathbb{K}) = 0$.

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Example 1. Set $S := \mathbf{P}^1 \times \mathbf{P}^1$. Let $\pi_1 : S \rightarrow \mathbf{P}^1$ and $\pi_2 : S \rightarrow \mathbf{P}^1$ denote the two projections. We have $\text{Pic}(S) \cong \mathbb{Z}^{\oplus 2}$ with, as generators, the class $\mathcal{O}_S(1, 0)$ of a fiber of π_1 and the class $\mathcal{O}_S(0, 1)$ of a fiber of π_2 . For all $P \in S$ let $2P$ denote the first infinitesimal neighborhood of P in S , i.e. the closed subscheme of S with $\mathcal{I}_{P, S^2}$ as its ideal sheaf. Hence $(2P)_{\text{red}} = \{P\}$, and $\text{length}(2P) = 3$ and $\text{length}(3P) = 6$. Fix any integral curve $C \in |\mathcal{O}_S(u, v)|$ and set $A(C) := \text{Sing}(C)$. Let $\nu : X \rightarrow C$ be the normalization map and $B(C) \subset S$ the conductor of X in S . Hence $p_a(X) = uv - u - v + 1 - \text{length}(B(C))$. We recall that $B(C) = A(C)$ if C has only ordinary nodes and ordinary cusps as its singularities. Since $\omega_S \cong \mathcal{O}_S(-2, -2)$, we have $h^0(S, \omega_S) = h^1(S, \omega_S) = 0$. Hence the classical adjunction theory developed for plane curves works for curves in S and gives $H^0(X, \omega_X) \cong H^0(S, \mathcal{I}_{B(C)}(u-2, v-2))$. Set $L := \nu^*(\mathcal{O}_C(1, 0))$. Hence L is a degree u spanned line bundle on X . Fix an integer k such that $1 \leq k \leq v-3$. By adjunction theory we have $h^0(X, L^{\otimes k}) = k+1 + h^1(S, \mathcal{O}_S(u-2, v-2-k))$.

Proof of Theorem 1. The non-emptiness of Γ_A and its dimension easily follow from [1], Cor. 4.6, because ω_S^* is ample. Set $Z := \bigcup_{P \in A} 2P$. By [1], Cor. 4.6, we have $h^1(S, \mathcal{I}_Z(u, v)) = 0$. Since P is general in S and $h^0(S, \mathcal{I}_Z(u, v)) \neq 0$, we get $h^1(S, \mathcal{I}_{Z \cup \{P\}}(u, v)) = 0$. Fix a general $T \in |\mathcal{O}_S(1, 0)|$ and a general $P \in T$. Hence $T \cap A = \emptyset$. For any quasi-projective scheme Θ and any $Q \in \Theta_{\text{reg}}$ let $\{2Q, \Theta\}$ denote the first infinitesimal neighborhood of Q in Θ . Hence $\{2P, T\}$ is the degree two closed subscheme of T with P as support. Set $Z' = Z \cup \{2P, T\}$. Take another general $P' \in T$ and set $Z'' := Z' \cup \{2P', T\}$.

First Claim: $h^1(S, \mathcal{I}_{Z'}(u, v)) = 0$.

Proof of the First Claim: Set $W := \bigcup_{Q \in \pi_1(A)} \{2Q, \mathbf{P}^1\}$. Assume $h^1(S, \mathcal{I}_{Z'}(u, v)) \neq 0$. Since $h^0(S, \mathcal{I}_Z(u, v)) \geq 2$ and P is general in S , we get that the differential of the rational map induced on S by linear system $|\mathcal{I}_Z(u, v)|$ vanishes identically in the direction of the π_1 -fibering. Since $\text{char}(\mathbb{K}) = 0$, this implies that this rational map factors through π_1 . Hence $h^0(S, \mathcal{I}_Z(u, v)) = h^0(\mathbf{P}^1, \mathcal{I}_W(v)) = \max\{0, v+1 - \#(\pi_1(A))\}$. Since $\#(\pi_1(A)) = 0$, $h^0(S, \mathcal{I}_Z(u, v)) = (u+1)(v+1) - 3a$ and $u \geq 2$, we get a contradiction and hence prove the First Claim.

Second Claim: $h^1(S, \mathcal{I}_{Z''}(u, v)) = 0$.

Proof of the Second Claim: Here we use the assumption $(u+1)(v+1) - 3a \geq 4$ to get $h^0(S, \mathcal{I}_{Z'}(u, v)) = 0$. Then we may easily adapt the proof of the First Claim to get the Second Claim.

By the Second Claim we get that for a general $C \in |\mathcal{I}_Z(u, v)|$ the morphism $\pi_1 \circ \nu$ has at least one ordinary ramification point which is the only ramification point in its fiber of $\pi_1 \circ \nu$. $\square$

Proof of Theorem 2. Set $u := mk$. Assume the existence of a degree k multiple covering $f : X \rightarrow Y$ of a smooth curve of genus g such that there is $R \in \text{Pic}(Y)$ with $L := \nu^*(\mathcal{O}_C(1, 0)) \cong f^*(R)$ and $h^0(X, \nu^*(\mathcal{O}_C(1, 0))) = h^0(Y, R)$. Fix any integer t such that $1 \leq t \leq v-2$. By the last part of Example 1 we have $h^0(X, L^{\otimes t}) = (t+1) + h^1(S, \mathcal{I}_Z(u, v-2-t))$. Notice that $h^0(X, L^{\otimes t}) \geq h^0(Y, R^{\otimes t}) \geq tm + 1 - m$ and that $tm + 1 - m \geq t-2$ if $m(t-1) \geq g+1$. Hence $h^1(S, \mathcal{I}_Z(u-2, v-2-t)) \neq 0$ if $m(t-1) \geq g+1$. We have $h^1(S, \mathcal{I}_Z(u-2, v-2-t)) = 0$, if $a \leq \min\{u-1, v-1-t\}$. $\square$

REFERENCES

- [1] E. Arbarello and M. Cornalba, Footnotes to a paper of Beniamino Segre, *Math. Ann.* 256 (1981), no. 3, 341–362.

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