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Birationally very ample line bundles on smooth curves

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Abstract. For all integers $r \geq 2$ and any smooth and connected projective curve X, let $\rho_X(r)$ denote the minimal integer d such that there is a morphism $\phi : X \to \mathbf{P}^r$ birational onto its image and such that $\deg(\phi(X)) = d$ and $\phi(X)$ spans $\mathbf{P}^r$. Fix integers d, g such that $d \geq 8$ and $d^2/6 < g \leq d^2/4 - d$. Here we prove the existence of a smooth genus g curve X such that $\rho_X(3) = d$.

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1. Introduction

For all integers $r \geq 2$ and any smooth and connected projective curve X, let $\rho_X(r)$ denote the minimal integer d such that there is a morphism $\phi : X \to \mathbf{P}^r$ birational onto its image and such that $\deg(\phi(X)) = d$ and $\phi(X)$ spans $\mathbf{P}^r$. Suppose you have a morphism $\phi : X \to \mathbf{P}^r$ which is birational onto its image and whose associated base point free linear system is complete. Hence $\rho_X(r) \leq d$. It seems to be very hard to give conditions on X which give $\rho_X(r) = d$. In [6] the authors considered the function $\rho_X(2)$. In particular, they give an existence theorem of smooth genus g curves with prescribed $\rho_X(2)$ ([6], Prop. 2.2). We do not know if such an existence theorem (" no gaps for the integers $\rho_X(r)$ ") is true for some $r \geq 3$. Here we fill in an interval for the genus with prescribed $\rho_X(3)$: when the genus is very large with respect to the integer $d := \rho_X(3)$, near Castelnuovo's upper bound for space curves

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with fixed degree ([5], Ch. III). To state our result we define the functions $\pi(d,3)$ and $\pi_1(d,3)$ as in [5], Ch. III.

For all integers $d \geq 7$, set $\pi(d,3) := d^2/4 - d + 1$ if d is even, $\pi(d,3) := (d^2 - 1)/4 - d + 1$ if d is odd, $\pi_1(d,3) = d^2/6 - d/2 + 1$ if $d \equiv 0 \pmod 3$ and $\pi_1(d,3) = d^2/6 - d/2 + 1/3$ if $d \equiv 1, 2 \pmod 3$.

Theorem 1. *Fix integers d, g such that $d \geq 8$ and $\pi_1(d-1,3) < g \leq \pi(d,3)$. Then there exists a smooth genus g curve X such that $\rho_X(3) = d$.*

Indeed, we will also find " large " families of curves genus g curves with $\rho_X(3) = d$ (see Remarks 6 and 7).

We work over an algebraically closed field $\mathbb{K}$ with char$(\mathbb{K}) = 0$.

2. Proof of Theorem 1

Example 1. Set $S := \mathbf{P}^1 \times \mathbf{P}^1$. Let $\pi_1 : S \to \mathbf{P}^1$ and $\pi_2 : S \to \mathbf{P}^1$ denote the two projections. We have $\mathrm{Pic}(S) \cong \mathbb{Z}^{\oplus 2}$ with, as generators, the class $\mathcal{O}_S(1,0)$ of a fiber of π_2 and he class $\mathcal{O}_S(0,1)$ of a fiber of π_1. We have $\omega_S \cong \mathcal{O}_S(-2,-2)$. We have $h^0(S, \mathcal{O}_S(a,b)) = 0$ if either $a < 0$ or $b < 0$, $h^0(S, \mathcal{O}_S(a,b)) = (a+1)(b+1)$ if $a \geq 0$ and $b \geq 0$, $h^1(S, \mathcal{O}_S(a,b)) = 0$, if either $a \leq -1$ and $b \leq -1$ or $a \geq -1$ and $b \geq -1$, $h^1(S, \mathcal{O}_S(a,b)) = (a+1)(-1-b)$ if $a \geq 0$ and $b \leq -2$ and $h^1(S, \mathcal{O}_S(a,b)) = (-1-a)(b+1)$ if $a \leq -2$ and $b \geq 0$ (Künneth formula). For all integers $u \geq 0$ and $v \geq 0$, $\mathcal{O}_S(u,v)$ is spanned and $p_a(C) = uv - u - v + 1$ for all $C \in |\mathcal{O}_S(u,v)|$. Set $\gamma(u,v) := uv - u - v + 1$. For all $P \in S$ let $2P$ denote the first infinitesimal neighborhood of P in S, i.e. the closed subscheme of S with ${\mathcal{I}_{P,S}}^2$ as its ideal sheaf. Hence $(2P)_{red} = \{P\}$, and length$(2P) = 3$. Fix any integral nodal curve $C \in |\mathcal{O}_S(u,v)|$ and set $A := \mathrm{Sing}(C)$ and $a := \sharp(A)$. Let $\nu : X \to C$ be the normalization map. Hence $p_a(X) = \gamma(u,v) - a$. Since $\omega_S \cong \mathcal{O}_S(-2,-2)$, we have $h^0(S, \omega_S) = h^1(S, \omega_S) = 0$. Hence the classical adjunction theory developed for plane curves works for curves in S and gives $H^0(X, \omega_X) \cong H^0(S, \mathcal{I}_A(u-2, v-2))$.

Example 2. Let $T \subset \mathbf{P}^3$ be a quadric cone with vertex P and $u : M \to T$ the blowing-up of P. Hence $M \cong F_2$ and $\mathrm{Pic}(M)$ is freely generated by a fiber f of the ruling $\pi : M \to \mathbf{P}^1$ and by $h := u^{-1}(P)$ (the section of π with minimal self-intersection). We have $h^2 = -2$, $h \cdot f = 1$, $f^2 = 0$ and $\omega_M \cong \mathcal{O}_M(-2h - 4f)$. Furthermore $\mathcal{O}_M(ah + bf)$ is spanned (resp. spanned and big, resp. very ample) if and only if $b \geq 2a \geq 0$ (resp. $b \geq 2a > 0$, resp. $b > 2a > 0$). The map u is induced by the complete linear system $|\mathcal{O}_M(h + 2f)|$. Let X be a smooth and projective curve and $\phi : X \to \mathbf{P}^3$ be a non-degenerate morphism which is birational onto its image and such that $\phi(X) \subset T$. Set $d := \deg(\phi(X))$. Let μ be the multiplicity of $\phi(X)$ at P and C the strict transform of $\phi(X)$ in M. Then $d - \mu$ is even and $C \in |\mathcal{O}_M(((d-\mu)/2)h + df)|$. Hence X has gonality at most $(d-\mu)/2$. We have $\omega_C \cong \mathcal{O}_C((d - \mu - 4)/2)h + (d-4)f)$ and hence $p_a(C) = 1 + (d^2 - 4d - \mu^2)/4$.

Remark 1. Take the set-up of Example 2. Fix an integer $u \geq 4$, an integer $u' \geq u$, an integer $d > 2u'$ and an integer $d' \leq d - 1$. Let $A \subset M$ be an integral curve, $A \in |u'h + d'f|$. We have $p_a(A) = 1 + u'(d' - u') - d' \leq 1 + u(d - 1 - u) - d + 1 < 1 + u(d-u) - 2u = \gamma(u, d-u)$. Notice that $\gamma(u, d-u) \leq \gamma(x, d-x)$ for all x such that $u \leq x \leq \lfloor d/2 \rfloor$.

Remark 2. Let N be a smooth and connected projective surface, H an ample line bundle on N and F a vector bundle on N. For any torsion free sheaf G on N let $\mu(G,H) := (c_1(G)\cdot H)/\mathrm{rank}(G)$ denote its H-slope. Set $r := \mathrm{rank}(F)$. We recall that F is called μ-semistable with respect to H if $\mu(G,H) \le \mu(F,H)$ for all non-zero subsheaves G of F. Now assume that F is μ-semistable with respect to H. D. Gieseker proved in [3] the so-called Bogomolov-Gieseker's inequality: $c_1(F)^2 \le (2r/(r-1))\cdot c_2(F)$. Fix a zero-dimensional subscheme $Z \subset N$, $Z \ne \emptyset$, and $L \in \mathrm{Pic}(N)$. Let E be any rank two vector bundle on M fitting in an exact sequence

$$0 \to \mathcal{O}_N \to E \to \mathcal{I}_{Z,N} \otimes L \to 0 \tag{1}$$

Assume that L is nef and big. By [8], Lemma 2.3, (or see [1], Prop. 1.4, for a related result) E is semistable, unless there is an effective divisor D of N such that $Z \subset D$ and

$$L\cdot D - \deg(Z) \le D^2 < (L\cdot D)/2 < \deg(Z) \tag{2}$$

Furthermore, if $(L\cdot D)^2 = (L^2)(D^2)$ then L and λD are linearly equivalent for some $\lambda \in \mathbb{Q}$ such that $2 < \lambda \le 1 + \deg(Z)/D^2$ ([1], Prop. 1.4).

Lemma 1. *Fix integers u, v, a, b such that $v \ge u \ge 3$, $4u > v$, $0 \le a \le v-u$ and $v \ge u \ge c \ge 0$. Let $A \subset S$ be a general subset such that $\sharp(A) = a$. Then there is no $J \subset S$ such that $\sharp(J) = c$, $h^1(S, \mathcal{I}_{A\cup J}(u-2, v-2)) \ne 0$, and $\sharp(\pi_2(A\cup J)) = a+c$, except that if $v \le u+1$ (and hence $a \le 1$) we also require $\sharp(\pi_1(A\cup J)) = a+c$.*

Proof. The assumption $\sharp(\pi_2(A\cup J)) = a+c$ implies $J\cap A = \emptyset$. The result is known and easy if $a = 0$ (see below for the cases $1 \le a \le 3$ and [7] for our main application of Lemma 1 when $a = 0$), Hence we will assume $a > 0$ and that the result is true for the integer $a' := a-1$. For this fixed a we may also assume $c > 0$ and that the result is true for all $0 \le c' < c$. Assume the existence of $J \subset S$ such that $\sharp(J) = c$, $\sharp(\pi_2(A\cup J)) = a+c$, $\sharp(\pi_1(A\cup J)) = a+c$ and $h^1(S, \mathcal{I}_{A\cup J}(u-2, v-2)) \ne 0$. By our inductive assumptions we have $h^1(S, \mathcal{I}_{(A\cup J)\setminus\{P\}}(u-2, v-2)) = 0$ for all $P \in A\cup J$, i.e $h^0(S, \mathcal{I}_{(A\cup J)\setminus\{P\}}(u-2, v-2)) = h^0(S, \mathcal{I}_{A\cup J}(u-2, v-2))$ for all $P \in A\cup J$. Hence the set $A\cup J$ has the Cayley-Bacharach property with respect to the line bundle $\mathcal{O}_S(u,v)$ ([4], p. 671 and p. 731, or [2] or [1]). Hence there is an exact sequence

$$0 \to \mathcal{O}_S \to E \to \mathcal{I}_{A\cup J}(u,v) \to 0 \tag{3}$$

with E locally free, $c_1(E)^2 = 2uv$ and $c_2(E) = a+c$. Since $a+c \le v$ and $u \ge 3$, we have $2uv > 4(a+c)$. Since $v \ge u > 0$, the line bundle $L := \mathcal{O}_S(u,v)$ is nef and big. By Remark 2 there is an effective divisor D on S, say $D \in |\mathcal{O}_S(x,y)|$, such that $A\cup J \subset D$ and the inequalities (2) are satisfied, i.e.

$$yu + xv - a - c \le 2xy < (yu+xv)/2 < a+c \tag{4}$$

Since A is general, we have $(x+1)(y+1) \ge a+1$. The inequalities

$$yu + vx < 2(a+c) \le 2v \tag{5}$$

imply $x \le 1$. We also have $2x \le u$ and $2y \le v$, because $L - 2D$ is effective ([1], 0.2). First assume $x = 0$. Since $\sharp(\pi_2(A\cup J)) = a+c$, we have $y \ge a+c$. Hence from the first inequality in (5) we get $(a+c)u < v$. Hence $a+c \le 3$ (here we use the assumption $4u > v$). To complete this case we will do below (see parts (i), (ii) and (iii)) all cases with $a \le 3$ assuming only the weaker inequality $y \ge a$. Now assume

$x = 1$. From (5) we get $yu < v$, while we also have $2y \geq a - 1$. Hence from (5) we get $(a-1)u < 4v$. Hence $a \leq 16$, because $4u > v$. We take y minimal and in this case we must distinguish the case " D irreducible " (see part (iv) below) and the case " D reducible " (see part (v) below).

(i) Assume $a = 1$. From (4) we get $2 + 2u > yu + xv$. Hence it is sufficient to check the following cases:

(i1) $y = 0$, $x = 1$;
(i2) $y = x = 1$, $c = u$, $u \leq v \leq v + 1$;
(i3) $y = 0$, $x = 2$, $v = u$.

In case (i1) D is of type $(1, 0)$, i.e. the set $A \cup J$ is contained in a fiber F of π_2. Use the cohomology of $\mathcal{O}_F(u)$ and the surjectivity of the map $H^0(S, \mathcal{O}_S(u, v)) \to H^0(F, \mathcal{O}_F(u))$, true because $v \geq 0$. Similarly, in the case (i3) the set $A \cup J$ is contained in two fibers of π_2 and we conclude, just using that $v > 0$. In case (i2) the set $A \cup J$ is contained in a curve $D \in \mathcal{O}_S(1, 1)$. If D is reducible, then we conclude as above using only that $u > 0$ and $v > 0$. If D is irreducible, then $D \cong \mathbf{P}^1$ and we conclude because the restriction map $H^0(S, \mathcal{O}_S(u, v)) \to H^0(D, \mathcal{O}_D(u, v))$ is surjective, $\deg(\mathcal{O}_D(u, v)) = u + v$ and $a + c \leq v \leq u + v + 1$.

(ii) Assume $a = 2$. From (4) we get $4 + 2u > yu + xv$. Hence it is sufficient to check the following cases (here we use $v \geq u \geq 3$):

(ii1) $y = 0$, $0 \leq x \leq 2$, $u \leq v \leq u + 1$;
(ii2) $y = 0$, $x = 1$;
(ii3) $y = 1$, $x = 1$, $u \leq v \leq u + 3$.

Case (ii2) is impossible, because two general points of A have different images by the map π_2. Case (ii1) (resp. (ii3)) is done as case (i3) (resp. (i2)) of part (i).

(iii) Assume $a = 3$. From (4) we get $6 + 2u > yu + xv$. Furthermore, $(x+1)(y+1) \geq 4$ and $v \geq u \geq 3$. Hence it is sufficient to check the following cases:

(iii1) $y = 0$, $x = 2$, $u \leq v \leq u + 1$;
(iii2) $y = 1$, $x = 1$, $u \leq v \leq u + 5$.

Case (iii1) is done as cases (ii3). Case (iii2) is done as case (i2), distinguishing between the case " D reducible " and the case " D irreducible ". First assume D irreducible. Hence $D \cong \mathbf{P}^1$ and $\deg(\mathcal{O}_D(u, v)) = u+v+1$. Since $a+c \leq v \leq u+v+1$, the restriction map $H^0(D, \mathcal{O}_D(u, v)) \to H^0(A \cup J, \mathcal{O}_{A \cup J}(u, v))$ is surjective. If D is reducible, then a very weak form of our assumptions " $\sharp(\pi_2(A \cup J)) = a + c$ and $\sharp(\pi_1(A \cup J)) = a + c$ if $v \leq u + 1$ " gives $a + c \leq 2$, contradiction.

(iv) Here we assume $x = 1$ and D irreducible. Since $2y \leq v$ and $v \geq u \geq 4$ (the nefness of $L - 2D$ stated in [1], 0.2), we have $h^1(S, \mathcal{O}_S(u - 3, v - y - 2)) = 0$. Hence the restriction map $H^0(S, \mathcal{O}_S(u - 2, v)) \to H^0(S, \mathcal{O}_D(u - 2, v - 2))$ is surjective. We get $h^1(S, \mathcal{I}_{A \cup J, S}(u - 2, v - 2)) = 0$, contradiction.

(v) Here we assume $x = 1$ and D reducible. If D is a union of a divisor of type $(1, 0)$ and y divisors of type $(0, 1)$, then we have $y \geq \sharp(\pi_2(A \cup J)) = a + c$. Since $2y \leq v$, this is a very easy case. Now assume that D is the union of a divisor D' of type $(1, y')$, $0 < y' < y$, and $y - y'$ divisors of type $y - y'$. We easily get $A \subset A'$. Set $J' := J \cap D'$. By the minimality of y we have $J' \neq J$. By assumption (ii) and the minimality of y we have $c - \sharp(J') = y - y'$.Since $2y \leq v$ and $v \geq u \geq 4$, we get again $h^1(S, \mathcal{O}_S(u-3, v-y-2)) = 0$. Notice that D' contains exactly $a+c+y'-y$ points of $A \cup J$, while each of the other components of D contains exactly one point of $A \cup J$.

Since $D' \cong \mathbf{P}^1$, we get as in part (iv) that $h^1(S, \mathcal{O}_{(A\cup J)\cap D'}(u-2, v-2-y+y')) = 0$. Using $y - y'$ Mayer-Vietoris exact sequences and that each component of $D\backslash D'$ contains only one point of $(A \cup J)\backslash(A \cup J) \cap D'$ we get $h^1(S, \mathcal{I}_{A,S}(u-2, v-2)) = 0$, contradiction. □

Remark 3. Fix integers $v \geq u \geq 3$ and a such that $0 \leq a \leq v - u$. Let $A \subset S$ be a general subset with $\sharp(A) = a$. Set $Z := \bigcup_{P\in A} 2P$. It is very easy to check that $h^1(S, \mathcal{I}_Z(u, v)) = 0$, $h^0(S, \mathcal{I}_Z(u, v)) = (u + 1)(v + 1) - 3a$ and that a general $C \in |\mathcal{I}_Z(u, v)|$ is integral, nodal and with $A = \mathrm{Sing}(C)$.

Remark 4. Let $C \subset S$ be an integral nodal curve and $\nu : X \to C$ its normalization. Set $A := \mathrm{Sing}(C)$. Let W be any base point free pencil on X. Fix any finite set $\{D_i\}_{i\in I}$ of integral curves on S, say $D_i \in |\mathcal{O}_S(x_i, y_i)|$, such that for every $i \in I$ the curve D_i is the only element of $|\mathcal{O}_S(x_i, y_i)|$ containing the set $A \cap D_i$. Take a general $B \in W$. Since W is base point free and B is general in W we have $A \cap B = \emptyset$ (i.e. $\sharp(\nu(B)) = \sharp(B)$) and $\nu(B) \cap D_i = \emptyset$ for all $i \in I$. Now assume that any ramification point of $\pi_i|(C\backslash A)) : C\backslash A \to \mathbf{P}^1$, $i = 1, 2$, is an ordinary ramification point and that two such ramification points are mapped by π_i onto different points of $\mathbf{P}^1$. This implies that there is no morphism $\alpha : X \to \mathbf{P}^1$, such that $\deg(\alpha) \geq 3$ and $\pi_i \circ \nu$ factors through α. In this case either W is induced by a ruling of π_i or we have $\sharp(\pi_i(\nu(B)) = \sharp(B)$. Take the set-up of Remark 3. Our assumption on the ramification points of the map $\pi_i|(C\backslash A))$ is satisfied for a general $C \in \Gamma_A$.

Remark 5. Let $C \subset \mathbf{P}^3$ be an integral degree d curve not contained in any quadric surface. Then $p_a(C) \leq \pi_1(d, 3)$ ([5], Th. 3.13).

Remark 6. Take C and X as in Remarks 3 and 4. By Lemma 1 X has gonality exactly u. Hence the degree u pencil $(\pi_1|C) \circ \nu$ is complete. Condition (iii) implies that for any non-constant morphism $f : X \to \mathbf{P}^1$ such that $f \neq (\pi_1|C) \circ \nu$, the morphism $((\pi_1|C) \circ \nu, f) : X \to S$ is birational onto its image and hence its image has arithmetic genus at least $uv - u - v + 1 - a$.

Remark 7. Take v, u, a and A as in Lemma 1 and set $\Gamma := \bigcup_A \Gamma_A$, where the union is over all sufficiently general A. We get $\dim(\Gamma) = (u+1)(v+1) - a - 1$. This well-known equality could also be checked under far less restrictive assumptions on the integer u, v, a just studying the normal sheaf of a nodal curve in S.

Proof of Theorem 1. By our assumptions on d and g there are integers u, v such that $v \geq u \geq 3$, $u + v = d$, $4u > v$ and $\gamma(u - 1, v + 1) < g \leq \gamma(u, v)$. Set $a := p_a(u, v)$ and fix a general $A \subset S$ such that $\sharp(A) = a$. Notice that $\gamma(u, v) - \gamma(u - 1, v + 1) = v - u - 1$. Hence $a \leq v - u$. By Remarks 3 and 4 we may apply Lemma 1. Hence the normalization, X, of any integral nodal curve $C \in |\mathcal{O}_S(u, v)|$ with $\mathrm{Sing}(C) = A$ has gonality u and that any g^1_u on X is induced by a ruling of S. Hence there is no morphism $\psi : X \to S$ birational onto its image and with $\phi(X) \in |\mathcal{O}_S(u', v')|$ with either $u' < u$ or $v' < v$. Furthermore, since $p_a(\phi(X)) \geq p_a(X) = g > \gamma(u, t)$, for all $t < v$ there is no such morphisms with $u' = u$ and $v' < v$. Hence there is no morphism $\phi : X \to \mathbf{P}^3$ birational onto its image, with $\deg(\phi(X)) < d$ and with $\phi(X) \subset S$. By Remark 1 there is no morphism $\psi : X \to \mathbf{P}^3$ birational onto its image, with $\deg(\psi(X)) < d$ and with $\psi(X)$ contained in a quadric cone. Since $g > \pi_1(d - 1, 3)$, we get $\rho_X(3) = d$. □

References

[1] M. Beltrametti, P. Francia and A. J. Sommese, On Reider's method and higher order embeddings, Duke Math. J. 58 (1989), no. 2, 425–439.

[2] F. Catanese, Footnotes to a theorem of I. Reider, Algebraic Geometry, Proceedings, L'Aquila 1988, pp. 67–74, Lecture Notes in Mathematics 1417, Springer, Berlin, 1990.

[3] D. Gieseker, On a theorem of Bogomolov on Chern classes of stable bundles, Amer. J. Math. 101 (1979), no. 1, 79–85.

[4] P. Griffiths and J. Harris, Principles of Algebraic Surfaces, Wiley & Sons, New York, 1978.

[5] J. Harris, with the collaboration of D. Eisenbud, Curves in Projective Space, Les Presses de l'Université de Montréal, Montréal, 1982.

[6] C. Keem and G. Martens, Curves without plane model of small degree, preprint.

[7] G. Martens, The gonality of curves on a Hirzebruch surface, Arch. Math. (Basel) 67 (1996), no. 4, 349–352.

[8] T. Nakashima, On Reider's method for surfaces in positive characteristic, J. Reine Angew. Math. 438 (1993), 175-185.

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