On Countable \((\omega)\)Paracompactness

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Abstract

The notion of countable \((\omega)\)paracompactness is a generalization of the notion of \((\omega)\)paracompactness. In this paper, along with other results, we show that the product space \(X \times Y\) is \((\omega)\)normal if \(X\) is a strongly \((\omega)\)normal, countably \((\omega)\)paracompact space and \(Y\) is a compact metric space.

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1 Introduction

A set equipped with an increasing sequence of topologies on it is called an \((\omega)\)topological space [1]. The notion of \((\omega)\)topological paracompactness is studied in [1] and [2]. Countable \((\omega)\)paracompactness is introduced and studied in [4]. In this paper, we prove some more results on countable \((\omega)\)paracompactness.

2 Preliminaries

The set of natural numbers, the set of rational numbers and the set of real numbers are denoted by \(N\), \(Q\) and \(R\) respectively. We require the following definitions and theorems.
Definition 2.1 (Bose and Tiwari [1]). If \( \{ J_n \} \) is a sequence of topologies on a set \( X \) with \( J_n \subset J_{n+1} \) for all \( n \in N \), then the pair \( (X, \{ J_n \}) \) is called an \((\omega)\)topological space.

Definition 2.2 (Bose and Tiwari [1]). Let \( (X, \{ J_n \}) \) be an \((\omega)\)topological space. A set \( G \in \cap_n J_n \) is called an \((\omega)\)open set. If \( X - F \) is \((\omega)\)open then \( F \) is said to be \((\omega)\)closed.

Definition 2.3 (Bose and Tiwari [1]). An \((\omega)\)topological space \( (X, \{ J_n \}) \) is said to be \((\omega)\)normal if given two \((\omega)\)closed sets \( A \) and \( B \) with \( A \cap B = \emptyset \), there exists an \( n \in N \), such that for some \( U, V \in J_n \), we have \( A \subset U \), \( B \subset V \) and \( U \cap V = \emptyset \).

Definition 2.4 (Bose and Tiwari [4]). An \((\omega)\)topological space \( (X, \{ J_n \}) \) is said to be countably \((\omega)\)paracompact if every countable \((\omega)\)open cover of \( X \) has, for some \( n \in N \), a \((J_n)\)locally finite \((J_n)\)open refinement.

Definition 2.5 (Bose and Tiwari [4]). Let \( (X, \{ A_n \}) \) and \( (Y, \{ B_n \}) \) be two \((\omega)\)topological spaces and for each \( n \), \( \mathcal{P}_n \) be the product topology on \( X \times Y \) of the topologies \( A_n \) and \( B_n \) on \( X \) and \( Y \) respectively. Then \( (X \times Y, \{ \mathcal{P}_n \}) \) is called the product \((\omega)\)topological space of the spaces \( (X, \{ A_n \}) \) and \( (Y, \{ B_n \}) \).

Theorem 2.6 (Bose and Tiwari [4]). Let \( (X, \{ J_n \}) \) be an \((\omega)\)topological space. Suppose for each \( n \in N \), \( (X, J_n) \) is a normal topological space. Then the following statements are equivalent.

(a) \( X \) is countably \((\omega)\)paracompact.

(b) Every countable \((\omega)\)open cover of \( X \) has, for some \( n \in N \), a point finite \((J_n)\)open refinement.

(c) Every countable \((\omega)\)open cover \( \{U_i \mid i \in N\} \) has, for some \( n \in N \), a \((J_n)\)open refinement \( \{V_i \mid i \in N\} \) with \((J_n)\)cl\(V_i \subset U_i\).

Theorem 2.7 (Bose and Tiwari [4]). If an \((\omega)\)topological space \( (X, \{ J_n \}) \) is countably \((\omega)\)paracompact, then for any decreasing sequence \( \{F_i \mid i \in N\} \) of \((\omega)\)closed sets with \( \cap_{i=1}^\infty F_i = \emptyset \), there exists, for some \( n \in N \), a decreasing sequence \( \{G_i \mid i \in N\} \) of \((J_n)\)open sets satisfying \( G_i \supset F_i \) for each \( i \) and \( \cap_{i=1}^\infty (J_n)\)cl\(G_i = \emptyset \). If every countable \((\omega)\)open cover of \( X \), has, for some \( n \in N \), a \((J_n)\)open refinement, then the converse is also true.
3 Countable (ω)paracompactness

Definition 3.1 An (ω)topological space \((X, \{\mathcal{J}_n\})\) is said to be strongly (ω)normal if for each \(n \in \mathbb{N}\), the topological space \((X, \mathcal{J}_n)\) is normal.

Obviously if an (ω)topological space is strongly (ω)normal then it is (ω)normal but the converse is not true as shown by the following example.

Example 3.2 (Bose and Tiwari [3]). Taking \(X = [0, 1]\), let us define an (ω)topological space \((X, \{\mathcal{J}_n\})\) as follows:

Let \(I_n = [0, 1 - \frac{1}{n+1})\) and \(\mathcal{T}_n = \mathcal{U}/I_n\), where \(\mathcal{U}\) denotes the usual topology on \(R\). Now for any \(n \in \mathbb{N}\), define \(\mathcal{J}_n\) to be the topology generated by the base \(\mathcal{T}_n \cup \mathcal{S}\), where \(\mathcal{S}\) is the collection of all the left open subintervals of \(X\) having 1 as the right end point. Then it is easy to see that the (ω)topological space is (ω)normal but not strongly (ω)normal.

Definition 3.3 A real valued function \(f\) on an (ω)topological space \((X, \{\mathcal{J}_n\})\) is called upper (resp. lower) (ω)semicontinuous if for each \(r \in R\), the set \(\{x \in X \mid f(x) < r\}\) (resp. \(\{x \in X \mid f(x) > r\}\)) is (ω)open.

Theorem 3.4 If the (ω)topological space \((X, \{\mathcal{J}_n\})\) is countably (ω)paracompact and strongly (ω)normal and \((Y, d)\) is a compact metric space, then the product (ω)topological space \(X \times Y\) is (ω)normal.

Proof. Let \(\mathcal{D}\) be the topology on \(Y\) generated by the metric \(d\). Suppose \(\mathcal{P}_n\) be the product topology \(\mathcal{J}_n \times \mathcal{D}\) on \(X \times Y\). Then \((X \times Y, \{\mathcal{P}_n\})\) is the product (ω)topological space. We consider two (ω)closed sets \(A, B \subset X \times Y\) with \(A \cap B = \emptyset\). Then for some \(n \in \mathbb{N}\), \(A\) and \(B\) are \((\mathcal{P}_n)\)-closed. Let \(\{D_k \mid k \in \mathbb{N}\}\) be a countable base for \(\mathcal{D}\). Let \(\Gamma\) denote the class of all finite subsets \(\gamma\) of \(\mathbb{N}\). Then \(\Gamma\) is countable. Let \(\Gamma = \{\gamma_1, \gamma_2, \ldots\}\). For \(\gamma \in \Gamma\), we write \(H_\gamma = \bigcup_{k \in \gamma} D_k\).

For \(x \in X\), let \(A_x = \{y \in Y \mid (x, y) \in A\}\) and \(B_x = \{y \in Y \mid (x, y) \in B\}\). Then \(A_x\) and \(B_x\) are \((\mathcal{D})\)-closed and hence \((\mathcal{D})\)-compact subsets of \(Y\). Also we have \(A_x \cap B_x = \emptyset\). Let

\[ U_\gamma = \{x \in X \mid A_x \subset H_\gamma \subset (\mathcal{D})clH_\gamma \subset Y - B_x\}. \]

Let \(x_0 \in \{x \in X \mid A_x \subset H_\gamma\}\). So if \(y \in Y - H_\gamma\), then \(y \notin A_{x_0}\) and hence \((x_0, y) \notin A\). Since \(A\) is \((\mathcal{P}_n)\)-closed, there is a neighbourhood \(L_y \times M_y\) of \((x_0, y)\) which does not intersect \(A\). Since \(Y - H_\gamma\) is a \((\mathcal{D})\)-compact subset of \(Y\), a finite number of \((\mathcal{D})\)-open sets \(M_y\) cover \(Y - H_\gamma\). Suppose \(L_{x_0}\) is the intersection of the corresponding finite number of \((\mathcal{J}_n)\)-open sets \(L_y\). Then \(L_{x_0} \times (Y - H_\gamma)\) does not intersect \(A\). Therefore if \(x \in L_{x_0}\), then \(A_x \subset H_\gamma\). Thus \(\{x \in X \mid A_x \subset H_\gamma\}\) is \((\mathcal{J}_n)\)-open. Similarly \(\{x \in X \mid (\mathcal{D})clH_\gamma \subset Y - B_x\}\) is \((\mathcal{J}_n)\)-open. Hence \(U_\gamma\) is \((\mathcal{J}_n)\)-open.
For \(x \in X\), there exists a \((D)\)open subset \(D\) of \(Y\) such that
\[
A_x \subset D \subset (D)clD \subset Y - B_x.
\]
Since \(A_x \subset D\) and \(A_x\) is \((D)\)compact, there is a finite set \(\gamma\) of positive integers such that
\[
A_x \subset H_\gamma \subset D \implies A_x \subset H_\gamma \subset (D)clH_\gamma \subset Y - B_x.
\]
Hence \(x \in U_\gamma\). Therefore \(\{U_\gamma \mid \gamma \in \Gamma\}\) forms a countable \((\omega)\)open cover of \(X\). Since \(X\) is countably \((\omega)\)paracompact, the \((\omega)\)open cover \(\{U_\gamma \mid \gamma \in \Gamma\}\) has, for some \(m \in N\), a (\(J_m\))locally finite \((J_m)\)open refinement \(\{V_\alpha\}\). We write
\[
V_{\gamma_k} = \bigcup\{V_\alpha | V_\alpha \subset U_{\gamma_k}, V_\alpha \not\subset U_{\gamma_l} \text{ if } l < k\}.
\]
Then \(\mathcal{V} = \{V_\gamma \mid \gamma \in \Gamma\}\) is (\(J_m\))locally finite \((J_m)\)open refinement of \(\{U_\gamma \mid \gamma \in \Gamma\}\) with \(V_\gamma \subset U_\gamma, \gamma \in \Gamma\). By Theorem 2.6, \(\mathcal{V}\) has, for some \(l \in N\), a (\(J_l\))open refinement \(\mathcal{W} = \{W_\gamma\}\) with \((J_l)clW_\gamma \subset V_\gamma\). Then \(U = \bigcup_\gamma (W_\gamma \times H_\gamma) \in \mathcal{P}_l\). For any point \((x,y)\) of \(A\), we have \(x \in W_\gamma \subset U_\gamma\) for some \(\gamma\). Then \(y \in A_x \subset H_\gamma\). Therefore \((x,y) \in W_\gamma \times H_\gamma\).

Thus \(A \subset U\). For \(x \in X\), there exists a \((J_m)\)open neighbourhood \(N_x\) intersecting a finite number of sets of \(\{W_\gamma\}\). Therefore \(N_x \times Y\) is a \((P_m)\)open neighbourhood of \((x,y), y \in Y\) intersecting a finite number of sets of \(\{W_\gamma \times H_\gamma\}\). If \(m_0 = \max(l,m)\), then we have
\[
(P_{m_0})clU = \bigcup \{(P_{m_0})cl(W_\gamma \times H_\gamma)\}
\]
\[
= \bigcup \{(J_{m_0})clW_\gamma \times (D)clH_\gamma\}
\]
\[
\subset \bigcup \{V_\gamma \times (D)clH_\gamma\}
\]
\[
\subset \bigcup \{U_\gamma \times (D)clH_\gamma\}
\]
\[
\implies (P_{m_0})clU \subset Y - B.
\]
Thus \(X \times Y\) is \((\omega)\)normal.

**Theorem 3.5** Let \((X, \{J_n\})\) be an \((\omega)\)topological space. If \(X\) is countably \((\omega)\)paracompact and strongly \((\omega)\)normal then for a lower \((\omega)\)semitrivial function \(g : X \to R\) and a upper \((\omega)\)semitrivial function \(h : X \to R\) with \(h(x) < g(x)\) for all \(x \in X\), there exists, for some \(n \in N\), a \((J_n)\)continuous function \(f : X \to R\) such that \(h(x) < f(x) < g(x)\) for all \(x \in X\).

Conversely, if the above condition is satisfied and if every countable \((\omega)\)open cover of \(X\), has for some \(n \in N\), a \((J_n)\)open refinement, then \(X\) is countably \((\omega)\)paracompact and for some \(n \in N\), \(X\) is \((J_n)\)normal.

**Proof.** By the \((\omega)\)semitriviality of the functions \(g\) and \(h\), for \(r \in Q\), the sets \(\{x \in X \mid g(x) > r\}\) and \(\{x \in X \mid h(x) < r\}\) are \((\omega)\)open and hence the
set \( D_r = \{ x \in X \mid h(x) < r < g(x) \} \) is \((\omega)\)open. Since for any \( x \in X \), there exists a rational number \( r(x) \) with \( h(x) < r(x) < g(x) \), the collection \( \{ D_r \mid r \in \mathbb{Q} \} \) forms a countable \((\omega)\)open cover of \( X \). Therefore it has, for some \( m \in \mathbb{N} \), a \((\mathcal{J}_m)\)locally finite \((\mathcal{J}_m)\)open refinement \( \{ U_r \} \) with \( U_r \subset D_r \). By Theorem 2.6, there is, for some \( l \in \mathbb{N} \), a \((\mathcal{J}_l)\)open refinement \( \{ V_i \} \) with \( (\mathcal{J}_l)clV_r \subset U_r \). Suppose \( n = \max(l, m) \).

There is a \((\mathcal{J}_n)\)continuous function \( f_r \) with \(-\infty \leq f_r(x) \leq r \) such that \( f_r(x) = -\infty \) for \( x \not\in U_r \) and \( f_r(x) = r \) for \( x \in (\mathcal{J}_n)clV_r \). Let \( f(x) = \sup_{r} f_r(x) \).

Since \( \{ U_r \} \) is \((\mathcal{J}_n)\)locally finite, it follows that the function \( f : X \to R \) thus defined is \((\mathcal{n})\)continuous. Also we have \( h(x) < f(x) < g(x) \).

To prove the converse, we consider two \((\omega)\)closed sets \( A \) and \( B \) with \( A \cap B = \emptyset \). Suppose \( X = A + J_\ell \), \( X - B \in J_m \) and \( k = \max(l, m) \). Then \( A \) and \( B \) are \((J_k)\)closed. Let \( h(x) = 1(\text{resp.} \ g(x) = 1) \) for \( x \in A(\text{resp.} \ x \in B) \) and \( h(x) = 0(\text{resp.} \ g(x) = 2) \) for \( x \not\in A(\text{resp.} \ x \not\in B) \). Then \( h(\text{resp.} \ g) \) is upper(\text{resp.} \ lower \) \((\mathcal{J}_k)\)semitopological and \( h(x) < g(x) \). Hence there is, for some \( n \in N \) a \((\mathcal{J}_n)\)continuous function \( f \) on \( X \) such that \( h(x) < f(x) < g(x) \).

If \( U = \{ x \in X \mid f(x) > 1 \} \) and \( V = \{ x \in X \mid f(x) < 1 \} \), then \( U \) and \( V \) are \((\mathcal{J}_n)\)open, \( A \subset U \), \( B \subset V \) and \( U \cap V = \emptyset \). Here \( X \) is \((\mathcal{J}_n)\)normal. Using Theorem 2.7 and proceeding as Theorem 4 of [5], we can show that \( X \) is countably \((\omega)\)paracompact.

The following result is parallel to Theorem 1 of [7].

**Theorem 3.6** Let \((X, \{ \mathcal{J}_n \})\) be a strongly \((\omega)\)normal space. Then the space \((X, \{ \mathcal{J}_n \})\) is countably \((\omega)\)paracompact iff every countable \((\omega)\)open cover \( \{ U_i \} \) of \( X \) has for some \( m \in \mathbb{N} \), a \((\mathcal{J}_m)\)open refinement \( \{ V_i \} \) with \( V_i \subset U_i \), which again has, for some \( n \in \mathbb{N} \), a countable \((\mathcal{J}_n)\)closed refinement.

**Proof.** Firstly suppose that \( X \) is countably \((\omega)\)paracompact. Let \( \{ U_i \} \) be a countable \((\omega)\)open cover of \( X \). Then it has, for some \( m \in \mathbb{N} \), a \((\mathcal{J}_m)\)open refinement \( \mathcal{V} \). Let \( V_i = \cup \{ V \in \mathcal{V} \mid V \subset U_i \} \). Then \( V_i \subset U_i \) and \( V_i \in \mathcal{J}_m \) for each \( i \). By Theorem 2.6, \( \{ V_i \} \) has for some \( n \in \mathbb{N} \), a \((\mathcal{J}_n)\)open refinement \( \{ G_i \} \) with \((\mathcal{J}_n)clG_i \subset V_i \). So \((\mathcal{J}_n)clG_i \) is a \((\mathcal{J}_n)\)closed refinement of \( \{ V_i \} \).

Conversely, suppose the condition holds. We consider a countable \((\omega)\)open cover \( \{ U_i \} \) of \( X \). For \( m \in \mathbb{N} \), let \( \{ V_i \} \) be a \((\mathcal{J}_m)\)open refinement of \( \{ U_i \} \) with \( V_i \subset U_i \). Also let \( \{ F_j \} \), for some \( n \in \mathbb{N} \), be a countable \((\mathcal{J}_n)\)closed refinement of \( \{ V_i \} \). Let \( V_i \) be the first element in \( \{ V_i \} \) containing \( F_j \) and \( l = \max(m, n) \). By the normality of \((X, \mathcal{J}_l)\), there exists a \( W_{ij} \in \mathcal{J}_l \) such that \( F_j \subset W_{ij} \subset (\mathcal{J}_l)clW_{ij} \subset V_i \).

Then \( \{ W_{ij} \} \) is a countable \((\mathcal{J}_l)\)open refinement of \( \{ V_i \} \). We write \( \{ W_{ij} \} = \{ W_k \} \), \( k \in N \). For \( p \in N \), we write \( A_p = X - \bigcup_{k=1}^{p-1} W_k \) if \( p > 1 \) and \( A_1 = X \). Then each \( A_p \) is \((\mathcal{J}_l)\)closed and \( \{ A_p \} \) is \((\mathcal{J}_l)\)locally finite. If \( B_p = A_p \cap \)}
\((\mathcal{J}_l)\text{cl}W_p\), then \(\{B_p\}\) is \((\mathcal{J}_l)\text{locally finite } (\mathcal{J}_l)\text{closed cover of } X\). Hence if \(D_i = \bigcup \{B_p| B_p \subset V_i\}\), then \(\{D_i\}\) is \((\mathcal{J}_l)\text{closed}\) and \(D_i \subset V_i\). Again by the normality of \((X, \mathcal{J}_l)\), there exists an \(H_i \in \mathcal{J}_l\) such that
\[ D_i \subset H_i \subset (\mathcal{J}_l)\text{cl}H_i \subset V_i. \]
Then \(\{H_i\}\) is a \((\mathcal{J}_l)\text{open refinement of } \{V_i\}\) and hence of \(\{U_i\}\) with \((\mathcal{J}_l)\text{cl}H_i \subset V_i \subset U_i\). Therefore by Theorem 2.6, \((X, \{\mathcal{J}_n\})\) is countably \((\omega)\text{paracompact.}\)

**Theorem 3.7** Let the \((\omega)\text{topological space } (X, \{\mathcal{J}_n\})\) be strongly \((\omega)\text{normal.}\) Then \((X, \{\mathcal{J}_n\})\) is countably \((\omega)\text{paracompact iff every countable } (\omega)\text{open cover of } X,\) has for some \(n \in N,\) a \((\mathcal{J}_n)\text{open finite } (\mathcal{J}_n)\text{open refinement.}\)

*Proof.* Since star finiteness of a refinement implies its local finiteness, the sufficient part follows. So we prove only the necessary part.

Suppose \((X, \{\mathcal{J}_n\})\) is countably \((\omega)\text{paracompact.}\) If \(\mathcal{U} = \{U_i\}\) is a countable \((\omega)\text{open cover of } X\), then it has, for some \(l \in N,\) a \((\mathcal{J}_l)\text{locally finite } (\mathcal{J}_l)\text{open refinement } \mathcal{V} = \{V_i\}\) with \(V_i \subset U_i.\) Again by Theorem 2.6, there is, for some \(m \in N,\) a \((\mathcal{J}_m)\text{open refinement } \mathcal{W} = \{W_i\}\) with \((\mathcal{J}_m)\text{cl}W_i \subset V_i.\) Let \(n = \max(l, m)\). Then \((\mathcal{J}_n)\text{cl}W_i \subset V_i.\) Since the topological space \((X, \mathcal{J}_n)\) is normal, proceeding as in Iseki [6](Theorem, p. 350), we can prove that there exists a \((\mathcal{J}_n)\text{open star finite refinement of } \mathcal{U}.\)

**References**


[4] ______, On \((\omega)\text{compactness and } (\omega)\text{paracompactness, Communicated for publication.}


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