Generalized Derivation in $\Gamma$-Regular Ring

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Abstract

In this paper, we have defined Symmetric Martindale $\Gamma$– Regular ring and generalized derivation in $\Gamma$– Regular ring. Further, we have proved that the product of two generalized derivation in $\Gamma$– Regular ring is again a generalized derivation in $\Gamma$– Regular ring.

Mathematics Subject Classification: 15A33, 15A60

Keywords: Gamma ring, Generalized derivation, Regular ring, $\Gamma$– Regular ring, Symmetric Martindale $\Gamma$– Regular ring

1 Introduction

The notion of a generalized derivation in rings was introduced by Bojan Hvala in 1998[2]. In 1936, the concept of a regular ring was introduced by Von-Neumann [6] and the characterization of a generalized derivation in regular ring was studied in [4]. $\Gamma$– ring was introduced by Nobusawa [5] and further Barnes [1] studied various properties in $\Gamma$– rings.

In 2009, Krishnaswamy introduced $\Gamma$– Regular ring and characterization of $\Gamma$– Regular ring was studied in [3]. In this paper, we study the generalized derivation in $\Gamma$– Regular ring.
In view of the generalized derivation in Regular ring, we now define the generalized derivation in $\Gamma$– Regular ring as follows:

A function $f : R \times \Gamma \times R \to R$ will be called a generalized derivation in $\Gamma$– Regular ring, if there exists a derivation $D$ of $R$ such that $f(axa) = ca + aD$ for $a \in R, x \in \Gamma$ and $c$ is a fixed element in $R$.

2 Preliminaries

**Definition 2.1** Let $R$ and $\Gamma$ be additive abelian groups. Then $R$ is called a $\Gamma$ ring if for any $x, y, z \in R$ and $\alpha, \beta \in \Gamma$, the following conditions are satisfied

(i) $x\alpha y \in R$
(ii) $(x + y)\alpha z = x\alpha z + y\alpha z; x(\alpha + \beta)z = x\alpha z + x\beta z$
(iii) $(x\alpha y)\beta z = x\alpha (y\beta z)$

**Definition 2.2** (2) Let $R$ be $\Gamma$– ring and $F : R \to R$ be an additive map. Then, $F$ is called a generalized derivation if there exists a derivation $D : R \to R$ such that $F(axa) = F(x)\alpha y + xaD(y)$ for all $x, y \in R$ and $\alpha \in \Gamma$.

**Definition 2.3** (6) An element $a$ of a ring $R$ is said to be regular if and only if for $x$ in $R$ such that $axa = a$. The ring $R$ is regular if and only if each element of $R$ is regular.

**Definition 2.4** (4) Let $R$ be an additive abelian group. A function $f : R \to R$ will be called a generalized derivation in Regular ring, if there exist a derivation $D$ of $R$ such that $f(axa) = ca + aD$ for all $x, a \in R$ and $c$ is a fixed element in $R$.

**Definition 2.5** (3) Let $R$ and $\Gamma$ be two additive abelian groups. An element $a \in R$ is said to be $\Gamma$– regular if and only if there exist an element $x \in R$ such that $axa = a$. The ring $R$ is said to be $\Gamma$– regular ring if and only if each element of $R$ is $\Gamma$– regular ring.

**Definition 2.6** The Left Martindale $\Gamma$– regular ring of quotients $Q_L R$ is characterized as the unique regular ring extension $Q$ of $R$ satisfying

(i) if $0 \neq q \in Q$ and $0 \neq I, A \subset R$ with $Aq \subset R$, then $Iq \neq 0$.
(ii) if $0 \neq A \subset R$ and $f : A \to R$, then there exist $axq \in Q$ with $afx = axq$ for all $a \in A$ and $x \in \Gamma$.

The Right Martindale $\Gamma$– regular ring of quotients $Q_R R$ can be defined in the same manner.

**Definition 2.7** The Symmetric Martindale $\Gamma$– regular ring of quotients $Q_S R$ is characterized as the unique regular ring extension $Q$ of $R$ satisfying
(i) If $0 \neq q \in Q$ and $0 \neq I, A, B \subseteq R$ with $Aq, qB \subseteq R$, then $Iq, qI \neq 0$.

and (ii) If $0 \neq A, B \subseteq R$ and $f : A \to R, g : B \to R$ and $(af)b = ax(gb)$ for all $a \in A, b \in B$ and $x \in \Gamma$, then there exist $axq \in Q$ with $axf = axq$ and $gbx = gxq$ for all $a \in A, b \in B$ and $x \in \Gamma$.

**Definition 2.8** The central closure $R_C$ is a $\Gamma$-regular ring which contains the linear identities of $R$ in the sense that if $0 = \neq a, b, c, d \in R$ with $axb = cxd$ for all $x \in \Gamma$ then there exist an element $0 \neq q \in C$ with $a = cxq$ and $b = qxd$.

**Definition 2.9** The extend centroid $C$ is a field and it is in the centre of both $Q_R(R)$ and $Q_S(R)$. The extend cetroid of $R_C$ is equal to $C$, where $R_C$ is equal to its central closure. Since $R_C$ is a $\Gamma$-regular ring as well one can construct the $\Gamma$-regular rings $Q_R(R)$ and $Q_S(R)$.

**Lemma 2.10** If $c_i, d_i \in A$ satisfying $\sum c_i x d_i = 0$ for all $x \in \Gamma$, then there exists $c_i$'s as well as $d_i$'s are $C$- dependent unless all $c_i = 0$ or $d_i = 0$.

**Lemma 2.11** Let $c, d \in A, x \in \Gamma$ and let $f : \Gamma \to A$ be defined by $f(x) = cxd$. If $f$ is a generalized derivation, then either $c \in C$ or $d \in C$.

**Proof:** Let $a \in R$ and $x \in \Gamma$ by the definition of 2.2, we have

$$caxd = caxd + ax\delta(a),$$

where $\delta$ is a derivation. We obtain

$$ax(ad - da) - ax(\delta(a)) = 0$$

for all $a \in R$ and $x \in \Gamma$. Using Lemma 2.10, it follows that either $c \in C$ or $d \in C$.

**Preposition 2.12** Suppose that $\sum_{i=1}^{n} f_i(ax)\alpha_d + \sum_{j=1}^{k} c_j \alpha h_j(ax) = 0$ for all $a, \alpha \in R, x \in \Gamma$ where $f_i, h_j \in R$ and $f_i : R \to A$ and $h_j : R \to R_C$ are any maps. If the sets $\sum_{i=1}^{n} d_i + \sum_{j=1}^{k} c_j$ are $C$- independent, then there exist $q_{ij} \in Q_R(R_C)$ for $i = 1, 2, ..., n$ and $j = 1, 2, ..., k$. Such that $f_i(ax) = -\sum_{j=1}^{k} c_j \alpha q_{ij}$ and $h_j(ax) = \sum_{i=1}^{n} a q_{ij} \alpha d_i$ for all $a, \alpha \in R, x \in \Gamma$ and $i = 1, 2, ..., n$ and $j = 1, 2, ..k$.

**Lemma 2.13** Let $f : R \to R_C$ be an additive map satisfying

$$f(axa) = f(ax)a$$

for all $a \in R$ and $x \in \Gamma$. Then there exists $q \in Q_R(R)$ such that $f(ax) = qax$ for all $a \in R$ and $x \in \Gamma$.

**Proof:** Let $\bar{f} : R \to R_C$ be an extension of $f$ according to

$$\bar{f}(\sum a_i x_i \lambda_i) = \sum \bar{f}(a_i x_i) \lambda_i$$

where $a_i \in R, x_i \in \Gamma$ and $\lambda_i \in C$. Let $I$ be a non-zero ideal in $R$ such that $\lambda_i I \subseteq R$ for every $i$. Take $a \in I$ and the factors in the sum $\sum a_i x_i (\lambda_i \alpha)$ lie in $R$. Therefore, we have $0 = \sum \bar{f}(a_i x_i)(\lambda_i a) = \sum \bar{f}(a_i x_i \lambda_i)(a)$ since this is true for all $a \in I$. We have $\sum \bar{f}(a_i x_i) \lambda_i = 0$. Thus $\bar{f}$ is well defined. This fact that $f(axa) = \bar{f}(axa)$ for all $f(axa) = f(ax)a + ax\delta(a)$, can be verified by a direct computation. This proves that
\( \bar{f} : R_C \to R_C \) is a right \( R_C \) module map, hence there exist \( q \in Q_R(R_C) \) such that \( \bar{f}(ax) = qax \), for all \( x \in \Gamma \). Since \( \bar{f} \) is an extension of \( f \). This proves the lemma.

3 Product of two Generalized derivation in \( \Gamma \) – Regular ring

The aim of this section is to prove that the product of two generalized derivation in \( \Gamma \) – Regular ring is again a generalized derivation in \( \Gamma \) – Regular ring.

**Theorem 3.1** Let \( R \) be a \( \Gamma \) – Regular ring and let \( f_1, f_2 : R \to R \) be the generalized derivation. Then, the product \( f_3 = f_1f_2 \) is again a generalized derivation if and only if the following conditions hold.

(i) there exists \( \alpha \in C \) such that either \( f_1(axa) = \alpha a \) and \( f_2(axa) = \alpha a \).

(ii) there exists \( p, q \in Q_R(R_C) \) such that \( f_1(axa) = pa \) and \( f_2(axa) = qa \).

(iii) there exists \( p, q \in Q_R(R_C) \) such that \( f_1(axa) = ap \) and \( f_2(axa) = aq \).

Proof : From the definition it is clear that the product \( f_3 = f_1f_2 \) is again a generalized derivation.

To prove the converse, let \( f_3 = f_1f_2 \) is a generalized derivation. We have \( f_3(axa) = f_1(axa)a + axd_i(a) \), for \( i = 1, 2, 3 \) and \( a \in R \) and \( x \in \Gamma \) for some derivations \( d_i \).

Now \( f_3(axa) = f_1[f_2(axa)] \), we obtain
\[
f_2(ax)d_1(a) + f_1(ax)d_2(a) + ax(d_1d_2 - d_3)(a) = 0 \quad \text{for all } a \in R \text{ and } x \in \Gamma.
\]
Replacing \( ax \) by \( axa \), we obtain
\[
f_2(ax)d_1(a) + f_1(ax)ad_2(a) + ax[d_1(\alpha)d_2(a) + d_1(a)d_2(\alpha) + \alpha(d_1d_2 - d_3)(a)] = 0 \quad (1)
\]
for all \( a, \alpha \in R \) and \( x \in \Gamma \).

Fix \( \alpha \) and from proposition 2.12, we have the following two possibilities.

(i) \( d_1(a) \) and \( d_2(a) \) are \( \mathbb{C} \) – dependent for all \( a \in R \).

(ii) there exist \( p_1, p_2 \in Q_R(R_C) \) such that \( f_1(ax) = -ap_1 \) and \( f_2(ax) = -ap_2 \).

From case (II), write \( p = -p_1 \) and \( q = -p_2 \), it follows that the condition (iii) holds.

From case (I), since \( d_1(a) \) and \( d_2(a) \) are \( \mathbb{C} \) – dependent, we have \( [d_1(a), d_2(a)] = 0 \) for all \( a \in R \). It follows that
\[ \alpha_1 d_1(a) + \alpha_2 d_2(a) = 0 \] for some \( \alpha_1, \alpha_2 \in C \) \quad \rightarrow (2)

When \( d_1 = 0 \), implies that \( f_1(axa) = f_1(ax) a \) for all \( a \in R \) and \( x \in \Gamma \). It follows from lemma 2.13, there exist \( p \in Q_R(R_C) \) such that \( f_1(axa) = pax \). Equation (1) can be written in the form

\[ pax \alpha d_2(a) - ax \alpha d_3(a) = 0 \] for all \( a \in R, \alpha \in C \) and \( x \in \Gamma \). \quad \rightarrow (3)

In a similar manner, when \( d_2 = 0 \), yields \( f_2(ax) = qax \) for some \( q \in Q_R(R_C) \) for all \( a \in R \) and \( x \in \Gamma \) it follows that the condition (ii) holds. If \( d_2 \neq 0 \) and \( d_1 = 0 \), we choose \( a \in R \) and \( x \in \Gamma \) from (3) and \( axd_2(a) \neq 0 \) and hence equation (2) together with lemma 2.10 yields \( f_1(axa) = \alpha a \), for some \( \alpha \in C \). Similarly, if \( d_2 = 0 \) and \( d_1 \neq 0 \), yields \( f_2(axa) = \alpha a \) for all \( a \in R, x \in \Gamma \) and hence (i) holds.

Finally, let us consider \( d_1 \neq 0 \) and \( d_2 \neq 0 \) so that \( d_2 = \alpha d_1 \), for some non-zero \( \alpha \in C \). Define \( F(ax) = f_2(ax) - \alpha f_1(ax) \) and note that \( F : R \to R_C \) and \( F(axa) = F(axa) a \) for all \( a \in R \) and \( x \in \Gamma \). Thus, Lemma 2.13 yields that there exists \( r \in Q_R(R_C) \) such that \( F(axa) = rax \) and therefore \( f_2(ax) = rax + \alpha f_1(ax) \) for all \( a \in R \) and \( x \in \Gamma \). \quad \rightarrow (4)

Using equation (1), we obtain the result for the function

\[ [2\alpha f_1(ax) + pax] \alpha d_1(a) + ax[d_1(a)g(a) + \alpha h(a)] = 0 \] where \( g(a) = 2\alpha d_1(a) \) and \( h(a) = (d_1d_2 - d_3)(a) \). Pick \( a \in R \) such that \( d_1(a) \neq 0 \) and apply proposition 2.12 there exists \( p \in Q_R(R_C) \) such that

\[ 2\alpha f_1(ax) + pax = -ax \gamma. \] This gives us \( f_1(ax) = pax + axq \) where \( p = -(2\alpha)^{-1}p \) and \( q = -(2\alpha)^{-1}q \) for all \( p, q \in Q_R(R_C) \).

Using equation (4), we obtain a similar result for the other function \( f_2(ax) = (r + \alpha p)ax + axq \alpha \). Now we compute the product \( f_1f_2 \) and obtain \( f_3(ax) = f_1[f_2(ax)] = (pr + \alpha p^2)a + aq^2 + (r + 2\alpha p)aq \). Since both the maps \( ax \to (pr + \alpha p^2)ax + axq^2 \alpha \) and \( f_3 \) are generalized derivations. So, is their difference i.e., the map \( ax \to (r + 2\alpha pa)xq \). By using lemma 2.11, we have \( (r + 2\alpha pa) \in C \) or \( q \in C \). It is easy to notice that the condition (ii) leads to the condition (iii) of the Theorem. So, we choose \( (r + 2\alpha pa) = \beta \in C \). Write \( \gamma = -\alpha \) and \( r = \beta + 2p\gamma \) and we obtain \( f_2(ax) = \beta ax + \gamma(pax - axq) \). Hence, the condition (iv) holds.

**Corollary 3.2** Let \( f_1, f_2 : R \to R \) be the generalized derivations. Then, the product \( f_1f_2 = 0 \) if and only if one of the possibilities holds.

(i) either \( f_1 = 0 \) or \( f_2 = 0 \).

(ii) there exists \( p, q \in Q_R(R_C) \) such that \( f_1(axa) = pa \), \( f_2(axa) = qa \) and \( qp = 0 \).

(iii) there exists \( p, q \in Q_R(R_C) \) such that \( f_1(axa) = ap \), \( f_2(axa) = aq \) and \( pq = 0 \).

(iv) there exists \( p, q \in Q_R(R_C) \) and \( \beta, \gamma \in C \) such that \( f_1(axa) = pa + aq \), \( f_2(axa) = \beta a + \gamma(pa - aq) \) and \( \beta p + \gamma p^2 = \gamma q^2 - \beta q \in C \).
Proof: In all cases it is clear that $f_1f_2 = 0$.

To prove the converse, if for $f_1$ and $f_2$ are non-zero generalized derivation, one of the conditions of Theorem 3.1 must hold.

The condition (i) of the theorem 3.1 hold, $f_1(axa) = ca$ and $f_2(axa) = ca$ for all $a \in R$, $x \in \Gamma$ and $c \in C$ and therefore either $f_1 = 0$ or $f_2 = 0$.

If the condition (ii) of the Theorem 3.1 hold, there exist $p, q \in Q_R(R_C)$ such that $f_1(axa) = ap$ and $f_2(axa) = qa$. We have $f_3 = f_1f_2(axa) = qpa = 0$ for all $a \in R$ and $x \in \Gamma$ and therefore $qp = 0$.

If the condition (iii) of the Theorem 3.1 hold, there exist $p, q \in Q_R(R_C)$ such that $f_1(axa) = ap$ and $f_2(axa) = qa$. We have $f_3 = f_1f_2(axa) = apq = 0$ for all $a \in R$ and $x \in \Gamma$ and therefore $pq = 0$.

Finally from (iv) of the Theorem 3.1, we have $f_1[f_2(axa)] = (\beta p + \gamma p^2)a + a(\beta q - \gamma q^2) = 0$ for all $a \in R$.

where $\beta p + \gamma p^2 = \gamma q^2 - \beta q \in C$ follow from Lemma 2.11.

**Corollary 3.3** Let $f : R \rightarrow R$ be a generalized derivation and $c, d \in C$, if we have $cf(axa) + f(axa)d = 0$ for all $a \in R$, $x \in \Gamma$, then one of the following conditions hold.

(i) $c, d \in C$ and $c + d = 0$. 

(ii) $c \in C$ and there exist $p \in Q_R(R_C)$ such that $f(axa) = pa$ and $(c + d)p = 0$.

(iii) $d \in C$ and there exist $p \in Q_R(R_C)$ such that $f(axa) = ap$ and $p(c + d) = 0$.

(iv) there exists $p, q \in Q_R(R_C)$ and $\beta, \gamma \in C$ such that $f(axa) = \beta a + \gamma (pa - aq)$ and $\beta p + \gamma p^2 = \gamma q^2 - \beta q \in C$.

Proof: Define $g(axa) = ca + ad$ for all $a \in R$ and $x \in \Gamma$. So, we have $gf = 0$ according to corollary 3.2, this is possible in one of the four cases. The possibility $g = 0$ is equivalent to $c, d \in C$ and $c + d = 0$. The cases (ii) and (iii) of corollary 3.2 can be easily transformed in case (ii) and (iii) of corollary 3.3. Similarly, the case (iv) of corollary 3.2 gives us $f(axa) = \beta p + \gamma (pa - aq)$. Once we know this, write $cf(axa) + f(axa)d = 0$ and use lemma 2.11 to derive $\beta p + \gamma p^2 = \gamma q^2 - \beta q \in C$.

**Corollary 3.4** Let $f : R \rightarrow R$ be a generalized derivation. Then $f^2$ is a generalized derivation if and only if there exist $a \in Q_R(R_C)$ such that either $f(axa) = ax$, $x \in \Gamma$ or $f(axa) = xa$, $x \in \Gamma$.

Proof: Suppose that $f^2$ is a generalized derivation by applying theorem 3.1 for $f(axa) = f_1(axa) = f_2(axa)$. The conditions (i) - (iii) of Theorem 3.1 can be unified in saying that $f$ is either left or right multiplication by an element of $Q_R(R_C)$.
References


Received: September, 2010