On Pseudo-Projective and Essentially Pseudo Injective Modules

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Abstract

Pseudo injectivity is a generalization of quasi injectivity. In this paper; certain properties of $M$-pseudo-injective modules, essentially pseudo injective modules and essentially pseudo stable submodules have been investigated. Dually some results on $M$-pseudo projective and mutually pseudo projective-modules have been obtained.

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1 Introduction

In this paper the basic ring $R$ is supposed to be ring with unity and all modules are supposed to be unitary left $R$-modules.

Given two $R$-modules $N$ and $M$, $N$ is called $M$-projective if for every submodule $A$ of $M$, any homomorphism $\alpha : N \to M/A$ can be lifted to a homomorphism $\beta : N \to M$. A module $N$ is called projective if it is $M$-projective for every $R$-module $M$. On the other hand, $N$ is called quasi-projective if $N$ is $N$-projective. Recall that an epimorphism $f : M \to N$ is said to split if there exists a homomorphism $g : N \to M$ with $fog = I_N$. A module $N$ is called $M$-pseudo-projective(or pseudo-projective relative to $M$) if for every submodule $A$ of $M$, any epimorphism $\alpha : N \to M/A$ can be lifted to a homomorphism $\beta : N \to M$. Moreover $N$ is called pseudo-projective if $N$ is $N$-pseudo-projective. Any two modules $N$ and $M$ are called mutually (pseudo-)projective if $N$ is $M$-(pseudo-)projective and $M$ is $N$-(pseudo-)projective.
Moreover, we say $N$ is $M$-pseudo-injective (or pseudo-injective relative to $M$) if for every submodule $A$ of $M$, any monomorphism $\alpha : A \to N$ can be extended to a homomorphism $\beta : M \to N$. Any two modules $M$ and $N$ are called relatively (pseudo-)injective if $M$ is $N$-(pseudo-)injective and $N$ is $M$-(pseudo-)injective. A module $M$ is said to be essentially pseudo injective if for any module $A$, any essential monomorphism $g : A \to M$ and monomorphism $f : A \to M$ there exists $h \in \text{End}(M)$ such that $f = hog$.

A submodule $T$ of a module $M$ is said to be essentially pseudo stable if for any essential monomorphism $g : A \to M$ and monomorphism $f : A \to M$ with $\text{Img} + \text{Im}f \subseteq T$, there exists $h \in \text{End}(M)$ such that $f = hog$ then $h(T) \subseteq T$. Let $M$ and $N$ be any two modules then $M$ is said to be essentially pseudo-$N$-injective if for any essential submodule $A$ of $N$, any monomorphism $f : A \to M$ can be extended to some $g \in \text{Hom}(N, M)$. If $\alpha \in \text{Hom}_R(M, N)$ then $\alpha$ is called regular if there exists $\beta \in \text{Hom}_R(M, N)$, such that $\alpha \circ \beta \circ \alpha = \alpha$. We call $\text{Hom}_R(M, N)$ regular if every $\alpha \in \text{Hom}_R(M, N)$ is regular. An $R$-module $M$ is called completely reducible if every submodule of $M$ is a direct summand of $M$. $P \subseteq \oplus N$ denotes that $P$ is a direct summand of $N$.

## 2 Main Results

**Proposition 1.** (a) If $N$ is $M$-pseudo-projective module, $P \subseteq \oplus N$ and $Q \subseteq \oplus M$ then any epimorphism $\alpha : Q \to P$ splits.

(b) If $N$ is $M$-pseudo-projective module and $P \subseteq \oplus N$ then every epimorphism $\alpha : M \to P$ splits.

(c) If $N$ is $M$-pseudo-projective module, $P \subseteq \oplus N$ and if $\alpha : M \to P$ is epic and $\pi_P : N \to P$ is any projection on $P$ then there exists $\beta : N \to M$ with $\alpha \circ \beta = \pi_P$.

(d) If $N$ is $M$-pseudo-projective module and $S$ is a submodule of $M$ such that $M/S$ is isomorphic to a direct summand of $N$ then $S$ is a direct summand of $M$.

**Proof.** (a) $P \subseteq \oplus N$ implies that $P$ is $M$-pseudo-projective by [6, Theorem 3.3]. Also $Q \subseteq \oplus M$ implies that $P$ is $Q$-pseudo-projective by [6, Theorem 3.9]. So by [6, Theorem 3.1] any epimorphism $\alpha : Q \to P$ splits.

(b) $P \subseteq \oplus N$ implies that $P$ is $M$-pseudo-projective by [6, Theorem 3.3]. So by [6, Theorem 3.1] any epimorphism $\alpha : M \to P$ splits.

(c) By (a) we get $\alpha : M \to P$ splits so there exists $\psi : P \to M$ satisfying $\alpha \circ \psi = I_P$, if we define $\beta = \psi \circ \pi \pi_P$ then $\alpha \circ \beta = \alpha \circ \psi \circ \pi \pi_P = I_P \circ \pi_P = \pi_P$. 

(d) Let \( P \subseteq^\oplus N \) such that \( P \) is isomorphic to \( M/S \) and \( \alpha : P \to M/S \) be the corresponding isomorphism. Let \( \pi_P : N \to P \) be the projection map, \( \psi : M \to M/S \) be the natural map. By \( M \)-pseudo-projectivity of \( N \) there exists a homomorphism \( h : N \to M \) such that \( \alpha \circ \pi_P = \psi \circ h \). Let \( \phi : P \to M \) be given by \( \phi = hoJ_P \) and \( g : M/S \to M \) by \( g(x + S) = \phi \circ \alpha^{-1}(x + S) \). Now, \( \psi \circ g = \psi \circ \phi \circ \alpha^{-1} = \psi \circ h \circ \pi_P \circ \alpha^{-1} = \alpha \circ \pi_P \circ J_P \circ \alpha^{-1} = \alpha \circ \alpha^{-1} = I_{M/S}. \) So the sequence \( 0 \to S \to M \to M/S \to 0 \) splits, which implies that \( S \) is a direct summand of \( M \).

\( \square \)

Dually we have,

**Proposition 2.** (a) If \( M \) is \( N \)-pseudo-injective module, \( P \subseteq^\oplus M \) and \( Q \) is any submodule of \( N \) then every monomorphism \( \alpha : P \to Q \) splits.

(b) If \( M \) is \( N \)-pseudo-injective module and \( P \subseteq^\oplus M \) then every monomorphism \( \alpha : P \to N \) splits.

(c) If \( M \) is \( N \)-pseudo-injective module, \( P \subseteq^\oplus M \) and if \( \alpha : P \to N \) is monic and \( i : P \to M \) is the inclusion map, then there exist \( \beta : N \to M \) with \( \beta \circ \alpha = i \).

(d) If \( M \) is \( N \)-pseudo-injective module and \( K \) is a submodule of \( N \) such that \( K \) is isomorphic to a direct summand of \( M \) then \( K \) is a direct summand of \( N \).

*Proof.* (a) \( P \subseteq^\oplus M \) implies \( P \) is \( N \)-pseudo-injective by [3, Proposition 2.1(4)]. Also \( Q \) is any submodule of \( N \) implies \( P \) is \( Q \)-pseudo-injective by [3, Proposition 2.1(3)]. So by [3, Proposition 2.1(1)] any monomorphism \( \alpha : P \to Q \) splits.

(b) \( P \subseteq^\oplus M \) implies \( P \) is \( N \)-pseudo-injective by [3, Proposition 2.1(4)]. So by [3, Proposition 2.1(1)] any monomorphism \( \alpha : P \to N \) splits.

(c) \( P \subseteq^\oplus M \) implies \( P \) is \( N \)-pseudo-injective by [3, Proposition 2.1(4)]. So \( \alpha : P \to N \) splits by (b), so there exists \( \phi : N \to P \) satisfying \( \phi \circ \alpha = I_P \), if we define \( \beta = i \circ \phi \) then \( \beta \circ \alpha = i \circ \phi \circ \alpha = i \circ I_P = i \).

(d) Let \( P \subseteq^\oplus M \) such that \( P \) is isomorphic to \( K \) and \( \alpha : K \to P \) be the corresponding isomorphism. Let \( J_P : P \to M \) be injection map, \( \psi : K \to N \) be inclusion map. As \( M \) is \( N \)-pseudo-injective there exists homomorphism \( h : N \to M \) such that \( J_P \circ \alpha = h \circ \psi \). Define \( \phi : N \to K \) by \( \phi = \alpha^{-1} \circ \pi_P \circ h \). Now \( \phi \circ \psi = \alpha^{-1} \circ \pi_P \circ h \circ \psi = \alpha^{-1} \circ \pi_P \circ J_P \circ \alpha = \alpha^{-1} \circ \alpha = I_K \). So \( K \) is a direct summand of \( N \).

\( \square \)
Proposition 3. If $M$ and $N$ are mutually pseudo-projective modules, $P$ is a direct summand of $N$ and $Q$ is a direct summand of $M$ then $P$ and $Q$ are mutually pseudo-projective.

Proof. Let $N$ be $M$-pseudo-projective and $Q \subseteq \oplus M$, which implies that $N$ is $Q$-pseudo-projective and $N = P \oplus P'$. Let $Q'$ be a submodule of $Q$ and $f : P \rightarrow Q/ Q'$ be any epimorphism. Define epimorphism $g : N \rightarrow Q/Q'$ by $g = f o \pi_P$, where $\pi_P$ is the natural projection of $N$ onto $P$. Then $g(a,b) = f o \pi_P(a,b) = f(a) \forall a \in P, b \in P'$. As $N$ is $Q$-pseudo-projective there is $g^* : P \oplus P' \rightarrow Q$ lifting $g$. Then $f^* = g^*/P$ is a homomorphism which lifts $f$. So $P$ is $Q$-pseudo-projective. Similarly we can show that $Q$ is $P$-pseudo-projective. Thus $P$ and $Q$ are mutually pseudo-projective. □

Dually we have,

Proposition 4. If $M$ and $N$ are relatively pseudo-injective modules, $P$ is a direct summand of $M$ and $Q$ is a direct summand of $N$ then $P$ and $Q$ are relatively pseudo-injective.

Proposition 5. If $M = P \oplus N$ is pseudo projective then $P \oplus N$ is $P$-pseudo-projective as well as $N$-pseudo-projective.

Proof. Let $f : M \rightarrow P/X$ be any epimorphism where $X$ is a submodule of $P$, $\pi_P : M \rightarrow P$ be the projection map and $\nu : P \rightarrow P/X$ be the natural map. Then by pseudo projectivity of $M$ there exists $h : M \rightarrow M$ which lifts $f$. Then the mapping $\psi = \pi_P h : M \rightarrow P$ lifts $f$, which implies $M$ is $P$-pseudo-projective. Similarly we can show that $M$ is $N$-pseudo-projective. □

Proposition 6. If $P \oplus N$ is pseudo projective then $P$ and $N$ are mutually pseudo-projective.

Proof. We first show $P$ is $N$-pseudo-projective. Let $X$ be a submodule of $N$, $f : P \rightarrow N/X$ be an epimorphism. Define $g : P \oplus N \rightarrow N/X$ by $g = f o \pi_P$, where $\pi_P$ is the projection on $P$ then $g(p,n) = f o \pi_P(p,n) = f(p)$ for $p \in P, n \in N$. Let $\nu : N \rightarrow N/X$ be the natural map. As $P \oplus N$ is pseudo projective, we get by Proposition(5) that $P \oplus N$ is $N$-pseudo-projective. So $\exists g^* : P\oplus N \rightarrow N$ lifting $g$. Then $f^* = g^*/P$ is a homomorphism which lifts $f$. So $P$ is $N$-pseudo-projective. Similarly we can show that $N$ is $P$-pseudo-projective. So $N$ and $P$ are mutually pseudo-projective. □

Corollary(6.1): If $\oplus_{i \in I} M_i$ is pseudo projective, then $M_j$ is $M_k$-pseudo-projective for all distinct $j, k \in I$.

Proof. Straight forward from Proposition 6. □
Proposition 7. Let $N$ and $M$ be mutually pseudo-projective modules and $A$ be any submodule of $N$ such that $N/A$ is isomorphic to a direct summand of $M$, then $A$ is a direct summand of $N$.

Proof. Follows from Proposition 1(d) with suitable changes. 

Corollary 7.1 Let $N$ and $M$ be mutually pseudo-projective modules and $A$ be any submodule of $N$ such that $N/A$ is isomorphic to a direct summand of $M$, then $A$ is also $M$-pseudo-projective.

Proof. From Proposition 7, we get $A$ is a direct summand of $N$ which clearly implies that $A$ is $M$-pseudo-projective by [6, Proposition 3.3].

Proposition 8. If $N$ is $M$-pseudo-projective module and $\alpha(M) \subseteq^\oplus N$ for every $\alpha \in \text{Hom}_R(M, N)$ then $\ker(\alpha) \subseteq^\oplus M$.

Proof. Let $\alpha \in \text{Hom}_R(M, N)$ then $M/\ker(\alpha)$ is isomorphic to $\alpha(M)$ and $\alpha(M) \subseteq^\oplus N$. Since $N$ is $M$-pseudo-projective module, so by Proposition 1(d) we get, $\ker(\alpha)$ is a direct summand of $M$.

Dually we have,

Proposition 9. Let $M$ is $N$-pseudo-injective module and $\ker(\alpha) \subseteq^\oplus M$ for every $\alpha \in \text{Hom}_R(M, N)$ then $\alpha(M) \subseteq^\oplus N$.

Proposition 10. If $N$ is $M$-pseudo-projective module and $\alpha(M) \subseteq^\oplus N$ for every $\alpha \in \text{Hom}_R(M, N)$ then $\text{Hom}_R(M, N)$ is regular.


Dually we have,

Proposition 11. If $M$ is $N$-pseudo-injective module and $\ker(\alpha) \subseteq^\oplus M$ for every $\alpha \in \text{Hom}_R(M, N)$ then $\text{Hom}_R(M, N)$ is regular.

Corollary 11.1: Let $[M, N] = \text{hom}_R(M, N)$ then $[M, N]$ is regular if
(a) $N$ is $M$-pseudo-projective and $N$ is completely reducible.
(b) $M$ is $N$-pseudo-injective and $M$ is completely reducible.

Proposition 12. Let $M$ be an essentially pseudo injective module and $\phi : N \to M$ be an essential monomorphism. Then there exists a monomorphism $g$ in $\text{End}(M)$ extending $\phi$ such that $\text{Im} g$ is stable under $g$.

Proof. Follows from [2, Proposition 5].
We know that a $N$-pseudo-injective module is essentially pseudo-$N$-injective but converse is not true in general by [1, Example 1]. Here we mention a condition under which an essentially pseudo-$N$-injective module is $N$-pseudo-injective.

**Proposition 13.** For a uniform module $N$ the following conditions are equivalent:

(a) $M$ is $N$-pseudo-injective

(b) $M$ is essentially pseudo-$N$-injective.

*Proof.* (a)$\Rightarrow$(b) follows from the definition.

(b)$\Rightarrow$(a)

Let $M$ be an essentially pseudo-$N$-injective module, $A$ be any submodule of $N$. Let $f : A \rightarrow M$ be any monomorphism and $\alpha : A \rightarrow N$ be the inclusion map. $N$ being uniform implies $\alpha$ is an essential monomorphism. Since $M$ is essentially pseudo-$N$-injective $\exists h \in Hom(N,M)$ such that $f = ho\alpha$. Hence $M$ is $N$-pseudo-injective.

**Proposition 14.** Let $(T_i)_{i \in I}$ be a family of essentially pseudo stable submodules of an $R$-module $M$ then $\bigcap_{i \in I} T_i$ is also essentially pseudo stable.

*Proof.* Follows from [2, Proposition 6].

**Proposition 15.** Let $M$ be an essentially pseudo injective module and $N$ be an essential submodule of $M$ stable under monomorphisms of $End(M)$ then $N$ is essentially pseudo stable submodule of $M$.

*Proof.* Follows from [2, Theorem 10].

We now mention here a well known lemma which is dual of well known Homomorphism decomposition theorem:

**Lemma 16:** Let $A, B, C$ be modules and let $f : A \rightarrow B$ be a monomorphism and $g : C \rightarrow B$ be a homomorphism such that $Img \subseteq Imf$, then there exists a unique homomorphism $h : C \rightarrow A$ such that $g = foh$.

**Proposition 17:** Let $M$ be an essentially pseudo injective module and $K$ be an essential submodule of $M$, then $K$ is essentially pseudo injective if $K$ is stable under endomorphisms of $M$. 
Proof. Let $f : A \to K$ be an essential monomorphism, $g : A \to K$ be monomorphism and $\nu : K \to M$ be the inclusion map. Then by the essential pseudo injectivity of $M$ there exists $\psi \in \text{End}(M)$ such that $\nu g = \psi o \nu f$. Since $K$ is stable under $\text{End}(M)$, we have $\psi(K) \subseteq K$. Replacing $K$ by $\nu(K)$ we have $\psi o (K) \subseteq (K) \Rightarrow \text{Im}(\psi o) \subseteq \text{Im}(\nu)$. So by Lemma 16 there exists $\alpha \in \text{End}(K)$ such that $\psi o \nu = \nu o \alpha$. Since $\nu g = \psi o \nu f = \nu o \alpha f$ it implies that $g = \alpha o f$. Hence $K$ is essentially pseudo injective.

Proposition 18: If $M$ is an essentially pseudo injective module and $\phi : N \to M$ is any essential monomorphism then $I = \text{Im}\phi$ is an essentially pseudo stable submodule of $M$.

Proof. The monomorphism $\phi : N \to M$ induces an isomorphism $\phi^* : N \to \phi(N)$. Let $f : \phi(N) \to M$ be the inclusion map, then $\text{Im}(f) + \text{Im}(f\phi^*) \subseteq I$. As $M$ is essentially pseudo injective $\exists g \in \text{End}(M)$ such that $f o \phi^* = g o \phi$. Now, let $g(I) \not\subseteq I$, then there exists at least one $i \in I$ such that $g(i) \in g(I)$ and $g(i) \not\in I$, but $i \in I$ implies that $i \in \phi(N)$. So we have $\phi(n) = i$ for some $n \in N \Rightarrow g(\phi(n)) \not\in I \Rightarrow g o \phi(n) \not\in I \Rightarrow f o \phi^*(n) \not\in I \Rightarrow f o \phi^*(n) \not\in \text{Im}(\phi)$ which is a contradiction, since $\text{Im}(f o \phi^*) \subseteq I$. Thus $g(I) \subseteq I \Rightarrow I = \text{Im}\phi$ is an essentially pseudo stable submodule of $M$.

References


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