With Concern the Difficult Part of the General Berge Last Theorem and the Hadwiger Conjecture

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Abstract

The general Berge last Theorem (see [0] or [1] or [2] or [4] or [5] or [7] or [8]) was first proved by chudnovsky, Robertson, Seymour and Thomas in a paper of more than 145 pages long (see [0]); and an elementary proof of the general Berge last Theorem was given by Ikorong Anouk Nemron in a detailed and simplified article of 37 pages long (see [8]). The Hadwiger conjecture (see [3] or [4] or [6] or [7] or [9] or [10] or [11]) is well known (we recall that the famous four color problem is only a special case of the Hadwiger conjecture). In this paper, via two simple known Theorems, we present the very strong resemblance between the difficult part of the general Berge last Theorem and the Hadwiger conjecture. This strong resemblance immediately implies that: if the Hadwiger conjecture is true, then the difficult part of the general Berge last Theorem and the Hadwiger conjecture were exactly the same problems, and this confirm the conjecture stated in [7] that: the difficult part of the general Berge last Theorem and the Hadwiger conjecture are equivalent; and therefore, the Hadwiger conjecture is only a non obvious special case of the general Berge last Theorem.

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Preliminaries.

Recall that in a graph $G = [V(G), E(G), \chi(G), \omega(G), \bar{G}]$, $V(G)$ is the set of vertices, $E(G)$ is the set of edges, $\chi(G)$ is the chromatic number, $\omega(G)$ is the clique number and $\bar{G}$ is the complementary graph of $G$. We say that a graph
B is berge if every \( B' \in \{ B, \overline{B} \} \) does not contain an induced cycle of odd length \( \geq 5 \). A graph \( G \) is perfect if every induced subgraph \( G' \) of \( G \) satisfies \( \chi(G') = \omega(G') \). The general Berge last Theorem states that a graph \( H \) is perfect if and only if \( H \) is berge. Indeed the difficult part of the general Berge last Theorem consists to show that \( \chi(B) = \omega(B) \) for every berge graph \( B \). Briefly, the difficult part of the general Berge last Theorem will be called the Berge problem. In this topic, we present the very strong resemblance between the Berge problem and the Hadwiger conjecture. If the Hadwiger conjecture is true, then the Berge problem and the Hadwiger conjecture were exactly the same problems, and this confirm the conjecture stated in [7] that: the Berge problem and the Hadwiger conjecture are equivalent; and therefore, the Hadwiger conjecture is only a non obvious special case of the general Berge last Theorem. That being so, this paper is divided into five simple Sections. In Section 1, we present briefly some standard definitions known in Graph Theory. In Section 2, we introduce definitions that are not standard, and some elementary properties. In Section 3 we redefine the graph parameter \( \beta \) (\( \beta \) is explicit in [4] or in [5] or in [7], and is called the berge index), and we give some elementary properties of this parameter. In Section 4 we also redefine the graph parameter denoted by \( \tau \) (\( \tau \) is explicit in [4] or in [6] or in [7], and is is called the hadwiger index), and we also present elementary properties of this parameter. In Section 5, using the couple \( (\beta, \tau) \), we show, via two simple known Theorems, the very strong resemblance between the Hadwiger conjecture and the Berge problem. This very strong resemblance immediately implies that: if the Hadwiger conjecture is true, then the Berge problem and the Hadwiger conjecture were exactly the same problems, and this confirm the conjecture stated in [7] that: the Berge problem and the Hadwiger conjecture are equivalent; and therefore, the Hadwiger conjecture is only a non obvious special case of the general Berge last Theorem. Here, all results are completely different from all the investigations that have been done around the Berge problem and the Hadwiger conjecture in the past. In this paper, every graph is finite, is simple and undirected. We start.

1. Standard definitions.

We start by standard definitions (see [1] or [5] for instance). Recall that in a graph \( G = [V(G), E(G)] \), \( V(G) \) is the set of vertices and \( E(G) \) is the set of edges. \( \overline{G} \) is the complementary graph of \( G \) (recall \( \overline{G} \) is the complementary...
graph of $G$, if $V(G) = V(\bar{G})$ and two vertices are adjacent in $G$ if and only if they are not adjacent in $\bar{G}$). A graph $F$ is a subgraph of $G$, if $V(F) \subseteq V(G)$ and $E(F) \subseteq E(G)$. We say that a graph $F$ is an induced subgraph of $G$ by $Z$, if $F$ is a subgraph of $G$ such that $V(F) = Z, Z \subseteq V(G)$, and two vertices of $F$ are adjacent in $F$, if and only if they are adjacent in $G$. For $X \subseteq V(G), G \setminus X$ denotes the subgraph of $G$ induced by $V(G) \setminus X$. A clique of $G$ is a subgraph of $G$ that is complete; such a subgraph is necessarily an induced subgraph (recall that a graph $K$ is complete if every pair of vertices of $K$ is an edge of $K$); $\omega(G)$ is the size of a largest clique of $G$, and $\omega(G)$ is called the clique number of $G$. A stable set of a graph $G$ is a set of vertices of $G$ that induces a subgraph with no edges; $\alpha(G)$ is the size of a largest stable set, and $\alpha(G)$ is called the stability number of $G$. The chromatic number of $G$ (denoted by $\chi(G)$) is the smallest number of colors needed to color all vertices of $G$ such that two adjacent vertices do not receive the same color. It is easy to see:

Assertion 1.0. Let $G$ be a graph. Then $\omega(G) \leq \chi(G)$. □

The hadwiger number of a graph $G$ (denoted by $\eta(G)$), is the maximum of $p$ such that $G$ is contractible to the complete graph $K_p$. [[ Recall that, if $e$ is an edge of $G$ incident to $x$ and $y$, we can obtain a new graph from $G$ by removing the edge $e$ and identifying $x$ and $y$ so that the resulting vertex is incident to all those edges (other than $e$) originaly incident to $x$ or to $y$. This is called contracting the edge $e$. If a graph $F$ can be obtained from $G$ by a succession of such edge-contractions, then, $G$ is contractible to $F$. The maximum of $p$ such that $G$ is contractible to the complete graph $K_p$ is the hadwiger number of $G$, and is denoted by $\eta(G)$]]. The Hadwiger conjecture states that $\chi(G) \leq \eta(G)$, for every graph $G$. Clearly we have:

Assertion 1.1. Let $G$ and let $F$ be a subgraph of $G$. Then $\eta(F) \leq \eta(G)$. □

2. Non-standard definitions and some elementary properties.

In this section, we introduce definitions that are not standard. We say that a graph $B$ is berge (see Preliminaries), if every $B' \in \{B, \bar{B}\}$ does not contain an induced cycle of odd length $\geq 5$. A graph $G$ is perfect, if every induced subgraph $G'$ of $G$ is $\omega(G')$-colorable. The general Berge last Theorem states that a graph $G$ is perfect if and only if $G$ is berge. Indeed, the Berge problem (i.e. the difficult part of the general Berge last Theorem [see Preliminaries ]) consists to show that $\chi(B) = \omega(B)$, for every berge graph $B$. We will see in Section.5 that the Berge problem and the Hadwiger conjecture are strongly resembling.

We say that a graph $G$ is a true pal of a graph $F$, if $F$ is a subgraph of $G$ and
\( \chi(F) = \chi(G); \) \( \operatorname{trpl}(F) \) denotes the set of all true pals of \( F \) (so, \( G \in \operatorname{trpl}(F) \) means \( G \) is a true pal of \( F \)).

Recall that a set \( X \) is a stable subset of a graph \( G \), if \( X \subseteq V(G) \) and if the subgraph of \( G \) induced by \( X \) has no edges. A graph \( G \) is a complete \( \omega(G) \)-partite graph (or a complete multipartite graph), if there exists a partition \( \Xi(G) = \{Y_1, ..., Y_{\omega(G)}\} \) of \( V(G) \) into \( \omega(G) \) stable sets such that \( x \in Y_j \in \Xi(G), y \in Y_k \in \Xi(G) \) and \( j \neq k \), \( \Rightarrow x \) and \( y \) are adjacent in \( G \). It is immediate that \( \chi(G) = \omega(G) \), for every complete \( \omega(G) \)-partite graph. \( \Omega \) denotes the set of graphs \( G \) which are complete \( \omega(G) \)-partite. So, \( G \in \Omega \) means \( G \) is a complete \( \omega(G) \)-partite graph. Using the definition of \( \Omega \), then the following Assertion becomes immediate.

**Assertion 2.0.** Let \( H \in \Omega \) and let \( F \) be a graph. Then we have the following two properties.

(2.0.0) \( \chi(H) = \omega(H). \)

(2.0.1) There exists a graph \( P \in \Omega \) such that \( P \) is a true pal of \( F \).

**Proof.** Property (2.0.0) is immediate (use definition of \( \Omega \) and note \( H \in \Omega \)). Property (2.0.1) is also immediate (indeed, let \( F \) be graph and let \( \Xi(F) = \{Y_1, ..., Y_{\chi(F)}\} \) be a partition of \( V(F) \) into \( \chi(F) \) stable sets (it is immediate that such a partition \( \Xi(F) \) exists). Now let \( Q \) be a graph defined as follows: (i) \( V(Q) = V(F) \), (ii) \( \Xi(Q) = \{Y_1, ..., Y_{\chi(F)}\} \) is a partition of \( V(Q) \) into \( \chi(F) \) stable sets such that \( x \in Y_j \in \Xi(Q), y \in Y_k \in \Xi(Q) \) and \( j \neq k \), \( \Rightarrow x \) and \( y \) are adjacent in \( Q \). Clearly \( Q \in \Omega \), \( \chi(Q) = \omega(Q) = \chi(F) \), and \( F \) is visibly a subgraph of \( Q \); in particular \( Q \) is a true pal of \( F \) such that \( Q \in \Omega \) (because \( F \) is a subgraph of \( Q \) and \( \chi(Q) = \chi(F) \) and \( Q \in \Omega \)). Now put \( Q = P \); property (2.0.1) follows. \( \square \)

So, we say that a graph \( P \) is a parent of a graph \( F \), if \( P \in \Omega \cap \operatorname{trpl}(F) \). In other words, \( P \) is a parent of \( F \), if \( P \) is a complete \( \omega(P) \)-partite graph and \( P \) is also a true pal of \( F \) (observe that such a \( P \) exists, via property (2.0.1) of Assertion 2.0). parent\((F)\) denotes the set of all parents of a graph \( F \) (so, \( P \in \operatorname{parent}(F) \) means \( P \) is a parent of \( F \)). Using the definition of a parent, then the following Assertion is immediate.

**Assertion 2.1.** Let \( F \) be a graph and let \( P \in \operatorname{parent}(F) \). We have the following two properties.

(2.1.0) Suppose that \( F \in \Omega \). Then \( \chi(F) = \omega(F) = \omega(P) = \chi(P) \).

(2.1.1) Suppose that \( F \notin \Omega \). Then \( \chi(F) = \omega(P) = \chi(P) \). \( \square \)

3. The berge index of a graph.

In this section, we define the graph parameter called the berge index and we also recall some elementary properties concerning the berge index. Now using the definition of the berge graph (see Preliminaries or see Section 2) and the definition of \( \Omega \) (see Section 2), then the following Assertion becomes immediate.
Assertion 3.0. Let \( G \in \Omega \). Then, \( G \) is berge. \( \square \)

Assertion 3.0 says that the set \( \Omega \) is an obvious example of berge graphs.

Now, we define the berge index of a graph \( G \). Let \( G \) be a graph. Then the berge index of \( G \) (denoted by \( \beta(G) \)) is defined as follows. \( \beta(G) = \min_{F \in B(G)} \omega(F) \) if \( G \in \Omega \); and \( \beta(G) = \min_{P \in \text{parent}(G)} \beta(P) \) if \( G \notin \Omega \). Recall \( B(G) = \{ F; G \in \text{parent}(F) \text{ and } F \text{ is berge} \} \), and \( \text{parent}(G) \) is the set of all parents of \( G \). Using the previous, it is immediate to see.

Proposition 3.1 (see [4] or [5]). Let \( G \) be a graph. Then the berge index \( \beta(G) \) exists. \( \square \)

We recall (see Section.1) that \( \eta(G) \) is the hadwiger number of \( G \), and we clearly have.

Proposition 3.2. Let \( K \) be a complete graph and let \( G \in \Omega \). Then, we have the following three properties.

(3.2.0) Suppose that \( \omega(G) \leq 1 \). then \( \beta(G) = \omega(G) = \chi(G) = \eta(G) \).
(3.2.1) \( \beta(K) = \omega(K) = \chi(K) = \eta(K) \).
(3.2.2) \( \omega(G) \geq \beta(G) \).

Proof. Property (3.2.0) is immediate. We prove property (3.2.1). Indeed let \( B(K) = \{ F; K \in \text{parent}(F) \text{ and } F \text{ is berge} \} \), recall \( K \) is complete, and clearly \( B(K) = \{ K \} \); observe \( K \in \Omega \), so \( \beta(K) = \min_{F \in B(K)} \omega(F) \) (use definition of parameter \( \beta \) and note \( K \in \Omega \)), and we easily deduce that \( \beta(K) = \omega(K) = \chi(K) \).

Note \( \eta(K) = \chi(K) \) (since \( K \) is complete), and using the previous, we clearly have \( \beta(K) = \omega(K) = \chi(K) = \eta(K) \). Property (3.2.1) follows.

Now we prove property (3.2.2). Indeed, let \( B(G) = \{ F; G \in \text{parent}(F) \text{ and } F \text{ is berge} \} \), recall \( G \in \Omega \), and so \( \beta(G) = \min_{F \in B(G)} \omega(F) \) (use definition of parameter \( \beta \) and note \( G \in \Omega \)); observe \( G \) is berge (use Assertion 3.0), so \( G \in B(G) \), and the previous equality implies that \( \omega(G) \geq \beta(G) \). \( \square \)

We will see in Section.5 that the berge index helps to obtain an original version of the Berge problem.

4. The hadwiger index of a graph.

Here, we define the hadwiger index and we give some elementary properties related to the hadwiger index. We recall (see Section.1) that \( \eta(G) \) is the hadwiger number of \( G \). Using the definition of a true pal (see Section.2), then the following assertion is immediate.

Assertion 4.0. Let \( G \) be a graph. Then, there exists a graph \( S \) such that \( G \) is a true pal of \( S \) and \( \eta(S) \) is minimum for this property. \( \square \)
Now we define the **hadwiger index** of a graph $G$. Let $G$ be a graph and put $\mathcal{A}(G) = \{H; G \in \text{trpl}(H)\}$; clearly $\mathcal{A}(G)$ is the set of all graphs $H$, such that $G$ is a true pal of $H$. The **hadwiger index** of $G$ is denoted by $\tau(G)$, where $\tau(G) = \min_{F \in \mathcal{A}(G)} \eta(F)$. In other words, $\tau(G) = \eta(F'')$, where $F'' \in \mathcal{A}(G)$, and $\eta(F'')$ is minimum for this property. Via Assertion 4.0, it is immediate to see that $\tau(G)$ exists, for every graph $G$.

**Proposition 4.1.** Let $G \in \Omega$. We have the following three properties.

(4.1.0) If $\omega(G) \leq 1$, then $\eta(G) = \omega(G) = \tau(G) = \chi(G)$.

(4.1.1) If $G$ is a complete graph, then $\eta(G) = \omega(G) = \tau(G) = \chi(G)$.

(4.1.2) $\omega(G) \geq \tau(G)$.

**Proof.** Properties (4.1.0) and (4.1.1) are immediate. Now we show property (4.1.2). Indeed, recall $G \in \Omega$, and clearly $\chi(G) = \omega(G)$. Now, put $\mathcal{A}(G) = \{H; G \in \text{trpl}(H)\}$ and let $K$ be a complete graph such that $\omega(K) = \omega(G)$ and $V(K) \subseteq V(G)$; clearly $K$ is a subgraph of $G$ and

$$\chi(G) = \omega(G) = \chi(K) = \omega(K) = \eta(K) = \tau(K) \quad (4.1.2.0).$$

In particular $K$ is a subgraph of $G$ with $\chi(G) = \chi(K)$, and therefore, $G$ is a true pal of $K$. So $K \in \mathcal{A}(G)$ and clearly

$$\tau(G) \leq \eta(K) \quad (4.1.2.1).$$

Note $\omega(G) = \eta(K)$ (use (4.1.2.0)), and inequality (4.1.2.1) immediately becomes $\tau(G) \leq \omega(G)$. □

Observe Proposition 4.1 resembles to Proposition 3.2.

We will see in Section 5 that the hadwiger index helps to obtain the surgical reformulation of the Hadwiger conjecture. Now, we are ready to present the very strong resemblance between the Berge problem and the Hadwiger conjecture.

5. **The very strong resemblance between the Berge problem and the Hadwiger conjecture.**

In this section, we use two simple known Theorems which immediately imply that the Hadwiger conjecture and the Berge problem are very strongly resembling; so resembling that they seem identical. Using the berge index $\beta$, then the following first simple Theorem is known as the original version of the Berge problem (see Preliminaries or see Section 2 for the definition of the Berge problem).

**Theorem 5.1** (see [4] or [5]) (The original version of the Berge problem). The following are equivalent.
(1) The Berge problem is true.
(2) $\omega(G) = \beta(G)$, for every $G \in \Omega$. □

We recall (see Preliminaries or see Section 1) that the Hadwiger conjecture states that $\chi(G) \leq \eta(G)$, for every graph $G$. Using the hadwiger index $\tau$, then the following is the surgical reformulation of the Hadwiger conjecture.

**Theorem 5.2** (see [4] or [6]). (The surgical reformulation of the Hadwiger conjecture). The following are equivalent.
(1) The Hadwiger conjecture is true.
(2) $\omega(G) = \tau(G)$, for every $G \in \Omega$. □

Visibly, Theorem 5.1 and Theorem 5.2 are resembling; so resembling that they seem identical. More precisely, Theorem 5.1 and Theorem 5.2 clearly say that the Berge problem and the Hadwiger conjecture are implicitly the same problem. That being so, remarking via [0] or via [8] that the Berge problem is true, then, using Theorem 5.1 and Theorem 5.2, it becomes trivial to deduce that: **if the Hadwiger conjecture is true, then the Berge problem and the Hadwiger conjecture were exactly the same problems**; and this confirm the conjecture stated in [7] that: **the Berge problem and the Hadwiger conjecture are equivalent; and therefore, the Hadwiger conjecture is only a non obvious special case of the general Berge last Theorem**. So, the following two conjectures are natural, reasonable and not surprising.

**Conjecture.1**: (Tribute to Claude Berge). The Berge problem and the Hadwiger conjecture are exactly the same problem. □

**Conjecture.2** (Render homage to Claude Berge). The Hadwiger conjecture is only a non-obvious special case of the general Berge last Theorem. □

**References.**


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