Semi-Totally Continuous Functions
in Topological Spaces

S. S. Benchalli

Department of Mathematics
Karnatak University, Dharwad-580 003
Karnataka State, India
benchalliss@gmail.com

Umadevi I Neeli

Department of Mathematics
BVB College of Engineering and Technology
Hubli-580031
Karnataka State, India
uineeli@bvb.edu

Abstract

In this paper, a new generalization of strong continuity of Stone called semi-total continuity, which is stronger than total continuity, is presented. Further, some properties of these functions are investigated. Also semi-totally open functions in topological spaces are introduced and studied.

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1 Introduction

In the study of general topological spaces, semi-open sets [10] play an important role in terms of generalizations of continuous functions. By using these sets many researchers have introduced and studied various types of generalizations of continuity. In 1963, N. Levine [10] introduced the class of semi-continuous functions. In 1980, Jain [9] introduced totally continuous functions.
T. M. Nour [14] introduced the concept of totally semi-continuous functions as a generalization of totally continuous functions and several properties of totally semi-continuous functions were obtained.

In this paper, a new generalization of strong continuity of M. Stone [16] called semi-total continuity, which is stronger than total continuity is introduced and studied. Furthermore, basic properties of these functions and preservation theorems of semi-totally continuous functions are investigated and relationships between semi-totally continuous function and graphs are investigated. Also semi-totally open functions in topological spaces are introduced and studied.

2 Preliminaries

Throughout the present paper, spaces always mean topological spaces on which no separation axioms are assumed unless explicitly stated. Let $X$ be a space and $A$ be a subset of $X$. The closure of $A$ and interior of $A$ are denoted by $\text{Cl}(A)$ and $\text{Int}(A)$ respectively. A subset $A$ is said to be regular open if $A = \text{Int}(\text{Cl}(A))$ and regular closed if $A = \text{Cl}(\text{Int}(A))$. A subset $A$ of $X$ is said to be semi-open [10] if $A \subseteq \text{Cl}(\text{Int}(A))$. The complement of a semi-open set is called semi-closed set. The family of semi-open sets of space $X$ is denoted by $\text{SO}(X)$.

The intersection of all semi-closed sets containing $A$ is called semi-closure [4] of $A$ and is denoted by $\text{sCl}(A)$ and the union of all semi-open sets contained in $A$ is called semi-interior [4] of $A$ and is denoted by $\text{sInt}(A)$. The set which is both open and closed is called clopen set. The family of all clopen sets is denoted by $\text{CO}(X)$.

**Definition 2.1** A function $f : X \rightarrow Y$ is said to be

(i) semi-continuous [10] if the inverse image of each open subset of $Y$ is semi-open in $X$.

(ii) totally continuous [9] if the inverse image of every open subset of $Y$ is clopen subset of $X$.

(iii) strongly continuous [16] if the inverse image of every subset of $Y$ is a clopen subset of $X$.

(iv) totally semi-continuous [14] if the inverse image of every open subset of $Y$ is semi-clopen in $X$.

(v) strongly semi-continuous [14] if the inverse image of every subset of $Y$ is semi-clopen in $X$.

(vi) pre semi-open [4] if the image of every semi-open set in $X$ is semi-open in $Y$.

**Definition 2.2** A topological space $X$ is said to be

(i) semi-$T_0$ [11] if for each pair of distinct points in $X$, there exists a semi-open
set containing one point but not the other.

(ii) semi-$T_1$ [11] (resp. clopen $T_1$ [7]) if for each pair of distinct points $x$ and $y$ of $X$, there exist semi-open (resp. clopen) sets $U$ and $V$ containing $x$ and $y$ respectively such that $x \in U$, $y \notin U$ and $x \notin V$, $y \in V$.

(iii) semi-$T_2$ [11] (resp. ultra Hausdorff or $UT_2$ [17]) if every two distinct points of $X$ can be separated by disjoint semi-open (resp. clopen) sets.

(iv) s-normal [12] (resp. ultra-normal [17]) if each pair of non-empty disjoint closed sets can be separated by disjoint semi-open (resp. clopen) sets.

(v) s-regular [11] (resp. ultra regular [8]) if for each closed set $F$ of $X$ and each $x \notin F$, there exist disjoint semi-open (resp. clopen) sets $U$ and $V$ such that $F \subset U$ and $x \in V$.

(vi) semi-normal [6] (resp. clopen normal[7]) if for each pair of disjoint semi-closed (resp. clopen) sets $U$ and $V$ of $X$, there exist two disjoint semi-open (resp. open) sets $G$ and $H$ such that $U \subset G$ and $V \subset H$.

(vii) semi-regular [5] (clopen regular [7]) if for each semi-closed (resp. clopen) set $F$ of $X$ and each $x \notin F$, there exist disjoint semi-open (resp. open) sets $U$ and $V$ such that $F \subset U$ and $x \in V$.

(viii) locally indiscrete [15] if every open set of $X$ is closed in $X$.

(ix) s-connected [13] if $X$ is not the union of two nonempty disjoint semi-open subsets of $X$.

(x) semi-open [1] if $f(U)$ is semi-open in $Y$ for each open set $U$ in $X$.

(xi) semi-closed [2] if $f(F)$ is semi-closed in $Y$ for each closed set $F$ in $X$.

**Definition 2.3 (14)**: Let $X$ be a topological space and $x \in X$. Then the set of all points $y$ in $X$ such that $x$ and $y$ cannot be separated by semi-separation of $X$ is said to be the quasi semi-component of $x$.

A quasi semi-component of a point $x$ in a space $X$ means the intersection of all semi-clopen sets containing $x$.

### 3 Main Results

In this section, the notion of semi-totally continuous functions is introduced. Characterizations and some relationships between semi-totally continuous functions and other similar functions are obtained. Also some basic properties of semi-totally continuous functions are investigated.

**Definition 3.1** A function $f : X \to Y$ is said to be semi-totally continuous function if the inverse image of every semi-open subset of $Y$ is clopen in $X$.

**Example 3.2** Let $X = Y = \{a, b, c\}, \tau = \{X, \phi, \{a\}, \{b, c\}\}$ and $\sigma = \{Y, \phi, \{a\}\}$. Then $SO(Y) = \{Y, \phi, \{a\}, \{a, b\}, \{a, c\}\}$. Define $f(b) = f(c) = a$ and $f(a) = c$. Clearly the inverse image of each semi-open is clopen in $X$. Therefore $f$ is a semi-totally continuous function.
**Theorem 3.3** A function \( f : X \to Y \) is semi-totally continuous if and only if the inverse image of every semi-closed subset of \( Y \) is clopen in \( X \).

**Proof.** Let \( F \) be any semi-closed set in \( Y \). Then \( Y - F \) is semi-open set in \( Y \). By definition \( f^{-1}(Y - F) \) is clopen in \( X \). That is \( X - f^{-1}(F) \) is clopen in \( X \). This implies \( f^{-1}(F) \) is clopen in \( X \).

On the other hand, if \( V \) is semi-open in \( Y \), then \( Y - V \) is semi-closed in \( Y \). By hypothesis, \( f^{-1}(Y - V) = X - f^{-1}(V) \) is clopen in \( X \), which implies \( f^{-1}(V) \) is clopen in \( X \). Thus, inverse image of every semi-open set in \( Y \) is clopen in \( X \). Therefore \( f \) is semi-totally continuous function.

**Theorem 3.4** Every semi-totally continuous function is a totally continuous function.

**Proof.** Suppose \( f : X \to Y \) is semi-totally continuous and \( U \) is any open subset of \( Y \). Since every open set is semi-open, \( U \) is semi-open in \( Y \) and \( f : X \to Y \) is semi-totally continuous, it follows \( f^{-1}(U) \) is clopen in \( X \). Thus inverse image of every open set in \( Y \) is clopen in \( X \). Therefore the function \( f \) is totally continuous. The converse of the above theorem need not be true, as shown by the following.

**Example 3.5** Let \( X = Y = \{a, b, c\} \), \( \tau = \{X, \phi, \{a\}, \{b, c\}\} \) and \( \sigma = \{Y, \phi, \{a\}\} \). Then \( SO(Y) = \{Y, \phi, \{a\}, \{a, b\}, \{a, c\}\} \). Define \( f : (X, \tau) \to (Y, \sigma) \) as \( f(a) = a, f(b) = b, f(c) = c \). Clearly the inverse image of every open set is clopen. Therefore \( f \) is totally continuous. But \( f \) is not semi-totally continuous, because for the semi-open set \( \{a, b\} \), \( f^{-1}(\{a, b\}) = \{a, b\} \) is not clopen in \( X \).

**Theorem 3.6** Every strongly continuous function is semi-totally continuous.

**Proof.** Suppose \( f : X \to Y \) is strongly continuous function and \( A \) be any semi-open set in \( Y \). By definition \( f^{-1}(A) \) is clopen in \( X \). Thus the inverse image of each semi-open set in \( Y \) is clopen in \( X \). Therefore \( f \) is semi-totally continuous. The converse of the above theorem need not be true, as shown by the following.

**Example 3.7** Let \( X = Y = \{a, b, c\} \), \( \tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, c\}\} \) and \( \sigma = \{Y, \phi, \{a\}, \{b, c\}\} \). Then \( SO(Y) = \{Y, \phi, \{a\}, \{b, c\}\} \). Define \( f : (X, \tau) \to (Y, \sigma) \) by \( f(a) = b, f(b) = a, f(c) = c \). Clearly \( f \) is semi-totally continuous function, since the inverse image of every semi-open set in \( Y \) is clopen in \( X \). But it is not strongly continuous, because for the set \( \{b\} \), \( f^{-1}(\{b\}) = \{a\} \) is not clopen in \( X \).
Theorem 3.8 Every semi-totally continuous function is totally semi-continuous.

Proof. Suppose \( f : X \to Y \) is semi-totally continuous function and \( A \) is any open set in \( Y \). Since every open set is semi-open and \( f : X \to Y \) is semi-totally continuous, it follows that \( f^{-1}(A) \) is clopen and hence semi-clopen in \( X \). Thus the inverse image of each open set in \( Y \) is semi-clopen in \( X \). Therefore \( f \) is totally semi-continuous.

The converse of the above need not be true, as shown by the following.

Example 3.9 Let \( X = Y = \{a, b, c\} \), \( \tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\} \) and \( \sigma = \{Y, \phi, \{a\}\} \). \( SO(X) = \{X, \phi, \{a\}, \{a, b\}, \{a, c\}, \{b, c\}\} \) and \( SO(Y) = \{Y, \phi, \{a\}, \{a, b\}, \{a, c\}\} \). Define \( f : (X, \tau) \to (Y, \sigma) \) as \( f(a) = a \) and \( f(b) = f(c) = b \). Clearly the inverse image of every open set is semi-clopen. Therefore \( f \) is totally continuous. But \( f \) is not semi-totally continuous, because for the semi-open set \( \{a\} \), \( f^{-1}\{a\} = \{a\} \) is not clopen in \( X \).

Theorem 3.10 Every semi-totally continuous function is semi-continuous.

Proof. Suppose \( f : X \to Y \) is a semi-totally continuous function and \( A \) is any open set in \( Y \). Since \( f \) is semi-totally continuous function, \( f^{-1}(A) \) is clopen and hence semi-clopen in \( X \). This implies \( f^{-1}(A) \) is semi-open in \( X \). Thus the inverse image of an open set in \( Y \) is semi-open in \( X \). Therefore \( f \) is semi-continuous function.

The converse of the above need not be true, as shown by the following.

Example 3.11 Let \( X = \{a, b, c\} \), \( \tau = \{X, \phi, \{a\}\} \) and \( \sigma = \{X, \phi, \{a\}, \{a, b\}\} \). Then \( SO(X) = SO(Y) = \{X, \phi, \{a\}, \{a, b\}, \{a, c\}\} \). Define \( f : (X, \tau) \to (Y, \sigma) \) as \( f(a) = a \), \( f(b) = b \) and \( f(c) = c \). Clearly \( f \) is a semi-continuous function. But it is not semi-totally continuous, because for the set \( \{a\} \), \( f^{-1}\{a\} = \{a\} \) is not clopen in \( X \).

Thus we have the following interrelationship. Strong continuity \( \Rightarrow \) semi-total continuity \( \Rightarrow \) total continuity \( \Rightarrow \) total semi-continuity \( \Rightarrow \) semi-continuity. The converses are not true in general.

Theorem 3.12 Let \( f : X \to Y \) be a function, where \( X \) and \( Y \) are topological spaces. Then following are equivalent:

(i) \( f \) is semi-totally continuous

(ii) for each \( x \in X \) and each semi-open set \( V \) in \( Y \) with \( f(x) \in V \), there is a clopen set \( U \) in \( X \) such that \( x \in U \) and \( f(U) \subseteq V \).

Proof. (i) \( \Rightarrow \) (ii): Suppose \( f : X \to Y \) is semi-totally continuous and \( V \) be any semi-open set in \( Y \) containing \( f(x) \) so that \( x \in f^{-1}(V) \). Since \( f \) is semi-totally continuous, \( f^{-1}(V) \) is clopen in \( X \). Let \( U = f^{-1}(V) \), then \( U \) is clopen set in \( X \) and \( x \in U \). Also \( f(U) = f(f^{-1}(V)) \subseteq V \). This implies \( f(U) \subseteq V \).
(ii) $\Rightarrow$ (i): Let $V$ be semi-open in $Y$. Let $x \in f^{-1}(V)$ be any arbitrary point. This implies $f(x) \in V$. Therefore by (ii) there is a clopen set $f(G_x) \subset X$ containing $x$ such that $f(G_x) \subset V$, which implies $G_x \subset f^{-1}(V)$. We have $x \in G_x \subset f^{-1}(V)$. This implies $f^{-1}(V)$ is clopen neighbourhood of $x$. Since $x$ is arbitrary, it implies $f^{-1}(V)$ is clopen neighbourhood of each of its points. Hence it is clopen set in $X$. Therefore $f$ is semi-totally continuous.

**Theorem 3.13** Every semi-totally continuous function in to a finite $T_1$ space is strongly continuous.

**Proof.** Suppose $f : X \to Y$ is a semi-totally continuous function, $Y$ be a finite $T_1$ space and $B \subset Y$. Since $Y$ is finite $T_1$, $Y$ must be a discrete space. Therefore $B$ is an open set and hence semi-open set in $Y$. Since $f$ is semi-totally continuous, $f^{-1}(B)$ is clopen in $X$. Thus the inverse image of every subset $B$ of $Y$ is clopen in $X$. Therefore $f$ is strongly continuous.

**Definition 3.14** A topological space $X$ is said to be $sT_{1/2}$ if every semi closed set of $X$ is closed in $X$.

**Theorem 3.15** For a function $f : X \to Y$ the following properties hold
(i) If $f$ is continuous and $X$ is locally indiscrete then $f$ is totally continuous.
(ii) If $f$ is totally continuous and $Y$ is $sT_{1/2}$ then $f$ is semi-totally continuous.

**Proof.** (i) Suppose $f$ is continuous and $V$ is open in $Y$. Since $f$ is continuous and $X$ is locally indiscrete, $f^{-1}(V)$ is open and closed in $X$. Hence $f^{-1}(V)$ is clopen in $X$. Therefore $f$ is totally continuous.

(ii) Suppose $f$ is totally continuous and $V$ is semi-open in $Y$. Then $Y - V$ is semi-closed in $Y$. Since $Y$ is $sT_{1/2}$, $Y - V$ is closed in $Y$, which implies $V$ is open in $Y$. Since $f$ is totally continuous $f^{-1}(V)$ is clopen in $X$. Thus the inverse image of each semi-open set in $Y$ is clopen in $X$. Therefore $f$ is semi-totally continuous.

**Theorem 3.16** Let $f : X \to Y$ be a semi-totally continuous function from a space $X$ into a semi-$T_1$ space $Y$. Then $f$ is constant on each quasi semi-component of $X$.

**Proof.** Let $a$ and $b$ be two points of $X$ that lie in the same quasi semi-component of $X$. Then $f(a)$ and $f(b)$ are elements in $Y$. Assume $f(a) = \alpha \neq \beta = f(b)$. Since $Y$ is semi-$T_1$, $\{\alpha\}$ is semi-closed in $Y$ and so $Y - \{\alpha\}$ is semi-open. Since $f : X \to Y$ is semi-totally continuous $f^{-1}(\{\alpha\})$ and $f^{-1}(Y - \{\alpha\})$ are disjoint clopen subsets of $X$. Further $a \in f^{-1}(\{\alpha\})$ and $b \in f^{-1}(Y - \{\alpha\})$, which is a contradiction in view of the fact that $b$ belongs to the quasi semi-component of $a$ and hence $b$ must belong to every clopen set containing $a$. Hence the result.
Theorem 3.17 If $f : X \to Y$ is semi-totally continuous function from an $s$-connected space $X$ onto any space $Y$, then $Y$ is an indiscrete space.

Proof. Suppose $f : X \to Y$ is a semi-totally continuous function from an $s$-connected space $X$ onto any space $Y$. If possible, suppose $Y$ is not indiscrete. Let $A$ be a proper non empty semi-open subset of $Y$. Then $f^{-1}(A)$ is a proper non-empty clopen and hence semi-clopen subset of $X$. This implies $f^{-1}(A)$ is a proper non-empty semi-open subset of $X$, which is a contradiction to the fact that $X$ is $s$-connected. Therefore $Y$ must be indiscrete.

Theorem 3.18 If $f : X \to Y$ is semi-totally continuous and $A$ is clopen subset of $X$, then the restriction $f|_A : A \to Y$ is semi-totally continuous.

Proof. Consider the function $f|_A : A \to Y$ and $V$ be any semi-open set in $Y$. Since $f$ is semi-totally continuous, $f^{-1}(V)$ is clopen subset of $X$. Since $A$ is clopen subset of $X$ and $(f|_A)^{-1}(V) = A \cap f^{-1}(V)$ is clopen in $A$, it follows $(f|_A)^{-1}(V)$ is clopen in $A$. Hence $f|_A$ is semi-totally continuous.

Theorem 3.19 The composition two semi-totally continuous functions is semi-totally continuous.

Proof. Let $f : X \to Y$ and $g : Y \to Z$ be any two semi-totally continuous functions. Let $V$ be a semi-open set in $Z$. Since $g$ is semi-totally continuous $g^{-1}(V)$ is clopen and hence open in $Y$. Since every open set is semi-open, $g^{-1}(V)$ is semi-open in $Y$. Further, since $f$ is semi-totally continuous, $f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V)$ is clopen in $X$. Hence $g \circ f : X \to Z$ is semi-totally continuous.

Theorem 3.20 If $f : X \to Y$ is semi-totally continuous and $g : Y \to Z$ is irresolute, then $g \circ f : X \to Z$ is semi-totally continuous.

Proof. Let $f : X \to Y$ be semi-totally continuous and $g : Y \to Z$ be irresolute. Let $V$ be semi-open in $Z$. Since $g$ is irresolute, $g^{-1}(V)$ is semi-open in $Y$. Now since $f$ is semi-totally continuous, $f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V)$ is clopen in $X$. Hence $g \circ f : X \to Z$ is semi-totally continuous.

Theorem 3.21 If $f : X \to Y$ is semi-totally continuous and $g : Y \to Z$ is semi-continuous, then $g \circ f : X \to Z$ is totally continuous.

Proof. Let $V$ be open in $Z$. Since $g$ is semi-continuous, $g^{-1}(V)$ is semi-open in $Y$. Now since $f$ is semi-totally continuous, $f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V)$ is clopen in $X$. Hence $g \circ f : X \to Z$ is totally continuous.
Theorem 3.22 Let $f : X \to Y$ be semi-totally continuous and $g : Y \to Z$ be any function. Then $g \circ f : X \to Z$ is semi-totally continuous if and only if $g$ is irresolute.

Proof. Let $g : Y \to Z$ be irresolute. Then the proof follows from theorem 3.20. Conversely, let $g \circ f : X \to Z$ be semi-totally continuous. Let $V$ be semi-open set in $Z$. Since $g \circ f : X \to Z$ is semi-totally continuous, $(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$ is clopen in $X$. Since $f$ is semi-totally continuous, $g^{-1}(V)$ is semi-open in $Y$. Thus the inverse image of each semi-open set in $Z$ is semi-open in $Y$. Hence $g$ is irresolute.

Theorem 3.23 (10) : Let $X_1$ and $X_2$ be topological spaces and $X = X_1 \times X_2$ be the topological product space. Let $A_1 \in SO(X_1)$ and $A_2 \in SO(X_2)$. Then $A_1 \times A_2 \in SO(X_1 \times X_2)$.

Definition 3.24 Let $f : X \to Y$ be a function. Then the graph function of $f$ is defined by $g(x) = (x, f(x))$ for each $x \in X$.

Theorem 3.25 A function $f : X \to Y$ is semi-totally continuous, if its graph function is semi-totally continuous.

Proof. Let $g : X \to X \times Y$ be the graph function of $f : X \to Y$. Suppose $g$ is semi-totally continuous and $F$ be a semi-open set in $Y$. Then $X \times F$ is a semi-open set of $X \times Y$. Since $g$ is semi-totally continuous, $g^{-1}(X \times F) = f^{-1}(F)$ is clopen in $X$. Thus the inverse image of every semi-open set in $Y$ is clopen in $X$. Therefore $f$ is semi-totally continuous.

Let $\{X_\lambda : \lambda \in \Lambda\}$ be a family of topological spaces. Then the product space of $\{X_\lambda : \lambda \in \Lambda\}$ is denoted by $\Pi \{X_\lambda : \lambda \in \Lambda\}$ or simply $\Pi X_\lambda$.

Theorem 3.26 If a function $f : X \to \Pi Y_\lambda$ is semi-totally continuous, then $p_\lambda \circ f : X \to Y_\lambda$ is semi-totally continuous for each $\lambda \in \Lambda$, where $p_\lambda$ is the projection of $\Pi Y_\lambda$ on to $Y_\lambda$.

Proof. For $\lambda \in \Lambda$, suppose $V_\lambda$ is any semi-open set in $Y_\lambda$. Then $p_\lambda^{-1}(V_\lambda)$ is semi-open in $\Pi Y_\lambda$. Since $f$ is semi-totally continuous, $f^{-1}(p_\lambda^{-1}(V_\lambda)) = (p_\lambda \circ f)^{-1}(V_\lambda)$ is clopen in $X$. Therefore $f : X \to Y_\lambda$ is semi-totally continuous.

In the sequel, the relationships between semi-totally continuous functions and separation axioms are investigated.

Theorem 3.27 If $f : X \to Y$ is semi-totally continuous injection and $Y$ is semi-$T_1$, then $X$ is clopen-$T_1$. 
Proof. Let $x$ and $y$ be any two distinct points in $X$. Since $f$ is injective we have $f(x)$ and $f(y) \in Y$ such that $f(x) \neq f(y)$. Since $Y$ is semi-$T_1$, there exist semi-open sets $U$ and $V$ in $Y$ such that $f(x) \in U$, $f(y) \notin U$, $f(y) \in V$ and $f(x) \notin V$. Therefore we have $x \in f^{-1}(U)$, $y \notin f^{-1}(U)$, $y \in f^{-1}(V)$ and $x \notin f^{-1}(V)$, where $f^{-1}(U)$ and $f^{-1}(V)$ are clopen subsets of $X$ because $f$ is semi-totally continuous. This shows that $X$ is clopen-$T_1$.

**Theorem 3.28** If $f : X \to Y$ is semi-totally continuous injection and $Y$ is semi-$T_0$, then $X$ is ultra-Hausdorff.

**Proof.** Let $a$ and $b$ be any pair of distinct points of $X$ and $f$ be injective. Then $f(a) \neq f(b)$ in $Y$. Since $Y$ is semi-$T_0$, there exists a semi-open set $U$ containing say $f(a)$ but not $f(b)$. Then, we have $a \in f^{-1}(U)$ and $b \notin f^{-1}(U)$. Since $f$ is semi-totally continuous $f^{-1}(U)$ is clopen in $X$. Also $a \in f^{-1}(U)$ and $b \in X \setminus f^{-1}(U)$. This implies every pair of distinct points of $X$ can be separated by disjoint clopen sets in $X$. Therefore $X$ is ultra-Hausdorff.

**Theorem 3.29** If $f : X \to Y$ is semi-totally continuous injection and $Y$ is semi-$T_2$, then $X$ is ultra-Hausdorff.

**Proof.** Let $x_1, x_2 \in X$ and $x_1 \neq x_2$. Then, since $f$ is injective, $f(x_1) \neq f(x_2)$ in $Y$. Further, since $Y$ is semi-$T_2$, there exist $V_1$ and $V_2 \in SO(Y)$ such that $f(x_1) \in V_1$, $f(x_2) \in V_2$ and $V_1 \cap V_2 = \phi$. This implies $x_1 \in f^{-1}(V_1)$ and $x_2 \in f^{-1}(V_2)$. Since $f$ is semi-totally continuous, $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are clopen sets in $X$. Also $f^{-1}(V_1) \cap f^{-1}(V_2) = f^{-1}(V_1 \cap V_2) = \phi$. Thus every two distinct points of $X$ can be separated by disjoint clopen sets. Therefore $X$ is ultra-Hausdorff.

**Theorem 3.30** If $f : X \to Y$ is semi-totally continuous, closed injection and $Y$ is $s$-normal, then $X$ is ultra-normal.

**Proof.** Let $F_1$ and $F_2$ be disjoint closed subsets of $X$. Since $f$ is closed and injective, $f(F_1)$ and $f(F_2)$ are disjoint closed subsets of $Y$. Since $Y$ is $s$-normal, $f(F_1)$ and $f(F_2)$ are separated by disjoint semi-open sets $V_1$ and $V_2$ respectively. Therefore we obtain, $F_1 \subset f^{-1}(V_1)$ and $F_2 \subset f^{-1}(V_2)$. Since $f$ is semi-totally continuous, $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are clopen sets in $X$. Also, $f^{-1}(V_1) \cap f^{-1}(V_2) = f^{-1}(V_1 \cap V_2) = \phi$. Thus each pair of non-empty disjoint closed sets in $X$ can be separated by disjoint clopen sets in $X$. Therefore $X$ is ultra-normal.

**Theorem 3.31** If $f : X \to Y$ is semi-totally continuous surjection and $X$ is connected then $Y$ is $s$-connected.
Proof. Suppose $Y$ is not $s$-connected. Let $A$ and $B$ form disconnection of $Y$. Then $A$ and $B$ are semi-open sets in $Y$ and $Y = A \cup B$ where $A \cap B = \emptyset$. Also $X = f^{-1}(Y) = f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$, where $f^{-1}(A)$ and $f^{-1}(B)$ are non-empty clopen sets in $X$, because $f$ is semi-totally continuous. Further $f^{-1}(A) \cap f^{-1}(B) = f^{-1}(A \cap B) = \emptyset$. This implies $X$ is not connected, which is a contradiction. Hence $Y$ is $s$-connected.

Theorem 3.32 If $f : X \to Y$ is totally continuous injective semi-open function from a clopen regular space $X$ onto a space $Y$, then $Y$ is $s$-regular.

Proof. Let $F$ be a closed set in $Y$ and $y \notin F$. Take $y = f(x)$. Since $f$ is totally continuous, $f^{-1}(F)$ is clopen in $X$. Let $G = f^{-1}(F)$. Then we have $x \notin G$. Since $X$ is clopen regular, there exist disjoint open sets $U$ and $V$ such that $G \subset U$ and $x \in V$. This implies $F = f(G) \subset f(U)$ and $y = f(x) \in f(V)$. Further, since $f$ is injective and semi-open, we have $f(U) \cap f(V) = f(G) = \phi$ and $f(U)$ and $f(V)$ are semi-open sets in $Y$. Thus, for each closed set $F$ in $Y$ and each $y \notin F$, there exist disjoint semi-open sets $f(U)$ and $f(V)$ in $Y$ such that $F \subset f(U)$ and $y \in f(V)$. Therefore $Y$ is $s$-regular.

Theorem 3.33 If $f : X \to Y$ is semi-totally continuous injective semi-open function from a clopen regular space $X$ onto a space $Y$, then $Y$ semi-regular.

Proof. Let $F$ be a semi-closed set in $Y$ and $y \notin F$. Take $y = f(x)$. Since $f$ is semi-totally continuous, $f^{-1}(F)$ is clopen in $X$. Let $G = f^{-1}(F)$. Then we have $x \notin G$. Since $X$ is clopen regular, there exist disjoint open sets $U$ and $V$ such that $G \subset U$ and $x \in V$. This implies $F = f(G) \subset f(U)$ and $y = f(x) \in f(V)$. Further, since $f$ is injective and semi-open, we have $f(U)$ and $f(V)$ are semi-open and $f(U) \cap f(V) = f(G) = \phi$. Thus, for each semi-closed set $F$ in $Y$ and each $y \notin F$, there exist disjoint semi-open sets $f(U)$ and $f(V)$ such that $F \subset f(U)$ and $y \in f(V)$. Therefore $Y$ is semi-regular.

Theorem 3.34 Let $f : X \to Y$ be a totally continuous, closed injection. If $Y$ is $s$-regular then $X$ is ultra-regular.

Proof. Let $F$ be a closed set not containing $x$. Since $f$ is closed, we have $f(F)$ is a closed set in $Y$ not containing $f(x)$. Since $Y$ is $s$-regular, there exists disjoint semi-open sets $A$ and $B$ such that $f(x) \in A$ and $f(F) \subset B$, which implies $x \in f^{-1}(A)$ and $F \subset f^{-1}(B)$, where $f^{-1}(A)$ and $f^{-1}(B)$ are clopen sets in $X$ because $f$ is totally continuous. Moreover, since $f$ is injective, we have $f^{-1}(A) \cap f^{-1}(B) = f^{-1}(A \cap B) = f^{-1}(\emptyset) = \emptyset$. Thus, for a pair of a point and a closed set not containing the point, they can be separated by disjoint clopen sets. Therefore $X$ is ultra-regular.
Theorem 3.35 Let \( f : X \to Y \) be a semi-totally continuous and semi-closed injection. If \( Y \) is semi-regular, then \( X \) is ultra-regular.

Proof. Let \( F \) be a closed set not containing \( x \). Since \( f \) is semi-closed, we have \( f(F) \) is a semi-closed set in \( Y \) not containing \( f(x) \). Since \( Y \) is semi-regular, there exists disjoint semi-open sets \( A \) and \( B \) such that \( f(x) \in A \) and \( f(F) \subset B \), which implies \( x \in f^{-1}(A) \) and \( F \subset f^{-1}(B) \), where \( f^{-1}(A) \) and \( f^{-1}(B) \) are clopen sets because \( f \) is semi-totally continuous function. Moreover, since \( f \) is injective, \( f^{-1}(A) \cap f^{-1}(B) = f^{-1}(A \cap B) = f^{-1}(\phi) = \phi \). Thus, for each pair of a point and a closed set not containing the point, they can be separated by disjoint clopen sets. Therefore \( X \) is ultra-regular.

Theorem 3.36 If \( f : X \to Y \) is a totally continuous, injective and semi-open function from a clopen normal space \( X \) onto a space \( Y \) then \( Y \) is s-normal.

Proof. Let \( F_1 \) and \( F_2 \) be any two disjoint closed sets in \( Y \). Since \( f \) is totally continuous, \( f^{-1}(F_1) \) and \( f^{-1}(F_2) \) are clopen subsets of \( X \). Take \( U = f^{-1}(F_1) \) and \( V = f^{-1}(F_2) \). Since \( f \) is injective, we have \( U \cap V = f^{-1}(F_1) \cap f^{-1}(F_2) = f^{-1}(F_1 \cap F_2) = f^{-1}(\phi) = \phi \). Since \( X \) is clopen normal there exist disjoint open sets \( A \) and \( B \) such that \( U \subset A \) and \( V \subset B \). This implies \( F_1 = f(U) \subset f(A) \) and \( F_2 = f(V) \subset f(B) \). Further, since \( f \) is injective semi-open, \( f(A) \) and \( f(B) \) are disjoint semi-open sets. Thus, each pair of disjoint closed sets in \( Y \) can be separated by disjoint semi-open sets. Therefore \( Y \) is s-normal.

Theorem 3.37 If \( f : X \to Y \) is semi-totally continuous injective semi-open function from a clopen normal space \( X \) onto a space \( Y \), then \( Y \) is semi-normal.

Proof. Let \( F_1 \) and \( F_2 \) be any two disjoint semi-closed sets in \( Y \). Since \( f \) is semi-totally continuous, \( f^{-1}(F_1) \) and \( f^{-1}(F_2) \) are clopen subsets of \( X \). Take \( U = f^{-1}(F_1) \) and \( V = f^{-1}(F_2) \). Since \( f \) is injective \( U \cap V = f^{-1}(F_1) \cap f^{-1}(F_2) = f^{-1}(F_1 \cap F_2) = f^{-1}(\phi) = \phi \). Since \( X \) is clopen normal there exist disjoint open sets \( A \) and \( B \) such that \( U \subset A \) and \( V \subset B \). This implies \( F_1 = f(U) \subset f(A) \) and \( F_2 = f(V) \subset f(B) \). Further, since \( f \) is injective semi-open, \( f(A) \) and \( f(B) \) are disjoint semi-open sets. Thus, each pair of disjoint semi-closed sets can be separated by disjoint semi-open sets. Therefore \( Y \) is semi-normal.

In the sequel, we introduce a new class of function called semi-totally open function and also study some of their basic properties.

Definition 3.38 A function \( f : X \to Y \) is said to be semi-totally open if the image of every semi-open set in \( X \) is clopen in \( Y \).

Theorem 3.39 If a bijective function \( f : X \to Y \) is semi-totally open, then the image of each semi-closed set in \( X \) is clopen in \( Y \).
Proof. Let $F$ be a semi-closed set in $X$. Then $X-F$ is semi-open in $X$. Since $f$ is semi-totally open, $f(X-F) = Y-f(F)$ is clopen in $Y$. This implies $f(F)$ is clopen in $Y$.

**Theorem 3.40** A surjective function $f : X \to Y$ is semi-totally open if and only if for each subset $B$ of $Y$ and for each semi-closed set $U$ containing $f^{-1}(B)$, there is a clopen set $V$ of $Y$ such that $B \subset V$ and $f^{-1}(V) \subset U$.

Proof. Suppose $f: X \to Y$ is a surjective semi-totally open function and $B \subset Y$. Let $U$ be semi-closed set of $X$ such that $f^{-1}(B) \subset U$. Then $V= Y-f(X-U)$ is clopen subset of $Y$ containing $B$ such that $f^{-1}(V) \subset U$.

On the other hand, suppose $F$ is a semi-closed set of $X$. Then $f^{-1}(Y-f(F)) \subset X-F$ and $X-F$ is semi-open. By hypothesis, there exists a clopen set $V$ of $Y$ such that $Y-f(F) \subset V$, which implies $f^{-1}(V) \subset X-F$. Therefore $F \subset X-f^{-1}(V)$. Hence $Y-V \subset f(F) \subset f(X-f^{-1}(V)) \subset Y-V$. This implies, $f(F) = Y-V$, which is clopen in $Y$. Thus, the image of a semi-open set in $X$ is clopen in $Y$. Therefore $f$ is a semi-totally open function.

**Theorem 3.41** For any bijective function $f : X \to Y$, the following statements are equivalent (i) Inverse of $f$ is semi-totally continuous. (ii) $f$ is semi-totally open.

Proof. (i)$\Rightarrow$(ii) : Let $U$ be a semi-open set of $X$. By assumption $(f^{-1})^{-1}(U) = f(U)$ is clopen in $Y$. So $f$ is semi-totally open.

(ii)$\Rightarrow$(i) : Let $F$ be semi-open in $X$. Then $f(V)$ is clopen in $Y$. That is $(f^{-1})^{-1}V$ is clopen in $Y$. Therefore $f^{-1}$ is semi-totally continuous. ‘

**Theorem 3.42** The composition of two semi-totally open functions is again semi-totally open.

Proof. Suppose $f : X \to Y$ and $g : Y \to Z$ are any two semi-totally open functions. Then their composition is $g \circ f : X \to Z$. Let $V$ be a semi-open set in $X$. Consider $(g \circ f)(V)= g(f(V))$. Since $f$ is semi-totally open, $f(V)$ is clopen in $Y$. Hence it is open in $Y$. But every open set is semi-open, which implies $f(V)$ is semi-open in $Y$. Since $g$ is semi-totally open, $g(f(V))$ is clopen in $Z$. Thus, the image of each semi-open set in $X$ is clopen in $Z$. Therefore $g \circ f : X \to Z$ is semi-totally open.

**Theorem 3.43** If $f : X \to Y$ is semi-totally continuous and semi-totally closed surjection from an $s$-normal space $X$ to a space $Y$, then $Y$ is ultra-Hausdorff.
Proof. Let $A$ and $B$ be disjoint closed sets of $Y$. Since $f : X \to Y$ is semi-totally continuous, $f^{-1}(A)$ and $f^{-1}(B)$ are clopen hence closed sets in $X$. Since $X$ is s-normal, there exist disjoint semi-open sets $U$ and $V$ such that $f^{-1}(A) \subset U$ and $f^{-1}(B) \subset V$. By theorem 4.2, there are clopen sets $G$ and $H$ such that $A \subset G$, $B \subset H$ and $f^{-1}(G) \subset U$, $f^{-1}(H) \subset V$. Then we have, $f^{-1}(G) \cap f^{-1}(H) \subset U \cap V = \emptyset$, which implies $G \cap H = \emptyset$. Thus every pair of non-empty disjoint closed sets can be separated by disjoint clopen sets. Therefore $Y$ is ultra-Hausdorff.

Theorem 3.44 If $f : X \to Y$ is pre semi-open and $g : Y \to Z$ is semi-totally continuous, then $g \circ f : X \to Z$ is semi-totally open.

Proof. Let $V$ be any semi-open set in $X$. Since $f$ is pre semi-open $f(V)$ is semi-open in $Y$. Since $g$ is semi-totally open, $g(f(V))$ is clopen in $Z$. That is $(g \circ f)(V)$ is clopen in $Z$. Hence $g \circ f$ is semi-totally open.

Theorem 3.45 Let $f : X \to Y$ and $g : Y \to Z$ be two functions such that $g \circ f : X \to Z$ is semi-totally open function. Then (i) If $f$ is irresolute and surjective, then $g$ is semi-totally open. (ii) If $g$ is totally continuous and injective, then $f$ is semi-totally open.

Proof. (i) Let $V$ be a semi-open set in $Y$. Then $f^{-1}(V)$ is semi-open in $X$, because $f$ is irresolute. Since $(g \circ f)$ is semi-totally continuous $(g \circ f)(f^{-1}(V)) = g(V)$ is clopen in $Z$. This shows that $g$ is semi-totally open. (ii) Since $g$ is injective, we have, $f(A) = g^{-1}(g \circ f)(A)$ is true for every subset $A$ of $X$. Let $U$ be any semi-open set in $X$. Therefore $(g \circ f)(U)$ is clopen and hence open in $Z$. Since $g$ is totally continuous, $g^{-1}(g \circ f)(U) = f(U)$ is clopen in $Y$. This shows that $f$ is semi-totally open.

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