An Error Estimate for Finite Volume Method for the Incompressible Navier-Stokes Equations

M. Atounti
Université Mohamed Premier Oujda
Faculté pluridisciplinaire Nador
Dépt de Mathématiques & Informatique
B.P:300, Selouane 62700, Nador, Morocco
atounti@hotmail.fr

A. Alami-Idrissi
Université Mohammed V-Agdal
Faculté des Sciences
Dépt de Mathématiques & Informatique
Avenue Ibn Batouta BP 1014 Rabat 10000 Morocco
alialami@hotmail.com

Abstract
We prove in this paper an error estimate for the stationary incompressible Navier-Stokes equations in two dimensions by the cell centered finite volume method. We use the compactness theorem and the discrete Poincaré inequality and we add some assumptions to give the result.

Mathematics Subject Classification: 74D05, 35D30, 76D05

Keywords: Finite volume method, Galerkin expansion, Navier-Stokes equations, Discrete Poincaré inequality, Cauchy Schwartz inequality

1 Introduction
We are interested in this paper in finding an error estimate for the incompressible Navier-Stokes, which write:

\[
\begin{align*}
-\nu \Delta u^1 + u^1 \frac{\partial u^1}{\partial x} + u^1 \frac{\partial u^1}{\partial y} + \frac{\partial p}{\partial x} &= f^1(x, y) \quad \forall (x, y) \in \Omega \\
-\nu \Delta u^2 + u^2 \frac{\partial u^2}{\partial x} + u^2 \frac{\partial u^2}{\partial y} + \frac{\partial p}{\partial y} &= f^2(x, y) \quad \forall (x, y) \in \Omega
\end{align*}
\]
\[
\frac{\partial u^1}{\partial x}(x, y) + \frac{\partial u^2}{\partial y}(x, y) = 0 \quad \forall (x, y) \in \Omega
\] (3)

With a homogeneous Dirichlet boundary conditions of \(\Omega\) :

\[u^1(x, y) = u^2(x, y) = 0 \quad \forall (x, y) \in \partial \Omega \] (4)

We make the following assumptions :

**Assumption 1**

i/\(\Omega\) is an open bounded, polygonal and connected subset of \(\mathbb{R}^2\).

ii/\(\nu > 0\) and \(f = (f^1, f^2) \in (L^2(\Omega))^2\).

In the above equations, \(u^i\) represents the \(i\)th component of the velocity of a fluid, \(\nu\) the kinematic viscosity and \(p\) the pressure.

Numerical schemes for the Stokes equations and the Navier-Stokes equations have been extensively studied, see [3, 4, 7, 8, 11, 12, 19, 21, 23-30] and also the references therein. Among different schemes, finite elements schemes and finite volume schemes are frequently used for mathematical or engineering studies, see [1, 2, 5, 9, 10, 13-18, 20, 22]. An advantage of finite volume schemes is that the unknowns are approximated by piecewise constant functions.

In this paper, we use the compactness theorem and the discrete Poincaré inequality and we add some assumptions on the mesh to give the error estimate.

This paper is organized as follows : In section 2, we introduce the numerical scheme. Finally, in section 4, the error estimate in a discrete \(H^1\)-norm is proved.

## 2 Numerical Scheme

Let \(T = (K_{i,j})_{i=1,j=1}^{N_1,N_2}\) be an admissible mesh of \(\Omega\) consisting of rectangles, that is satisfying the following assumptions :

Let \(N_1 \in IN^*, N_2 \in IN^*, h_1 - h_{N_1} > 0\) and \(k_1 - k_{N_2} > 0\) such that :

For \(i = 1 - N_1\), \(x_{i+\frac{1}{2}} = x_{i-\frac{1}{2}} + h_i\). For \(j = 1 - N_2\), \(y_{j+\frac{1}{2}} = y_{j-\frac{1}{2}} + k_j\).

For \(i = 1 - N_1\) and \(j = 1 - N_2\) : \(K_{i,j} = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}] \times [y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}]\).

Let \((x_i)_{i=0}^{N_1+1}\) and \((y_j)_{j=0}^{N_2+1}\), such that

\(x_i\) is a meddle of \([x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]\), for all \(i = 1 - N_1\); \(x_0 = x_{\frac{1}{2}}\) and \(x_{N_1+1} = x_{N_1+\frac{1}{2}}\).

\(y_j\) is a meddle of \([y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}]\), for all \(j = 1 - N_2\); \(y_0 = y_{\frac{1}{2}}\) and \(y_{N_2+1} = y_{N_2+\frac{1}{2}}\).

\(x_{i,j} = (x_i, y_j)\) for all \(i = 1 - N_1\) and \(j = 1 - N_2\).

\(h_i^- = x_i - x_{i-\frac{1}{2}}\), \(h_i^+ = x_{i+\frac{1}{2}} - x_i\) for all \(i = 1 - N_1\).

\(k_j^- = y_j - y_{j-\frac{1}{2}}\), \(k_j^+ = y_{j+\frac{1}{2}} - y_j\) for all \(j = 1 - N_2\).

\(h = \max \{ (h_i; i = 1 - N_1), (k_j; j = 1 - N_2) \}\) and \(h_{i,j} = h_i \times k_j\).

The principle of the finite volume method is to integrate the Navier-Stokes (1)
and (2) on each control volume, \( K_{i,j} \).
We denote by \( u^l_{i,j} \) the approximation of \( u \) on each control volume \( K_{i,j} \). Using the approximation of Laplace operator, one gets
\[
\int_{K_{i,j}} \Delta u^l dx dy \approx k_i \left( \frac{u^l_{i+1,j} - u^l_{i,j} - u^l_{i,j} - u^l_{i-1,j}}{h_{i+\frac{1}{2}}} + h_i \left( \frac{u^l_{i,j+1} - u^l_{i,j} - u^l_{i,j} - u^l_{i,j-1}}{k_{j+\frac{1}{2}}} \right) \right).
\]

We find the approximation of nonlinear terms
\[
\int_{K_{i,j}} \left( \frac{\partial (u^l u^l)}{\partial x} + \frac{\partial (u^l u^2)}{\partial y} \right) dx dy \approx k_i \left( u^l_{i+\frac{1}{2},j} u^l_{i+\frac{1}{2},j} - u^l_{i-\frac{1}{2},j} u^l_{i-\frac{1}{2},j} \right) + h_i \left( u^l_{i,j+\frac{1}{2}} u^l_{i,j+\frac{1}{2}} - u^l_{i,j-\frac{1}{2}} u^l_{i,j-\frac{1}{2}} \right).
\]

\( u^l_{i+\frac{1}{2},j} \) and \( u^l_{i,j+\frac{1}{2}} \) are the auxiliaries terms which will be eliminated from the numerical scheme. Hence
\[
u \left( F_{i+\frac{1}{2},j} - F_{i-\frac{1}{2},j} + F_{i,j+\frac{1}{2}} - F_{i,j-\frac{1}{2}} \right) + h_i \left( u^l_{i,j+\frac{1}{2}} u^l_{i,j+\frac{1}{2}} - u^l_{i,j-\frac{1}{2}} u^l_{i,j-\frac{1}{2}} \right)
+ k_j \left( (u^l_{i+\frac{1}{2},j})^2 - (u^l_{i-\frac{1}{2},j})^2 \right) + \sum_{s \in S_{i,j}} p_s \int_{K_{i,j}} \frac{\partial \phi_s}{\partial x} dxdy = h_{i,j} \nu f^l \quad \text{(6)}
\]
\[
u \left( F_{i+\frac{1}{2},j} - F_{i-\frac{1}{2},j} + F_{i,j+\frac{1}{2}} - F_{i,j-\frac{1}{2}} \right) + k_j \left( (u^l_{i+\frac{1}{2},j})^2 - (u^l_{i-\frac{1}{2},j})^2 \right) + \sum_{s \in S_{i,j}} p_s \int_{K_{i,j}} \frac{\partial \phi_s}{\partial y} dxdy = h_{i,j} \nu f^l \quad \text{(7)}
\]

We use a Galerkin expansion for the pressure in the finite volume scheme. Let \( \phi_{i+\frac{1}{2},j} \) or \( \phi_s \) be the shape function associated at the vertex \( s \) or \( (x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}}) \) piecewise bilinear functions and \( p_{i+\frac{1}{2},j} \) or \( p_s \) is an approximation of \( p \) at \( (x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}}) \). Let \( S_T \) and \( S_{i,j} \) are the sets of vertices of \( T \) and \( K_{i,j} \) respectively, so \( S_{i,j} \subset S_T \) and \( S_T = \cup_{i,j} S_{i,j} \).

As \( \text{div}u = 0 \), we have
\[
\nu \left( \sum_{j=1}^{N_2} \sum_{i=1}^{N_1} (u^l_{i,j}) \int_{K_{i,j}} \frac{\partial \phi_s}{\partial x} dxdy + u^l_{i,j} \int_{K_{i,j}} \frac{\partial \phi_s}{\partial y} dxdy \right) = 0.
\]

We denote \( f^l_{i,j} = \frac{1}{h_{i,j}} \int f \text{, } (x,y) dxdy \).

For \( i = 1 - N_1, j = 1 - N_2 \) and \( l = 1, 2 \), a finite volume scheme can be defined by the following set of equations, see [2] :
\[
u \left( F^1_{i+\frac{1}{2},j} - F^1_{i-\frac{1}{2},j} + F^1_{i,j+\frac{1}{2}} - F^1_{i,j-\frac{1}{2}} \right) + h_i \left( u^l_{i,j+\frac{1}{2}} u^l_{i,j+\frac{1}{2}} - u^l_{i,j-\frac{1}{2}} u^l_{i,j-\frac{1}{2}} \right)
+ k_j \left( (u^l_{i+\frac{1}{2},j})^2 - (u^l_{i-\frac{1}{2},j})^2 \right) + \sum_{s \in S_{i,j}} p_s \int_{K_{i,j}} \frac{\partial \phi_s}{\partial x} dxdy = h_{i,j} \nu f^l_{i,j} \quad \text{(8)}
\]
\[
u \left( F^2_{i+\frac{1}{2},j} - F^2_{i-\frac{1}{2},j} + F^2_{i,j+\frac{1}{2}} - F^2_{i,j-\frac{1}{2}} \right) + k_j \left( (u^l_{i+\frac{1}{2},j})^2 - (u^l_{i-\frac{1}{2},j})^2 \right) + \sum_{s \in S_{i,j}} p_s \int_{K_{i,j}} \frac{\partial \phi_s}{\partial y} dxdy = h_{i,j} \nu f^l_{i,j} \quad \text{(9)}
\]
\[ \sum_{j=1}^{N_2} \sum_{i=1}^{N_1} \left( u_{i,j}^1 \int_{K_{i,j}} \frac{\partial \phi}{\partial x}(x,y) dx \, dy + u_{i,j}^2 \int_{K_{i,j}} \frac{\partial \phi}{\partial y}(x,y) dx \, dy \right) = 0 \] (9)

With

\[ F^{l}_{i,j+\frac{1}{2}} = -\frac{k_j}{h_{i+\frac{1}{2}}}(u_{i+1,j}^l - u_{i,j}^l) \quad \forall i = 0 - N_1 \ \forall j = 1 - N_2 \ \forall l = 1, 2 \] (10)

\[ F^{l}_{i,j+\frac{1}{2}} = -\frac{h_i}{k_{j+\frac{1}{2}}}(u_{i,j+1}^l - u_{i,j}^l) \quad \forall i = 1 - N_1 \ \forall j = 0 - N_2 \ \forall l = 1, 2 \] (11)

And the boundary conditions of \( \Omega \) are:

\[ u^l_{i,\frac{1}{2}} = u^l_{i,N_2+\frac{1}{2}} = u^l_{i,0} = u^l_{i,N_2+1} = 0 \ \forall i = 1 - N_1 \ \forall l = 1, 2 \] (12)

\[ u^l_{\frac{1}{2},j} = u^l_{N_1+\frac{1}{2},j} = u^l_{0,j} = u^l_{N_1+1,j} = 0 \ \forall j = 1 - N_2 \ \forall l = 1, 2 \] (13)

The discrete unknowns of (7)-(13) are \( u_{i,j}^l, i = 1 - N_1, j = 1 - N_2, l = 1, 2 \) and \( p_s, s \in S_T \).

3 Study of the numerical scheme

3.1 Definitions and results

**Definition 3.1** Let \( \Omega \) be an open bounded polygonal subset of \( \mathbb{R}^2 \) and \( T \) be an admissible mesh. Defining \( X(T) \) as the set of functions form \( \Omega \) to \( \mathbb{R} \) which are constant on each control volume of the mesh.

Let \( v \in X(T) \), such that

\[ v_{0,j} = v_{N_1+1,j} = 0 \quad \forall j = 1 - N_2 \] and \( v_{i,0} = v_{i,N_2+1} = 0 \quad \forall i = 1 - N_1. \] (14)

One defines \( X_0(T) = \{ v \in X(T) \mid v \text{ verifies (14)} \} \). Let \( v \) and \( u \) in \( X_0(T) \), we define the scalar product \((.,.)_0\) and the discrete norm \( ||.||_{1,T} \) associated to \((.,.)_0\) on \( X_0(T) \) by

\[ (v, u)_0 = \sum_{j=0}^{N_2} \sum_{i=1}^{N_1} \frac{h_i}{k_{j+\frac{1}{2}}}(v_{i,j+1} - v_{i,j})(u_{i,j+1} - u_{i,j}) \]

\[ + \sum_{j=1}^{N_2} \sum_{i=0}^{N_1} \frac{k_j}{h_{i+\frac{1}{2}}}(v_{i+1,j} - v_{i,j})(u_{i+1,j} - u_{i,j}) \]

**Lemma 3.2** \( X_0(T) \) is a finite dimensional Hilbert space for the scalar product \((.,.)_0\).
Remark 3.3 Let \( v \in X_0(T) \), \( v \) can be written in the following shape
\[
v(x, y) = \sum_{j=1}^{N_2} \sum_{i=1}^{N_1} v_{i,j} \chi_{K_{i,j}}(x, y), \quad \text{where} \ \chi_{K_{i,j}} \text{ is the characteristic function of the control volume } K_{i,j}.
\]

Let \( v = (v^1, v^2) \in X_0(T) \times X_0(T) \), such that
\[
\sum_{j=1}^{N_2} \sum_{i=1}^{N_1} \left( v_{i,j}^1 \int_{K_{i,j}} \frac{\partial \phi_s(x, y)}{\partial x} \, dx \, dy + v_{i,j}^2 \int_{K_{i,j}} \frac{\partial \phi_s(x, y)}{\partial y} \, dx \, dy \right) = 0 \quad \forall s \in S_T. \quad (15)
\]

We define \( V(T) = \{ v = (v^1, v^2) \in X_0(T) \times X_0(T) / v \ \text{verifies} \ (15) \} \). For \( u \) and \( v \) in \( V(T) \), we set \( ((u, v)) = (u^1, v^1)_0 + (u^2, v^2)_0 \). It is easy to see that \( ((., .)) \) is a scalar product on \( V(T) \).

Lemma 3.4 \( V(T) \) is a finite dimensional Hilbert space for the scalar product \( ((., .)) \).

For all \( u \in V(T) \), one defines \( |u|_{1,T}^2 = \|u^1\|_{1,T}^2 + \|u^2\|_{1,T}^2 \).

Remark 3.5 Let \( v \in V(T) \), \( f \) can be written in the following form
\[
v(x, y) = \sum_{j=1}^{N_2} \sum_{i=1}^{N_1} v_{i,j}^1 \chi_{K_{i,j}}(x, y) + \sum_{j=1}^{N_2} \sum_{i=1}^{N_1} v_{i,j}^2 \chi_{K_{i,j}}(x, y).
\]

Let \( V(T) \) with the following scalar product \( [u, v] = \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} u_{i,j}^l v_{i,j}^l \). It’s clearly that \( V(T) \) is a closed subspace of \( X_0(T) \times X_0(T) \), hence
\[
X_0(T) \times X_0(T) = V(T) \oplus V(T)^\perp.
\]

Lemma 3.6
\[
V(T)^\perp = \{ A \in X_0(T) \times X_0(T) / A = (A_{i,j}^1, A_{i,j}^2) \}
\]

with
\[
A_{i,j}^1 = \sum_{s \in S_{i,j}} a_s \int_{K_{i,j}} \frac{\partial \phi_s(x, y)}{\partial x} \, dx \, dy \quad \forall i = 1 - N_1; j = 1 - N_2,
\]
\[
A_{i,j}^2 = \sum_{s \in S_{i,j}} a_s \int_{K_{i,j}} \frac{\partial \phi_s(x, y)}{\partial y} \, dx \, dy \quad \forall i = 1 - N_1; j = 1 - N_2,
\]

where \( a_s \in \mathbb{R} \).
Lemma 3.7 Discrete Poincaré inequality [13]: Let $\Omega$ be an open bounded polygonal subset of $\mathbb{R}^d$, $(d = 2$ or $3)$. $T$ an admissible finite volume mesh and $u$ in $X(T)$, then

$$\|u\|_{L^2(\Omega)} \leq \text{diam}(\Omega)\|u\|_{1,T}$$

Where $\|\cdot\|_{1,T}$ is the discrete norm.

Lemma 3.8 Discrete Sobolev inequality [13]: Let $\Omega$ be an open bounded polygonal subset of $\mathbb{R}^2$ and $q \in ]2, +\infty[$. Let $T$ be an admissible finite volume mesh of $\Omega$ such that, for some $\xi > 0$, $d_{K,\sigma} \leq \xi d_{\sigma}$ for all control volume $K$, and $u \in X(T)$, there exists $C > 0$ only depending on $\Omega$ and $\xi$, such that

$$\|u\|_{L^q(\Omega)} \geq C_q \|u\|_{1,T},$$

where $d_{\sigma}$ is the euclidean distance between $x_K$ and $x_L$ if $\overline{K} \subset \Omega$ and $\sigma = \overline{K} \cap \overline{T}$, $K$ and $L$ in $T$, is the distance from $x_K$ to $\sigma$ if $\sigma = \overline{K} \cap \partial \Omega$. $x_K \in \overline{K}$, $x_L \in \overline{L}$, $(x_K \neq x_L)$, and the straight line $(x_Kx_L)$ is orthogonal to $\sigma$. $d_{K,\sigma}$ is the distance from $x_K$ to $\sigma$.

Lemma 3.9 [29] Let $X$ be a finite dimensional Hilbert space with scalar product $[..]$ and norm $[..]$ let $P$ be a continuous mapping from $X$ into itself such that

$$[P(x), x] > 0 \quad \text{for } |x| = k > 0 \quad (16)$$

Then there exists $\xi \in X; |\xi| \leq k$, such that $P(\xi) = 0$

3.2 Theorems

Let $u$ and $v$ in $V(T)$, we define the mapping $L(u, v)$ on $V(T) \times V(T)$ by

$$L(u, v) = \nu \sum_{l=1}^2 \left( \sum_{j=1}^{N_2} \sum_{i=1}^{N_1} (F_{i+\frac{1}{2},j} - F_{i-\frac{1}{2},j} + F_{i,j+\frac{1}{2}} - F_{i,j-\frac{1}{2}}) v_{i,j}^l \right)$$

$$+ \sum_{l=1}^2 \left( \sum_{j=1}^{N_2} \sum_{i=1}^{N_1} h_i (u_{i,j+\frac{1}{2}}^l u_{i,j+\frac{1}{2}}^l - u_{i,j-\frac{1}{2}}^l u_{i,j-\frac{1}{2}}^l) v_{i,j}^l \right)$$

$$+ \sum_{l=1}^2 \left( \sum_{j=1}^{N_2} \sum_{i=1}^{N_1} k_j (u_{i+\frac{1}{2},j}^l u_{i+\frac{1}{2},j}^l - u_{i-\frac{1}{2},j}^l u_{i-\frac{1}{2},j}^l) v_{i,j}^l \right)$$

$$- \sum_{l=1}^2 \sum_{j=1}^{N_2} \sum_{i=1}^{N_1} h_{i,j} f_{i,j}^l v_{i,j}^l$$

The mapping $L$ is linear with respect to variable second $v$. By the Riesz-Fréchet theorem (see [6]) applied to $L(u, \cdot)$ there exists a unique $Pr(u)$ in
V(T), such that \(((Pr(u), v)) = L(u, v) \ \forall v \in V(T)\). One deduces that \(Pr\) is a mapping from \(V(T)\) into \(V(T)\).

Let the following problem:

\[
\begin{align*}
\{Q\} & \begin{cases}
\text{Find } u_T \in V(T), \text{ such that } \\
L(u_T, v) = 0 \ \forall v \in V(T)
\end{cases}
\end{align*}
\]

**Remark 3.10** If \(u_T\) is solution to (7)-(13) then \(u_T\) is solution to \(\{Q\}\) and also if \(u_T\) is solution to \(\{Q\}\) then there exists \(p_T\) such that \((u_T, p_T)\) is solution to (7)-(13). It’s sufficient to solve the problem \(\{Q\}\) and to conclude the solution of the numerical scheme.

The problem \(\{Q\}\) admits a solution \(u_T\) if only if \(Pu_T = 0\).

Now, we assume that there exists a constant \(\xi > 0\) such that

\[
\begin{align*}
h_i^+ & > \xi h_i+\frac{1}{2} \ \forall i = 1 - (N_1 - 1) \quad \text{and} \quad h_i^- > \xi h_i-\frac{1}{2} \ \forall i = 1 - N_1 \\
k_j^+ & > \xi k_j+\frac{1}{2} \ \forall j = 1 - (N_2 - 1) \quad \text{and} \quad k_j^- > \xi k_j-\frac{1}{2} \ \forall j = 1 - N_2
\end{align*}
\]

**Theorem 3.11** [2] Under assumption 1 and if \(Cdiam(\Omega) < \nu^2\) where \(C\) is a constant only depending on \(\xi, \Omega\) and \(f\) then the problem (7)-(13) has at least one solution.

**Theorem 3.12** [2] Under assumptions 1 and if

\[
\max \left(4kC_4, 2^2 \left(C_4diam(\Omega)\|f\|_{(L^2)^2}\right)^{\frac{1}{2}}\right) < \nu
\]

Then the problem \(\{Q\}\) has a unique solution.

We define \(u_T^i(x, y) = u_T^i \ \forall (x, y) \in K_{i,j}, \ \forall i = 1 - N_1, \forall j = 1 - N_2\).

**Lemma 3.13** [13] let \(\Omega\) be an open bounded set of \(\mathbb{R}^d\), \((d = 2, 3)\). Let \(T\) be an admissible finite volume mesh and \(u \in X(T)\). One defines \(\tilde{u}\) by \(\tilde{u} = u \ a.e\ \Omega\) and \(\tilde{u} = 0 \ a.e\ \mathbb{R}^d \setminus (\Omega)\), Then there exists a constant \(C > 0\), only depending on \(\Omega\), such that

\[
\|\tilde{u}(.) + t\|_{L^2(\mathbb{R}^d)}^2 \leq \|u_T\|_{1,T}^2|t|(|t| + C \text{size}(T)) \ \forall t \in \mathbb{R}^d
\]

**Theorem 3.14** [13] Let \(\Omega\) be an open bounded set of \(\mathbb{R}^d\), \(d \geq 1\) and \(\{u_n, n \in IN\}\) be a bounded sequence of \(L^2(\Omega)\). For \(n \in IN\), one defines \(\tilde{u}_n\) by \(\tilde{u}_n = u_n\ a.e\ \Omega\) and \(\tilde{u}_n = 0 \ a.e\ \mathbb{R}^d \setminus (\Omega)\). Assume that there exists \(C \in \mathbb{R}^+\) and \(\{h_n, n \in IN\} \subset \mathbb{R}^+\) such that \(h_n \to 0\) as \(n \to +\infty\) and

\[
\|\tilde{u}_n(\cdot + t) - \tilde{u}(\cdot)\|^2_{L^2(\mathbb{R}^d)} \leq C|t|(|t| + h_n) \ \forall n \in IN \ \forall t \in \mathbb{R}^d
\]

Then \(\{u_n, n \in IN\}\) is relatively compact in \(L^2(\Omega)\). Furthermore, if \(u_n \to u\) in \(L^2(\Omega)\) as \(n \to +\infty\) then \(u \in H^1_0(\Omega)\).
Theorem 3.15 [2] Under assumptions 1 and if the condition (17) holds, then

\[ |u_T|_{1,T} \leq 2 \frac{\text{diam}(\Omega)}{\nu} \| f \|_{(L^2)^2} \]

Furthermore the sequence \((u_T)_T\) is relatively compact in \((L^2(\Omega))^2\).

We assume that there exists \(\xi_1\) and \(\xi_2\) two positive constants, such that

\[ \xi_2 h^2 < \text{mes}(K_{i,j}) < \xi_1 h^2, \quad \xi_2 h < \xi_1 h, \quad \xi_2 h < k_j < \xi_1 h \ \forall i = 1 - N_1; \forall j = 1 - N_2. \]

Theorem 3.16 [2] Under assumption 1 if the condition (17) holds and if \(x_i\) and \(y_j\) are the middles of \([x_i-\frac{1}{2}, x_i+\frac{1}{2}]\) and \([y_j-\frac{1}{2}, y_j+\frac{1}{2}]\) respectively. Then \(u_T\) converges to \(u\) in \((L^2(\Omega))^2\), as \(h\) tends to 0, where \(u\) is the unique solution of Navier-Stokes equations.

4 Error estimate

We assume that there exists a constant \(\xi\) such that

\[ h_i^+ \geq \xi h_{i+\frac{1}{2}} \quad \forall i = 1 - (N_1 - 1) \quad \text{and} \quad h_i^- \geq \xi h_{i-\frac{1}{2}} \quad \forall i = 1 - N_1 \]

\[ k_j^+ \geq \xi k_{j+\frac{1}{2}} \quad \forall j = 1 - (N_2 - 1) \quad \text{and} \quad k_j^- \geq \xi k_{j-\frac{1}{2}} \quad \forall j = 1 - N_2. \]

Let \(e^l_{i,j} = u^l(x_{i,j}) - u^l_{i,j}\), for each \(i = 1 - N_1, \ j = 1 - N_2\) and \(l = 1, 2\), where \(u^l\) assumed to be the unique variational solution of the problem (1)-(4) satisfying \(u^l\) in \(C^2(\Omega)\) for all \(l = 1, 2\). We put \(e^l_T(x, y) = e^l_{i,j}\) for a.e \((x, y)\) in \(K_{i,j}, \forall i = 1 - N_1, \forall j = 1 - N_2\) and \(l = 1, 2\), then \(e_T\) in \(V(T)\).

Under adequate regularity assumptions on the solution of problem (1)-(4), one may establish the error between the variational solution and the approximate solution given by the finite volume scheme (7)-(13).

Theorem 4.1 Under assumption 1. Then there exists \(C > 0\), only depending on \(u, \nu\) and \(\Omega\) such that

\[ \beta |e_T|_{1,T} + \sum_{i=1}^{2} \sum_{j=0}^{N_2} k_j e^1_{i+\frac{1}{2},j} + \frac{1}{2} e^1_{i+\frac{1}{2},j} (e^1_{i+1,j} - e^1_{i-1,j}) \]

\[ + \sum_{i=1}^{2} \sum_{j=0}^{N_2} h_i e^2_{i,j+\frac{1}{2}} + \frac{1}{2} e^2_{i,j+\frac{1}{2}} (e^2_{i,j+1} - e^2_{i,j-1}) \leq C (1 + h) |e_T|_{1,T}, \]

where \(\beta = \nu - 8\frac{C}{\nu} \text{diam}(\Omega) \| f \|_{(L^2)^2} > 0\),

\[ e^l_{i+\frac{1}{2},j} = \frac{h_i^+ e^l_{i+1,j} - h_{i+1} e^l_{i,j}}{h_{i+\frac{1}{2}}} \quad \text{and} \quad e^l_{i, j+\frac{1}{2}} = \frac{k_j^+ e^l_{i,j+1} - k_{j+1} e^l_{i,j}}{k_{j+\frac{1}{2}}} \]
Error estimate for finite volume method

469

Proof: Let us write the flux balance, for any $K_{i,j}$

$$
\nu \left( F_{i+\frac{1}{2},j} - F_{i-\frac{1}{2},j} + F_{i,j+\frac{1}{2}} - F_{i,j-\frac{1}{2}} \right) + \int_{K_{i,j}} \frac{\partial p}{\partial x}(x, y) dx dy
\]

$$
+ \nabla_{i+\frac{1}{2},j} - \nabla_{i-\frac{1}{2},j} + W_{i,j+\frac{1}{2}} - W_{i,j-\frac{1}{2}} = \int_{K_{i,j}} f^1(x, y) dx dy
\]

$$
\nu \left( F_{i+\frac{1}{2},j}^2 - F_{i-\frac{1}{2},j}^2 + F_{i,j+\frac{1}{2}}^2 - F_{i,j-\frac{1}{2}}^2 \right) + \int_{K_{i,j}} \frac{\partial p}{\partial y}(x, y) dx dy
\]

$$
+ \nabla_{i,j+\frac{1}{2}} - \nabla_{i,j-\frac{1}{2}} + W_{i+\frac{1}{2},j} - W_{i-\frac{1}{2},j} = \int_{K_{i,j}} f^1(x, y) dx dy,
\]

(21)

where

$$
F_{i+\frac{1}{2},j}^l = - \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} \frac{\partial u^l}{\partial y}(x_{i+\frac{1}{2}}, y) dy \quad \text{and} \quad F_{i,j+\frac{1}{2}}^l = - \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \frac{\partial u^l}{\partial x}(x, y_{j+\frac{1}{2}}) dx
\]

$$
\nabla_{i+\frac{1}{2},j} = \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} (u^1(x_{i+\frac{1}{2}}, y))^2 dy \quad \text{and} \quad \nabla_{i,j+\frac{1}{2}} = \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} (u^2(x, y_{j+\frac{1}{2}}))^2 dx
\]

$$
W_{i+\frac{1}{2},j} = \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} u^1(x_{i+\frac{1}{2}}, y) u^2(x_{i+\frac{1}{2}}, y) dy
\]

and

$$
W_{i,j+\frac{1}{2}} = \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} u^1(x, y_{j+\frac{1}{2}}) u^2(x, y_{j+\frac{1}{2}}) dx.
\]

Let $F_{i+\frac{1}{2},j}^l$, $F_{i,j+\frac{1}{2}}^l$, $V_{i,j+\frac{1}{2}}^*$, $V_{i+\frac{1}{2},j}^*$, $W_{i,j+\frac{1}{2}}^*$, $W_{i+\frac{1}{2},j}^*$, $V_{i,j+\frac{1}{2}}^*$, $V_{i+\frac{1}{2},j}^*$ and $W_{i,j+\frac{1}{2}}^*$ be defined by

$$
F_{i+\frac{1}{2},j}^l = - \frac{k_j}{h_{i+\frac{1}{2}}^2} \left( u^l(x_{i+1}, y_j) - u^l(x_i, y_j) \right)
\]

and

$$
F_{i,j+\frac{1}{2}}^l = - \frac{h_i}{k_{j+\frac{1}{2}}^2} \left( u^l(x_i, y_{j+1}) - u^l(x_i, y_j) \right)
\]

$$
V_{i,j+\frac{1}{2}}^* = k_j \left( \frac{h_i^2 u^1(x_{i+1}, y_j) + h_{i+1} u^1(x_i, y_j)}{h_{i+\frac{1}{2}}} \right)^2
\]
and

\[ V_{i,j+\frac{1}{2}}^* = h_i \left( \frac{k_j^+ u^2(x_i, y_{j+1}) + k_j^- u^2(x_i, y_j)}{k_j + \frac{1}{2}} \right)^2 \]

\[ W_{i+\frac{1}{2},j}^* = h_j \left( \frac{h_i^+ u^1(x_{i+1}, y_j) + h_i^- u^1(x_i, y_j)}{h_i + \frac{1}{2}} \right) \left( \frac{h_i^+ u^2(x_{i+1}, y_j) + h_i^- u^2(x_i, y_j)}{h_i + \frac{1}{2}} \right) \]

\[ W_{i,j+\frac{1}{2}}^* = h_i \left( \frac{k_j^+ u^1(x_i, y_{j+1}) + k_j^- u^1(x_i, y_j)}{k_j + \frac{1}{2}} \right) \left( \frac{k_j^+ u^2(x_i, y_{j+1}) + k_j^- u^2(x_i, y_j)}{k_j + \frac{1}{2}} \right) \]

\[ V_{i+\frac{1}{2},j} = k_j \left( u_{i+\frac{1}{2},j}^1 \right)^2 \quad \text{and} \quad V_{i,j+\frac{1}{2}} = h_i \left( u_{i,j+\frac{1}{2}}^2 \right)^2 \]

\[ W_{i+\frac{1}{2},j} = k_j \left( u_{i+\frac{1}{2},j}^1 \right) \left( u_{i+\frac{1}{2},j}^2 \right) \quad \text{and} \quad W_{i,j+\frac{1}{2}} = h_i \left( u_{i,j+\frac{1}{2}}^1 \right) \left( u_{i,j+\frac{1}{2}}^2 \right). \]

Then, the consistency error may be defined as

\[ R_{i,j+\frac{1}{2}}^l = \frac{1}{k_j} \left( F_{i+\frac{1}{2},j}^l - F_{i+\frac{1}{2},j}^{l,*} \right) \quad \text{and} \quad R_{i,j+\frac{1}{2}}^l = \frac{1}{h_i} \left( F_{i,j+\frac{1}{2}}^l - F_{i,j+\frac{1}{2}}^{l,*} \right) \]

\[ r_{i+\frac{1}{2},j}^1 = \frac{1}{k_j} \left( V_{i+\frac{1}{2},j} - V_{i+\frac{1}{2},j}^* \right) \quad \text{and} \quad r_{i,j+\frac{1}{2}}^1 = \frac{1}{h_i} \left( V_{i,j+\frac{1}{2}} - V_{i,j+\frac{1}{2}}^* \right) \]

\[ r_{i+\frac{1}{2},j}^2 = \frac{1}{k_j} \left( W_{i+\frac{1}{2},j} - W_{i+\frac{1}{2},j}^* \right) \quad \text{and} \quad r_{i,j+\frac{1}{2}}^2 = \frac{1}{h_i} \left( W_{i,j+\frac{1}{2}} - W_{i,j+\frac{1}{2}}^* \right) \]

Thanks to the regularity of \( u \), there exists \( C_1^l, C_2^l, C_3, C_4, C_5 \) and \( C_6 \) depending on \( u \) such that

\[ |R_{i+\frac{1}{2},j}^l| \leq C_1^l h \quad \text{and} \quad |R_{i,j+\frac{1}{2}}^l| \leq C_1^l h \]  

(23)

\[ |r_{i+\frac{1}{2},j}^1| \leq C_3 (1 + h) h \quad \text{and} \quad |r_{i,j+\frac{1}{2}}^1| \leq C_4 (1 + h) h \]  

(24)

\[ |r_{i+\frac{1}{2},j}^2| \leq C_5 h \quad \text{and} \quad |r_{i,j+\frac{1}{2}}^2| \leq C_6 h \]  

(25)

Relations (23)-(25) are easy to check. We have

\[ F_{i+\frac{1}{2},j}^l - F_{i+\frac{1}{2},j}^{l,*} = F_{i+\frac{1}{2},j}^l - F_{i+\frac{1}{2},j} + F_{i+\frac{1}{2},j}^{l,*} - F_{i+\frac{1}{2},j} \]

\[ = - \frac{k_j}{h_i+\frac{1}{2}} (e_{i+1,j}^l - e_{i,j}^l) + k_j R_{i+\frac{1}{2},j}^l, \]  

(26)

\[ F_{i,j+\frac{1}{2}}^l - F_{i,j+\frac{1}{2}}^{l,*} = - \frac{h_i}{k_j+\frac{1}{2}} (e_{i,j+1}^l - e_{i,j}^l) + h_i R_{i,j+\frac{1}{2}}^l, \]  

(27)

\[ \nabla_{i+\frac{1}{2},j} - V_{i+\frac{1}{2},j}^* = \nabla_{i+\frac{1}{2},j} - V_{i+\frac{1}{2},j} + V_{i+\frac{1}{2},j}^* - V_{i+\frac{1}{2},j} \]

\[ = V_{i+\frac{1}{2},j} - V_{i+\frac{1}{2},j} + k_j r_{i+\frac{1}{2},j}^1 \]

\[ = k_j \left( (e_{i+\frac{1}{2},j}^1)^2 + 2e_{i+\frac{1}{2},j}^1 u_{i+\frac{1}{2},j}^1 + r_{i+\frac{1}{2},j}^1 \right) \]  

(28)

\[ \nabla_{i,j+\frac{1}{2}} - V_{i,j+\frac{1}{2}} = h_i \left( (e_{i,j+\frac{1}{2}}^2)^2 + 2e_{i,j+\frac{1}{2}}^2 u_{i,j+\frac{1}{2}}^2 + r_{i,j+\frac{1}{2}}^1 \right), \]  

(29)
\[
\begin{align*}
\mathbf{W}_{i,j+\frac{1}{2}} - W_{i,j+\frac{1}{2}} &= \mathbf{W}_{i,j+\frac{1}{2}} - W_{i,j+\frac{1}{2}}^* + W_{i,j+\frac{1}{2}}^* - W_{i,j+\frac{1}{2}} \\
&= W_{i,j+\frac{1}{2}}^* - W_{i,j+\frac{1}{2}} + h_i r_{i,j+\frac{1}{2}}^2 \\
&= h_i \left(e_{i,j+\frac{1}{2}}^1 e_{i,j+\frac{1}{2}}^2 + e_{i,j+\frac{1}{2}}^2 u_{i,j+\frac{1}{2}}^1 + r_{i,j+\frac{1}{2}}^2 \right) \\
\mathbf{W}_{i+\frac{1}{2},j} - W_{i+\frac{1}{2},j} &= k_j \left(e_{i+\frac{1}{2},j}^1 e_{i+\frac{1}{2},j}^2 + e_{i+\frac{1}{2},j}^2 u_{i+\frac{1}{2},j}^1 + r_{i+\frac{1}{2},j}^2 \right) \\
&+ k_j \left(e_{i+\frac{1}{2},j}^1 u_{i+\frac{1}{2},j}^2 + r_{i+\frac{1}{2},j}^2 \right)
\end{align*}
\]

(30)

Subtracting (7) and (8) to (21) and (22) respectively, using (26)-(31), multiplying by \(e_{i,j}^1\) and \(e_{i,j}^2\) and summing for \(i,j\), we obtain

\[
\begin{align*}
\nu \left( \sum_{i=1,j=1}^{N_1,N_2} \left( F_{i+\frac{1}{2},j}^l - F_{i+\frac{1}{2},j}^l - F_{i+\frac{1}{2},j}^l + F_{i+\frac{1}{2},j}^l \right) e_{i,j}^l \right) \\
+ \nu \left( \sum_{i=1,j=1}^{N_1,N_2} \left( F_{i+\frac{1}{2},j}^l - F_{i+\frac{1}{2},j}^l - F_{i+\frac{1}{2},j}^l + F_{i+\frac{1}{2},j}^l \right) e_{i,j}^l \right) \\
- \nu \left( \sum_{i=1,j=0}^{N_1,N_2} \left( F_{i+\frac{1}{2},j}^l - F_{i+\frac{1}{2},j}^l \right) \left( e_{i+1,j}^l - e_{i,j}^l \right) \right) \\
- \nu \left( \sum_{i=1,j=0}^{N_1,N_2} \left( F_{i+\frac{1}{2},j}^l - F_{i+\frac{1}{2},j}^l \right) \left( e_{i,j+1}^l - e_{i,j}^l \right) \right) \\
= \nu \| e^l_{i,j} \|_{1,\ell}^2 - \nu \sum_{i=0,j=1}^{N_1,N_2} k_j R_{i+\frac{1}{2},j} (e_{i+1,j}^l - e_{i,j}^l) \\
- \nu \sum_{i=1,j=0}^{N_1,N_2} h_i R_{i+\frac{1}{2},j} (e_{i+1,j}^l - e_{i,j}^l)
\end{align*}
\]

(32)

\[
\begin{align*}
\sum_{i=1,j=1}^{N_1,N_2} \left( \mathbf{V}_{i+\frac{1}{2},j} - V_{i+\frac{1}{2},j} - \mathbf{V}_{i-\frac{1}{2},j} + V_{i-\frac{1}{2},j} \right) e_{i,j}^1 = \\
- \sum_{i=0,j=1}^{N_1,N_2} \left( \mathbf{V}_{i+\frac{1}{2},j} - V_{i+\frac{1}{2},j} \right) (e_{i+1,j}^1 - e_{i,j}^1) \\
- \sum_{i=0,j=1}^{N_1,N_2} k_j (e_{i+\frac{1}{2},j}^1)^2 (e_{i+1,j}^1 - e_{i,j}^1) - 2 \sum_{i=0,j=1}^{N_1,N_2} k_j e_{i+\frac{1}{2},j}^1 u_{i+\frac{1}{2},j}^1 (e_{i+1,j}^1 - e_{i,j}^1) \\
- \sum_{i=0,j=1}^{N_1,N_2} k_j R_{i+\frac{1}{2},j} (e_{i+1,j}^1 - e_{i,j}^1)
\end{align*}
\]

(33)

\[
\begin{align*}
\sum_{i=1,j=1}^{N_1,N_2} \left( \mathbf{V}_{i,j+\frac{1}{2}} - V_{i,j+\frac{1}{2}} - \mathbf{V}_{i,j-\frac{1}{2}} + V_{i,j-\frac{1}{2}} \right) e_{i,j}^2 = 
\end{align*}
\]
\[ - \sum_{i=1,j=0}^{N_1,N_2} h_i (e_{i,j+\frac{1}{2}}^2 - e_{i,j+1}^2) + \sum_{i=1,j=0}^{N_1,N_2} h_i e_{i,j+\frac{1}{2}}^2 (e_{i,j+1}^2 - e_{i,j}^2) - 2 \sum_{i=1,j=0}^{N_1,N_2} h_i e_{i,j+\frac{1}{2}}^2 u_{i,j+\frac{1}{2}}^2 (e_{i,j+1}^2 - e_{i,j}^2) \\
- \sum_{i=1,j=0}^{N_1,N_2} h_i r_{i,j+\frac{1}{2}} (e_{i,j+1}^2 - e_{i,j}^2) \] (34)

\[ \sum_{i=1,j=1}^{N_1,N_2} (W_{i,j+\frac{1}{2}} - W_{i,j+\frac{1}{2}} - W_{i,j-\frac{1}{2}} + W_{i,j-\frac{1}{2}}) e_{i,j}^1 = \]

\[ - \sum_{i=1,j=0}^{N_1,N_2} (W_{i,j+\frac{1}{2}} - W_{i,j+\frac{1}{2}}) (e_{i,j+1}^1 - e_{i,j}^1) \]

\[ = - \sum_{i=1,j=0}^{N_1,N_2} h_i e_{i,j+\frac{1}{2}}^1 (e_{i,j+1}^1 - e_{i,j}^1) - \sum_{i=1,j=0}^{N_1,N_2} h_i e_{i,j+\frac{1}{2}}^1 u_{i,j+\frac{1}{2}}^1 (e_{i,j+1}^1 - e_{i,j}^1) \]

\[ - \sum_{i=1,j=0}^{N_1,N_2} h_i e_{i,j+\frac{1}{2}}^2 u_{i,j+\frac{1}{2}}^2 (e_{i,j+1}^2 - e_{i,j}^2) - \sum_{i=1,j=0}^{N_1,N_2} h_i r_{i,j+\frac{1}{2}}^2 (e_{i,j+1}^2 - e_{i,j}^2) \] (35)

\[ \sum_{i=1,j=1}^{N_1,N_2} (W_{i+\frac{1}{2},j} - W_{i+\frac{1}{2},j} - W_{i-\frac{1}{2},j} + W_{i-\frac{1}{2},j}) e_{i,j}^2 = \]

\[ - \sum_{i=0,j=1}^{N_1,N_2} k_j e_{i+\frac{1}{2},j}^1 (e_{i+1,j}^2 - e_{i,j}^2) - \sum_{i=0,j=1}^{N_1,N_2} k_j e_{i+\frac{1}{2},j}^1 u_{i+\frac{1}{2},j}^2 (e_{i+1,j}^2 - e_{i,j}^2) \]

\[ - \sum_{i=1,j=0}^{N_1,N_2} k_j e_{i+\frac{1}{2},j}^2 u_{i+\frac{1}{2},j}^1 (e_{i+1,j}^2 - e_{i,j}^2) - \sum_{i=0,j=1}^{N_1,N_2} k_j r_{i+\frac{1}{2},j}^2 (e_{i+1,j}^2 - e_{i,j}^2) \] (36)

\[ \sum_{i=1,j=1}^{N_1,N_2} \left( \int_{K_{i,j}} f^1(x,y) dx dy - h_{i,j} f_{1,i,j}^1 \right) e_{i,j}^1 = 0 \] (37)

As \( \text{div}(u) = 0 \), we deduce that

\[ \sum_{i=1,j=1}^{N_1,N_2} \left( \int_{K_{i,j}} \frac{\partial p}{\partial x}(x,y) dx dy - \sum_{s \in S_{i,j}} p_s \int_{K_{i,j}} \frac{\partial \phi_s}{\partial x}(x,y) dx dy \right) e_{i,j}^1 = 0 \] 

\[ + \sum_{i=1,j=1}^{N_1,N_2} \left( \int_{K_{i,j}} \frac{\partial p}{\partial y}(x,y) dx dy - \sum_{s \in S_{i,j}} p_s \int_{K_{i,j}} \frac{\partial \phi_s}{\partial y}(x,y) dx dy \right) e_{i,j}^2 = 0 \] (38)
For simplicity, we put

\[
Te_1 = \sum_{l=1}^{2} \sum_{i=0}^{N_1} \sum_{j=1}^{N_2} k_j \varepsilon_{i+\frac{1}{2},j} e_{l+\frac{1}{2},j} (e_{l+1,i,j} - e_{l,i,j})
\]

\[
Te_2 = \sum_{l=1}^{2} \sum_{i=0}^{N_1} \sum_{j=0}^{N_2} h_i e_{i,j+\frac{1}{2}} e_{l+\frac{1}{2},j} (e_{l,i,j+1} - e_{l,i,j})
\]

\[
T1 = \sum_{l=1}^{2} \sum_{i=0}^{N_1} \sum_{j=1}^{N_2} k_j \varepsilon_{i+\frac{1}{2},j} u_{l+\frac{1}{2},j} (e_{l+1,i,j} - e_{l,i,j})
\]

\[
T2 = \sum_{l=1}^{2} \sum_{i=0}^{N_1} \sum_{j=0}^{N_2} h_i e_{i,j+\frac{1}{2}} u_{l+\frac{1}{2},j} (e_{l,i,j+1} - e_{l,i,j})
\]

\[
Tu_1 = \sum_{l=1}^{2} \sum_{i=0}^{N_1} \sum_{j=1}^{N_2} k_j \varepsilon_{i+\frac{1}{2},j} u_{l+\frac{1}{2},j} (e_{l+1,i,j} - e_{l,i,j})
\]

\[
Tu_2 = \sum_{l=1}^{2} \sum_{i=0}^{N_1} \sum_{j=0}^{N_2} h_i e_{i,j+\frac{1}{2}} u_{l+\frac{1}{2},j} (e_{l,i,j+1} - e_{l,i,j})
\]

\[
TR_1 = \sum_{l=1}^{2} \sum_{i=0}^{N_1} \sum_{j=0}^{N_2} k_j R_{i+\frac{1}{2},j} (e_{l+1,i,j} - e_{l,i,j})
\]

\[
TR_2 = \sum_{l=1}^{2} \sum_{i=0}^{N_1} \sum_{j=1}^{N_2} h_i R_{i,j+\frac{1}{2}} (e_{l,i,j+1} - e_{l,i,j})
\]

\[
Tr_1 = \sum_{l=1}^{2} \sum_{i=0}^{N_1} \sum_{j=1}^{N_2} k_j r_{i+\frac{1}{2},j} (e_{l+1,i,j} - e_{l,i,j})
\]

\[
Tr_2 = \sum_{i=0}^{N_1} \sum_{j=1}^{N_2} h_i r_{i+\frac{1}{2},j} (e_{i,j+1}^2 - e_{i,j}^2) + \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} h_i r_{i,j+\frac{1}{2}} (e_{i,j+1}^2 - e_{i,j}^2)
\]

Summing (32)-(38), we obtain

\[
\nu |e|^2_{l+1,T} - Te_1 - Te_2 - T1 - T2 - Tu_1 - Tu_2 = \nu (TR_1 + TR_2) + Tr_1 + Tr_2. \tag{39}
\]

We have

\[
T1 + T2 + Tu_1 + Tu_2 = \begin{aligned}
\sum_{i=0}^{N_1} \sum_{j=1}^{N_2} k_j \varepsilon_{i+\frac{1}{2},j} u_{l+\frac{1}{2},j} (e_{l+1,i,j}^1 - e_{l,i,j}^1) \\
+ \sum_{i=1}^{N_1} \sum_{j=0}^{N_2} h_i e_{i,j+\frac{1}{2}} u_{l+\frac{1}{2},j} (e_{l,i,j+1}^2 - e_{l,i,j}^2) \\
+ \sum_{i=0}^{N_1} \sum_{j=1}^{N_2} k_j \varepsilon_{i+\frac{1}{2},j} u_{l+\frac{1}{2},j} (e_{l+1,i,j}^1 - e_{l,i,j}^1)
\end{aligned}
\]
Using the Cauchy-Schwartz inequality and the discrete Sobolev inequality, there exists constants \( C_{4,l}(\xi, \Omega) > 0 \), for \( l = 1, 2 \), such that

\[
T_1 + T_2 + Tu_1 + Tu_2 \leq 2(C_{4,1})^2 \| e_T^1 \|_{1,T} \| u_T^1 \|_{1,T} \left( \sum_{i=0, j=1}^{N_1, N_2} \frac{k_j}{h_{i+\frac{1}{2}}} (e_{i+1,j} - e_{i,j})^2 \right)^\frac{1}{2} \\
+ 2(C_{4,2})^2 \| e_T^2 \|_{1,T} \| u_T^2 \|_{1,T} \left( \sum_{i=1, j=0}^{N_1, N_2} \frac{h_i}{k_{j+\frac{1}{2}}} (e_{i,j+1} - e_{i,j})^2 \right)^\frac{1}{2} \\
+ (C_{4,1})^2 \| e_T^1 \|_{1,T} \| u_T^1 \|_{1,T} \left( \sum_{i=0, j=1}^{N_1, N_2} \frac{k_j}{h_{i+\frac{1}{2}}} (e_{i+1,j} - e_{i,j})^2 \right)^\frac{1}{2} \\
+ (C_{4,2})^2 \| e_T^2 \|_{1,T} \| u_T^2 \|_{1,T} \left( \sum_{i=0, j=1}^{N_1, N_2} \frac{k_j}{h_{i+\frac{1}{2}}} (e_{i+1,j} - e_{i,j})^2 \right)^\frac{1}{2} \\
+ (C_{4,1})^2 \| e_T^1 \|_{1,T} \| u_T^1 \|_{1,T} \left( \sum_{i=1, j=0}^{N_1, N_2} \frac{h_i}{k_{j+\frac{1}{2}}} (e_{i,j+1} - e_{i,j})^2 \right)^\frac{1}{2} \\
+ (C_{4,2})^2 \| e_T^2 \|_{1,T} \| u_T^2 \|_{1,T} \left( \sum_{i=1, j=0}^{N_1, N_2} \frac{h_i}{k_{j+\frac{1}{2}}} (e_{i,j+1} - e_{i,j})^2 \right)^\frac{1}{2}
\]

Put \( C_s = \max_{l=1,2} \{ C_{s,l} \} \), then

\[
T_1 + T_2 + Tu_1 + Tu_2 \leq C_s^2 |u_T|_{1,T} \left( \| e_T^1 \|_{1,T}^2 + \| e_T^2 \|_{1,T}^2 \right) \\
+ C_s^2 |u_T|_{1,T} \left( \sum_{i=0, j=1}^{N_1, N_2} \frac{k_j}{h_{i+\frac{1}{2}}} (e_{i+1,j} - e_{i,j})^2 \right) \\
+ C_s^2 |u_T|_{1,T} \left( \sum_{i=1, j=0}^{N_1, N_2} \frac{h_i}{k_{j+\frac{1}{2}}} (e_{i,j+1} - e_{i,j})^2 \right) \\
\leq 4C_s^2 |u_T|_{1,T} |e_T|^2_{1,T}.
\]
For the second term using the Cauchy-Schwartz inequality, we have

\[ TR1 \leq \sum_{l=1}^{2} C_l \sum_{i=0,j=1}^{N_1,N_2} k_j |e_{i+1,j}^l - e_{i,j}^l| \]

\[ \leq (mes(\Omega))^\frac{1}{2} \sum_{l=1}^{2} C_1 \left( \sum_{i=0,j=1}^{N_1,N_2} \frac{k_j}{h_i+\frac{1}{2}} (e_{i+1,j}^l - e_{i,j}^l)^2 \right)^{\frac{1}{2}} h \]

\[ \leq C_1 |e_T|_{1,T} h \]

(41)

Where \( C_1 \) depending on \( \Omega \) and \( \xi \). Similarly, we obtain

\[ TR2 \leq C_2 |e_T|_{1,T} h \]

(42)

Where \( C_2 \) depending on \( \Omega \) and \( \xi \). By using the Cauchy-Schwartz inequality to gether, (24) and (25), we deduce

\[ Tr1 \leq \left( C_3 \sum_{i=0,j=1}^{N_1,N_2} k_j |e_{i+1,j}^l - e_{i,j}^l| \right) (1 + h)h + \left( C_6 \sum_{i=0,j=1}^{N_1,N_2} k_j |e_{i+1,j}^2 - e_{i,j}^2| \right) h \]

(43)

\[ \leq C_3 (mes(\Omega))^\frac{1}{2} |e_T^l|_{1,T}(1 + h)h + C_6 (mes(\Omega))^\frac{1}{2} |e_T^2|_{1,T} h \]

\[ Tr2 \leq C_4 (mes(\Omega))^\frac{1}{2} |e_T^l|_{1,T}(1 + h)h + C_5 (mes(\Omega))^\frac{1}{2} |e_T^2|_{1,T} h \]

(44)

Then

\[ \nu(Tr1 + TR2) + Tr1 + Tr2 \leq C_7 |e_T|_{1,T}(1 + h)h \]

(45)

Where \( C_7 \) is depending on \( C_1, C_2, C_3, C_4, C_5, C_6, \nu \) and \( \Omega \). Hence

\[ (\nu \ - \ 4A^2 \ nu \ (mes(\Omega))^\frac{1}{2} |u||e_T|_{1,T} \ + \ \sum_{l=1}^{2} \sum_{i=0,j=1}^{N_1,N_2} k_j e_{i+1,j}^l e_{i,j}^l (e_{i,j}^l - e_{i+1,j}^l) \]

\[ + \ \sum_{l=1}^{2} \sum_{i=1,j=0}^{N_1,N_2} h_i e_{i,j+\frac{1}{2}} e_{i+\frac{1}{2},j} (e_{i,j}^l - e_{i,j+1}^l) \leq C_7(1 + h)h \]

(46)

Using the relation (20), we obtain:

\[ (\nu \ - \ 8C^2 \ nu \ diam(\Omega)||f||_{L^2}) |e_T|_{1,T} \ + \ \sum_{l=1}^{2} \sum_{i=0,j=1}^{N_1,N_2} k_j e_{i+1,j}^l e_{i,j}^l (e_{i,j}^l - e_{i+1,j}^l) \]

\[ + \ \sum_{l=1}^{2} \sum_{i=1,j=0}^{N_1,N_2} h_i e_{i,j+\frac{1}{2}} e_{i+\frac{1}{2},j} (e_{i,j}^l - e_{i,j+1}^l) \leq C_7(1 + h)h \]

Where \( \beta = \nu - 8C^2 \ nu \ diam(\Omega)||f||_{L^2} > 0 \), which completes the proof of theorem.

Remark 4.2 We have the follow formula

\[ -C^2 |e_T|_{1,T}^2 + (\nu - 8C^2 \ nu \ diam(\Omega)||f||_{L^2}) |e_T|_{1,T} \leq C_7(1 + h)h \]
References


Received: September, 2010