A Note on Some Higher-Order Iterative Methods Free from Second Derivative for Solving Nonlinear Equations

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Abstract

In the recent paper [M. A. Noor, W. A. Khan, K. I. Noor and Eisa Al-Said, Higher-order iterative methods free from second derivative for solving nonlinear equations, International Journal of the Physical Sciences, Vol. 6 (8) (2011), 1887-1893] several iterative methods for solving nonlinear equations are presented. One of the methods is the three-step iterative method given in Algorithm (2.10) in [1]. The authors claimed that the order of convergence of that method is nine. In this paper, we show that the method has only seventh order of convergence. Several numerical examples are presented to confirm our theoretical results.

Mathematics Subject Classification: 41A25, 65H05, 65K05

Keywords: Iterative methods, Nonlinear equations, Efficiency index, Order of convergence
1 Introduction

In the recent paper [M. A. Noor, W. A. Khan, K. I. Noor and Eisa Al-Said, Higher-order iterative methods free from second derivative for solving nonlinear equations, International Journal of the Physical Sciences, Vol. 6 (8) (2011), 1887-1893] several iterative methods for solving nonlinear equations of the form \( f(x) = 0 \) are presented, where \( f: I \subseteq R \to R \) is a real and sufficiently smooth function in \( I \), with \( I \) a real open interval and \( f \) has a simple zero at \( \alpha \) in \( I \).

One of these proposed methods is the three-step iterative method given in Algorithm (2.10) in [1], which comes as next:

For a given \( x_0 \), compute the approximate solution \( x_{n+1} \) by the iterative schemes:

\[
y_n = x_n - \frac{f(x_n)}{f'(x_n)},
\]

\[
z_n = y_n - \frac{2f(y_n)f'(y_n)}{2f'(y_n)^2 - f(y_n)P_f(x_n, y_n)},
\]

\[
x_{n+1} = z_n - \frac{2f'(x_n)f'(y_n)}{(f'(x_n))^2 + 2f'(x_n)f'(y_n) - (f'(y_n))^2} \frac{f(z_n)}{f'(x_n)},
\]

where

\[
P_f(x_n, y_n) = \frac{2}{(y_n - x_n)^2}[2f'(y_n) + f'(x_n) - 3\frac{f(y_n) - f(x_n)}{(y_n - x_n)}].
\]

The authors claimed that the order of convergence of method (1) is nine. In this paper, we show that the method has only seventh order of convergence. Per one iteration, method (1) requires three evaluations of the function and two evaluations of its first derivative, so its efficiency index is \( 7^{1/5} \approx 1.4758 \) rather than \( 9^{1/9} \approx 1.5518 \), where the efficiency index of a method is defined to be \( \rho^{1/\theta} \) with \( \rho \) is the order of convergence and \( \theta \) is the total number of functional evaluations per one iteration.

In Section 2, the analysis convergence of the method will be presented. Several numerical examples are given in Section 3 to confirm our theoretical results. Finally, some conclusions are pointed in Section 4.
2 Convergence Analysis

The convergence analysis of the three-step iterative method (1) for solving nonlinear equation \( f(x) = 0 \) will be established in this section.

**Theorem 2.1**: Let \( \alpha \in I \) be a simple zero of a sufficiently differentiable function \( f : I \subseteq R \to R \) for an open interval \( I \). If \( x_0 \) is sufficiently close to \( \alpha \), then method (1) has seventh order of convergence.

**Proof**: Let \( \alpha \) be a simple zero of \( f(x) = 0 \) and \( x_n = \alpha + e_n \). By Taylor expansion, we have

\[
f(x_n) = f'(\alpha)[e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + c_5e_n^5 + c_6e_n^6 + c_7e_n^7 + c_8e_n^8 + O(e_n^9)], \tag{3}
\]

\[
f'(x_n) = f'(\alpha)[1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + 5c_5e_n^4 + 6c_6e_n^5 + 7c_7e_n^6 + 8c_8e_n^7 + O(e_n^8)], \tag{4}
\]

where \( c_k = \frac{f^{(k)}(\alpha)}{k!f'(\alpha)}, k = 2, 3, \ldots \)

Substituting (3) and (4) into \( y_n \) in (1), to obtain

\[
y_n = x_n - \frac{f(x_n)}{f'(x_n)} = \alpha + c_2e_n^2 + 2(c_3 - c_2^2)e_n^3 + (3c_4 - 7c_2c_3 + 4c_2^2)e_n^4 + \ldots + (7c_8 + 64c_2^7) \\
- 19c_2c_7 + 118c_5c_2c_3 - 348c_4c_3c_2^2 - 31c_5c_4 - 92c_5c_3^2 - 27c_6c_3 + 4c_6c_2 + 64c_2c_4 \\
+ 75c_4c_3^2 + 176c_4c_2^3 - 135c_2c_3^2 + 408c_3c_2^3 - 304c_3c_2^5)e_n^8 + O(e_n^9). \tag{5}
\]

By expanding \( f(y_n) \) about \( \alpha \), we obtain

\[
f(y_n) = f'(\alpha)[c_2e_n^2 + 2(c_3 - c_2^2)e_n^3 + (3c_4 - 7c_2c_3 + 5c_3^2)e_n^4 + \ldots + (7c_8 + 144c_2^7 - 19c_2c_7 \\
+ 134c_5c_2c_3 - 455c_4c_3c_2^2 - 31c_5c_4 - 134c_5c_3^2 - 27c_6c_3 + 54c_6c_2^2 + 73c_2c_4^2 + 75c_3c_3^2 \\
+ 297c_4c_2^2 - 147c_2c_4^3 + 582c_3c_2^3 - 552c_3c_2^5)e_n^8 + O(e_n^9)]. \tag{6}
\]

Expanding \( f'(y_n) \) about \( \alpha \), to obtain

\[
f'(y_n) = f'(\alpha)[1 + 2c_2e_n^2 + 4(c_3 - 4c_2^2)e_n^3 + (6c_2c_4 - 11c_3c_2^2 + 8c_2^2)e_n^4 + \ldots + (-28c_7c_2c_3 \\
+ 14c_2c_8 - 38c_2c_2^2 + 88c_3c_2^2 + 164c_4c_2^2 + 448c_3c_2^2 - 179c_4c_2^2 - 62c_2c_5c_2 - 24c_3c_2c_6 \\
+ 100c_3c_2c_6^2 - 516c_3c_4c_2^2 - 72c_4^2 + 128c_8 + 387c_3c_2^2 - 368c_3c_2^6 + 48c_3c_5 + 27c_3c_4^2 \\
- 150c_3c_4c_2)e_n^8 + O(e_n^9)]. \tag{7}
\]

Substituting (3)-(7)into \( P_f(x_n, y_n) \) in (2), to obtain

\[
P_f(x_n, y_n) = \frac{2}{(y_n - x_n)}[2f'(y_n) + f'(x_n) - 3f(y_n) - f(x_n)] = f'(\alpha)[2c_2 + (6c_2c_3 - 2c_4)e_n^2 \\
+ (28c_4c_3c_2^2 - 60c_5c_2c_3 - 10c_8 - 2c_2c_7 + 22c_5c_4 - 10c_3c_2^3 + 30c_6c_3 + 6c_6c_2^2 \\
+ 20c_2c_4^2 - 86c_4c_2^2 + 88c_4c_2^4 + 198c_2c_3^3 - 312c_3c_2^3 + 96c_3c_2^5)e_n^6 + O(e_n^7)]. \tag{8}
\]
Substituting (5)-(8) into \( z_n \) in (1), to get

\[
z_n = y_n - \frac{2f(y_n)f'(y_n)}{2f'^2(y_n) - f(y_n)F'(x_n, y_n)} = \alpha + (-c_3c_2^2 + c_4c_2^2 + c_5^2)e_n^6 + \alpha + (21c_7^2 + 8c_5c_2c_3 \ldots
\]

Now, expand \( f(z_n) \) about \( \alpha \) to get

\[
f(z_n) = f'(\alpha)[(-c_3c_2^2 + c_4c_2^2c_5^2)e_n^6 + (4c_3c_2c_4 - 6c_3^2c_2^2 + 12c_4c_2^4 - 6c_4c_2^2 + 2c_5c_2^2 - 6c_5^2)e_n^7
\]

Substituting (4),(7),(9) and (10) into \( x_{n+1} \) in (1), to obtain

\[
x_{n+1} = \alpha + (4c_4c_2^3 - 4c_3c_2^4 + 4c_2^6)e_n^6 + (-40c_2^7 + 22c_4c_3c_2^2 + 8c_5c_2^3
\]

Thus, we have

\[
e_{n+1} = x_{n+1} - \alpha = (4c_4c_2^3 - 4c_3c_2^4 + 4c_2^6)e_n^6 + (-40c_2^7 + 22c_4c_3c_2^2 + 8c_5c_2^3
\]

Which shows that the order of convergence of method (1) is seven. This completes the proof.

3 Numerical examples

In this section, the obtained theoretical results are confirmed by numerical experiments. The test functions and their roots, found up to the 28th decimal places, are as follows:

<table>
<thead>
<tr>
<th>Example</th>
<th>the approximate zero ( \alpha )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f_1(x) = x^3 + 4x^2 - 10 )</td>
<td>1.365230013414096845760806829</td>
</tr>
<tr>
<td>( f_2(x) = \sin^2 x - x^2 + 1 )</td>
<td>1.404491648215341226035086818</td>
</tr>
<tr>
<td>( f_3(x) = x^2 - e^x - 3x + 2 )</td>
<td>0.2575302854398607604553673050</td>
</tr>
<tr>
<td>( f_4(x) = \cos x - x )</td>
<td>0.7390851332151606416553120877</td>
</tr>
<tr>
<td>( f_5(x) = (x - 1)^3 - 1 )</td>
<td>2.00000000000000000000000000</td>
</tr>
<tr>
<td>( f_6(x) = 3x - 10 )</td>
<td>2.154434690031883721759293567</td>
</tr>
<tr>
<td>( f_7(x) = e^{x^2} + 7x - 30 - 1 )</td>
<td>3.00000000000000000000000000</td>
</tr>
<tr>
<td>( f_8(x) = e^{-x} + \cos x )</td>
<td>1.746139530408012417650703089</td>
</tr>
<tr>
<td>( f_9(x) = \sin x - x/2 )</td>
<td>1.8954914267033980947144035738</td>
</tr>
<tr>
<td>( f_{10}(x) = 10xe^{-x^2} - 1 )</td>
<td>1.67963061042849940674920339</td>
</tr>
</tbody>
</table>
All computations were done using MATLAB 7.6 with 200 digit floating arithmetic (VPA=200). The following criteria

$$|x_n - x_{n-1}| < \varepsilon \quad \text{and} \quad |f(x_n)| < \varepsilon,$$

are used for stopping computer programmes. Displayed in Table 1 are the number of iterations (IT), such that the stopping criteria satisfied, where $\varepsilon$ is taken to be $10^{-15}$, the value of $|f(x_n)|$ after the required iterations. Moreover, displayed is the distance of two consecutive approximations $\delta = |(x_n - x_{n-1})|$ and the computational order of convergence (COC) which can be approximated using the formula

$$COC \approx \frac{\ln |(x_n - x_{n-1})/(x_{n-1} - x_{n-2})|}{\ln |(x_{n-1} - x_{n-2})/(x_{n-2} - x_{n-3})|}.$$ 

### 4 Conclusion

In this paper, we have shown that the order of convergence of the three-step iterative method (Algorithm 2.10) presented recently by Noor et al. [1] has order of convergence seven not nine as claimed by the authors, therefore the efficiency index of this method is $7^{1/5} \approx 1.4758$ instead of $9^{1/5} \approx 1.5518$. Also, we have presented several numerical examples to confirm our theoretical results.

### References


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Table 1:

| Function $f_i(x)$, $x_0$ | Method (1) | IT | $|f(x_n)|$ | $\delta$ | COC |
|--------------------------|-----------|----|----------|------|-----|
| $f_1(x)$, $x_0 = 1.0$   | 3         |    | 0.13405e-186 | 0.21246e-026 | 7.26 |
| $f_2(x)$, $x_0 = 1.3$   | 3         |    | 0.10000e-198  | 0.18047e-047 | 7.06 |
| $f_3(x)$, $x_0 = 2$     | 3         |    | 0.23868e-183  | 0.19389e-025 | 6.84 |
| $f_4(x)$, $x_0 = 1.7$   | 3         |    | 0.14303e-043  |                | 6.63 |
| $f_5(x)$, $x_0 = 2.5$   | 3         |    | 0.26754e-138  | 0.11882e-019  | 6.53 |
| $f_6(x)$, $x_0 = 2$     | 3         |    | 0.20000e-198  | 0.26191e-050  | 7.04 |
| $f_7(x)$, $x_0 = 3.2$   | 4         |    | 0.10000e-198  | 0.14954e-033  | 6.90 |
| $f_8(x)$, $x_0 = 2$     | 3         |    | 0.10000e-199  | 0.33233e-041  | 7.12 |
| $f_9(x)$, $x_0 = 2$     | 3         |    | 0.40000e-199  | 0.13769e-056  | 6.96 |
| $f_{10}(x)$, $x_0 = 1.8$ | 3     |    | 0.40000e-199  | 0.33233e-041  | 7.12 |