Quasihyperbolic Metrics, $Lip_G$-Extension Domains and John Disks\textsuperscript{1}

Weimao Qian

Huzhou Broadcast and TV University
Huzhou 313000, P. R. China
qwm661977@126.com

Weiming Gong

Department of Mathematics
Hunan City University
Yiyang 413000, P. R. China

Abstract

Suppose that $D$ is a Jordan proper subdomain of $\mathbb{R}^2$. In this paper, we prove that (1) $D$ is a John disk if and only if there exists a constant $c \geq 1$ such that $k_D(x_1, x_2) \leq c G_D(x_1, x_2)$ for all $x_1, x_2 \in D$; (2) $D$ is a John disk if and only if $D$ is a Lip\textsubscript{$G$}-extension domain. Here, $k_D$ is the quasihyperbolic metric in $D$, $G_D(x_1, x_2) = \frac{1}{2} \log(1 + \frac{l(\gamma)}{d(x_1, \partial D)}(1 + \frac{l(\gamma)}{d(x_2, \partial D)}))$, $\gamma \subset D$ is the quasihyperbolic geodesic with endpoints $x_1$ and $x_2$, and $l(\gamma)$ is the Euclidean length of $\gamma$.

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1. Introduction

In this paper, we shall adopt the notation and terminology as in paper [20], $\mathbb{R}^2$ denotes the 2-dimensional Euclidean space, $\bar{\mathbb{R}}^2 = \mathbb{R}^2 \cup \{\infty\}$, $D$ is a proper subdomain of $\mathbb{R}^2$, denotes $\partial D$ the boundary of $D$. For non-empty sets $A$ and $B$, denotes $d(A, B)$ the Euclidean distance between $A$ and $B$. For $x \in \mathbb{R}^2$ and

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0 < r < ∞, let $B^2(x, r) = \{ z \in \mathbb{R}^2 : |z - x| < r \}$, $\bar{B}^2(x, r)$ be the closure of $B^2(x, r)$. For a rectifiable arc $\gamma \subseteq D$, denotes $l(\gamma)$ the Euclidean length of $\gamma$.

A Jordan proper subdomain $D \subset \mathbb{R}^2$ is said to be a $b-$John disk if each pair of points $x_1, x_2 \in D$ can be joined by a rectifiable arc $\gamma \subset D$ for which

$$\min_{j=1,2} l(\gamma(x_j, x)) \leq bd(x_j, \partial D) \quad (1.1)$$

for all $x \in \gamma$, where $\gamma(x_j, x)$ is the subarc of $\gamma$ with endpoints $x_j$ and $x$. $D$ is said to be a John disk if it is a $b-$John disk for some constants $b$. A rectifiable arc satisfying (1.1) is called a double $b-$cone arc.

The concept of John disks was introduced in 1961 by F. John [12] in connection with his work in elasticity, the term is due to O. Martio and J. Sarvas [17] who employed John disks in certain approximation and injectivity questions. John disks arise also naturally in distortion problems of conformal and quasiconformal mappings, they appear in many contexts in analysis [1-5, 10, 11, 14, 15]. John disks enjoy many properties similar to those characteristic for quasidisks, i. e. images of disks or half planes under quasiconformal self maps of $\bar{\mathbb{R}}^2$. Indeed John disks may be regarded as one sided quasidisks [18, 19].

For each $x_1, x_2 \in D$, we set

$$k_D(x_1, x_2) = \inf_{\gamma} \int_{\gamma} d(x, \partial D)^{-1} ds , \quad (1.2)$$

where the infimum in (1.2) is taken over all rectifiable arcs $\gamma$ joining $x_1$ and $x_2$ in $D$. We call $k_D$ the quasihyperbolic metric in $D$ [8], and $\gamma$ the quasihyperbolic geodesic for which the infimum in (1.2) is attained [6, 9].

Next, for each $x_1, x_2 \in D$, we set

$$J_D(x_1, x_2) = \frac{1}{2} \log \left(1 + \frac{|x_1 - x_2|}{d(x_1, \partial D)} \right) \left(1 + \frac{|x_1 - x_2|}{d(x_2, \partial D)} \right) \quad (1.3)$$

and

$$G_D(x_1, x_2) = \frac{1}{2} \log \left(1 + \frac{l(\gamma)}{d(x_1, \partial D)} \right) \left(1 + \frac{l(\gamma)}{d(x_2, \partial D)} \right) , \quad (1.4)$$

where $\gamma \subset D$ is the quasihyperbolic geodesic with endpoints $x_1$ and $x_2$. P. W. Jones [13] studied the domains $D$ which there exist constants $c$ and $d$ such that

$$k_D(x_1, x_2) \leq c J_D(x_1, x_2) + d \quad (1.5)$$

for all $x_1, x_2 \in D$. It is precisely this class of domains $D$ for which each function $u$ with bounded mean oscillation in $D$ has an extension $v$ with bounded mean oscillation in $\mathbb{R}^2$. W. Li [16] proved that
Theorem A. A Jordan proper subdomain $D \subset \mathbb{R}^2$ is a quasidisk if and only if there exists a constant $c \geq 1$ such that

$$k_D(x_1, x_2) \leq cJ_D(x_1, x_2)$$

for all $x_1, x_2 \in D$.

One purpose of this paper is to prove the following result:

Theorem 1.1. A Jordan proper subdomain $D \subset \mathbb{R}^2$ is a John disk if and only if there exists a constant $c \geq 1$ such that

$$k_D(x_1, x_2) \leq cG_D(x_1, x_2)$$

for all $x_1, x_2 \in D$.

Suppose that $f : D \to \mathbb{R}^p$ is a mapping, $h(x, y)$ is a two-variable nonnegative real function defined in $D$. We say that $f \in \text{Lip}_h(D)$ if there exists a constant $m > 0$ such that

$$|f(x_1) - f(x_2)| \leq mh(x_1, x_2)$$

for all $x_1, x_2 \in D$. We say that $f \in \text{locLip}_h(D)$ if there exists a constant $m > 0$ such that (1.8) holds whenever $x_1$ and $x_2$ lie in any open disk $B$ which is contained in $D$.

In $\text{Lip}_h(D)$ and $\text{locLip}_h(D)$ we shall use the seminorms $\|f\|_h$ and $\|f\|_{h, \text{loc}}$ as follows:

$$\|f\|_h = \inf \{m : |f(x_1) - f(x_2)| \leq mh(x_1, x_2), x_1, x_2 \in D\},$$

$$\|f\|_{h, \text{loc}} = \inf \{m : |f(x_1) - f(x_2)| \leq mh(x_1, x_2), x_1, x_2 \in B \subset D\},$$

where $B$ ranges over all open disks which are contained in $D$.

We say that $D$ is a $\text{Lip}_h$-extension domain if there exists an absolute constant $a \geq 1$ such that

$$\|f\|_h \leq a \|f\|_{h, \text{loc}}$$

for all $f \in \text{locLip}_h(D)$.

In the case of $h(x, y) = |x - y|^\alpha$ with $0 < \alpha \leq 1$, F. W. Gehring and O. Martio [7] use $\text{Lip}_\alpha(D)$ to denote $\text{Lip}_h(D)$, and prove that

Theorem B. Quasidisks are $\text{Lip}_\alpha$-extension domains for all $0 < \alpha \leq 1$, and there exists a Jordan proper subdomain $D \subset \mathbb{R}^2$ which is $\text{Lip}_\alpha$-extension domain for all $0 < \alpha \leq 1$ but is not a quasidisk.

Theorem C. Suppose that $D \subset \mathbb{R}^2$ is a Jordan proper subdomain with $\infty \in \partial D$. If $D$ is a $\text{Lip}_\alpha$-extension domain for some $0 < \alpha \leq 1$, and $D' = \mathbb{R}^2 \setminus \overline{D}$ is a $\text{Lip}_\beta$-extension domain for some $0 < \beta \leq 1$, then $D$ is a quasidisk.

Another purpose of this paper is to prove the following result:

Theorem 1.2. A Jordan proper subdomain $D \subset \mathbb{R}^2$ is a John disk if and only if $D$ is a $\text{Lip}_G$-extension domain.
2. Lemmas

In order to prove our Theorems 1.1 and 1.2 we need four lemmas, which we present in this section.

**Lemma 2.1.** If \( c \geq 1 \), then
\[
\log(1 + cx) \leq c \log(1 + x) \tag{2.1}
\]
for all \( x \geq 0 \).

**Proof.** Let \( f(x) = c \log(1 + x) - \log(1 + cx) \), \( c \geq 1 \) and \( x \geq 0 \). Then
\[
f'(x) = \frac{cx(c-1)}{(1+x)(1+cx)} \geq 0.
\]
Hence, \( f(x) \) is strictly increasing in \([0, \infty)\), which implies that
\[
f(x) \geq f(0) = 0, \quad \log(1 + cx) \leq c \log(1 + x).
\]

**Lemma 2.2.** If \( x \geq 0 \), then
\[
\log(1 + x) < (1 + x)^{\frac{1}{2}}. \tag{2.2}
\]

**Proof.** Let \( f(x) = (1 + x)^{\frac{1}{2}} - \log(1 + x) \), then
\[
f'(x) = \frac{\sqrt{1 + x} - 2}{2(1 + x)}. \tag{2.3}
\]
Equation (2.3) implies that
\[
f'(x) > 0 \quad \text{for} \quad x > 3,
\]
\[
f'(x) < 0 \quad \text{for} \quad 0 \leq x < 3
\]
and
\[
f'(x) = 0 \quad \text{for} \quad x = 3.
\]
Therefore, we get
\[
\min_{x \in [0, \infty)} f(x) = f(3) = 2(1 - \log 2) > 0,
\]
which leads to
\[
\log(1 + x) < (1 + x)^{\frac{1}{2}}, \quad x \geq 0.
\]

**Lemma 2.3.** Suppose that \( D \) is a Jordan proper subdomain of \( R^2 \), \( x_1, x_2 \in D \). If there exists a double \( b \)-cone arc \( \gamma \subset D \) with endpoints \( x_1 \) and \( x_2 \), then
\[
k_D(x_1, x_2) \leq c \log \left( 1 + \frac{l(\gamma)}{d(x_1, \partial D)} \right) \left( 1 + \frac{l(\gamma)}{d(x_2, \partial D)} \right)
\]
with \( c = 3b + 2 \) is a constant which depends only on \( b \).
**Proof.** Choose \( x_0 \in \gamma \) such that \( l(\gamma(x_0, x_1)) = l(\gamma(x_2, x_0)) \), there are following two cases:

**Case 1.**

\[
l(\gamma(x_1, x_0)) \leq \frac{b}{b+1} d(x_1, \partial D). \tag{2.4}
\]

Then \( x_0 \in B^2(x_1, \frac{b}{b+1} d(x_1, \partial D)) \subset D \), denotes \([x_1, x_0]\) the closed segment with endpoints \( x_1 \) and \( x_0 \). For any \( x \in [x_1, x_0] \), making use of the triangle inequality and (2.4) we get

\[
d(x, \partial D) \geq d(x_1, \partial D) - |x_1 - x|
\geq d(x_1, \partial D) - |x_1 - x_0|
\geq d(x_1, \partial D) - l(\gamma(x_1, x_0))
\geq \frac{1}{b+1} d(x_1, \partial D) \tag{2.5}
\]

and

\[
|x_1 - x| \leq l(\gamma(x_1, x_0))
\leq \frac{b}{b+1} d(x_1, \partial D)
\leq bd(x, \partial D). \tag{2.6}
\]

Inequalities (2.5) and (2.6) lead to

\[
|x_1 - x| + d(x_1, \partial D) \leq (2b + 1)d(x, \partial D). \tag{2.7}
\]

It follows from (2.7) that

\[
k_D(x_1, x_0) \leq \int_{[x_1, x_0]} \frac{ds}{d(x, \partial D)}
\leq (2b + 1) \int_{[x_1, x_0]} \frac{|x - x_1| + d(x_1, \partial D)}{s + d(x_1, \partial D)}
= (2b + 1) \int_0^{x_1 - x_0} \frac{ds}{s + d(x_1, \partial D)}
= (2b + 1) \log \left( 1 + \frac{|x_1 - x_0|}{d(x_1, \partial D)} \right)
\leq (2b + 1) \log \left( 1 + \frac{l(\gamma)}{d(x_1, \partial D)} \right). \tag{2.8}
\]

**Case 2.**

\[
l(\gamma(x_1, x_0)) > \frac{b}{b+1} d(x_1, \partial D). \tag{2.9}
\]
Choose $y_1 \in \gamma(x_1, x_0)$ such that

$$l(\gamma(x_1, y_1)) = \frac{b}{b + 1} d(x_1, \partial D),$$

(2.10)

making use of the same method as in Case 1 we have

$$k_D(x_1, y_1) \leq (2b + 1) \log \left(1 + \frac{l(\gamma)}{d(x_1, \partial D)}\right).$$

(2.11)

For any $x \in \gamma(y_1, x_0)$, inequality (1.1) leads to

$$d(x, \partial D) \geq \frac{1}{b} l(\gamma(x_1, x)).$$

(2.12)

From (2.10) and (2.12) together with Lemma 2.1 we clearly see that

$$k_D(y_1, x_0) \leq \int_{\gamma(y_1, x_0)} \frac{ds}{d(x, \partial D)} \leq b \int_{\gamma(y_1, x_0)} \frac{ds}{l(\gamma(x_1, x))} \leq b \int_0^{l(\gamma(y_1, x_0))} \frac{ds}{s + l(\gamma(x_1, y_1))} = b \log \left(1 + \frac{b + 1}{b} \frac{l(\gamma(y_1, x_0))}{d(x_1, \partial D)}\right) \leq (b + 1) \log \left(1 + \frac{l(\gamma(y_1, x_0))}{d(x_1, \partial D)}\right).$$

(2.13)

It follows from (2.11) and (2.13) together with the triangle inequality that

$$k_D(x_1, x_0) \leq (3b + 2) \log \left(1 + \frac{l(\gamma)}{d(x_1, \partial D)}\right).$$

(2.14)

From Cases 1 and 2 we know that (2.14) holds. Making use of the same argument as above we get

$$k_D(x_2, x_0) \leq (3b + 2) \log \left(1 + \frac{l(\gamma)}{d(x_2, \partial D)}\right).$$

(2.15)

Inequalities (2.14) and (2.15) together with the triangle inequality lead to the conclusion that

$$k_D(x_1, x_2) \leq c \log \left(1 + \frac{l(\gamma)}{d(x_1, \partial D)}\right) \left(1 + \frac{l(\gamma)}{d(x_2, \partial D)}\right).$$
Lemma 2.4. If $D$ is a Jordan proper subdomain of $R^2$, then
\[ k_D(x_1, x_2) \geq G_D(x_1, x_2) \]
for all $x_1, x_2 \in D$.

**Proof.** For any $x_1, x_2 \in D$, let $\gamma \subset D$ be the quasihyperbolic geodesic with endpoints $x_1$ and $x_2$, and $\gamma(s)$ be the parameterization of $\gamma$ with respect to arc length measured from $x_1$, then
\[ k_D(x_1, x_2) = \int_{\gamma} \frac{ds}{d(x, \partial D)} \geq \int_{0}^{l(\gamma)} \frac{ds}{s + d(x_1, \partial D)} \]
\[ = \log \left( 1 + \frac{l(\gamma)}{d(x_1, \partial D)} \right). \quad (2.16) \]
Making use of the same argument as above we have
\[ k_D(x_1, x_2) \geq \log \left( 1 + \frac{l(\gamma)}{d(x_2, \partial D)} \right). \quad (2.17) \]
Inequalities (2.17) and (2.17) lead to
\[ k_D(x_1, x_2) \geq G_D(x_1, x_2). \]

3. Proof of Theorems 1.1 and 1.2

**Proof of Theorem 1.1.** Necessity. If $D$ is a $b$–John disk, $x_1, x_2 \in D$, and $\gamma \subset D$ is a quasihyperbolic geodesic with endpoints $x_1$ and $x_2$, then from [6, Theorem 4.1] we know that $\gamma$ is a double $b'$–cone arc, where $b'$ is a constant which depends only on $b$. It follows from Lemma 2.3 that
\[ k_D(x_1, x_2) \leq cG_D(x_1, x_2) \]
with $c = 6b' + 4$ is a constant which depends only on $b$.

Sufficiency. For any $x_1, x_2 \in D$, suppose that $\gamma \subset D$ is a quasihyperbolic geodesic with endpoints $x_1$ and $x_2$ such that (1.7) holds. Without loss of generality, we assume that $d(x_1, \partial D) \geq d(x_2, \partial D)$.

Suppose first that
\[ 2l(\gamma) \leq d(x_1, \partial D). \quad (3.1) \]
For any $x \in \gamma$, from the triangle inequality and (3.1) one has
\[ d(x, \partial D) \geq d(x_1, \partial D) - l(\gamma(x_1, x)) \]
\[ \geq 2l(\gamma) - l(\gamma) \]
\[ \geq \min_{j=1,2} l(\gamma(x_j, x)). \quad (3.2) \]
Next suppose that (3.1) does not hold. By compactness there exists a point $x_0 \in \gamma$ such that
\[ d(x_0, \partial D) = \sup_{x \in \gamma} d(x, \partial D). \] (3.3)

Let $m$ be the largest integer for which
\[ 2^m d(x_1, \partial D) \leq d(x_0, \partial D), \] (3.4)
and $y_0$ be the first point of $\gamma(x_1, x_0)$ with
\[ d(y_0, \partial D) = 2^m d(x_1, \partial D) \] (3.5)
as we traverse $\gamma$ from $x_1$ towards $x_0$. Clearly
\[ d(y_0, \partial D) \leq d(x_0, \partial D) < 2d(y_0, \partial D). \] (3.6)

Let $y_1 = x_1$ and choose points $y_2, y_3, \cdots, y_{m+1} \in \gamma(x_1, x_0)$ such that $y_i$ is the first point of $\gamma(x_1, x_0)$ for which
\[ d(y_i, \partial D) = 2^{i-1} d(y_1, \partial D) \] (3.7)
as we traverse $\gamma$ from $x_1$ towards $x_0$.

Clearly $y_{m+1} = y_0$, let $y_{m+2} = x_0$, then for $i = 1, 2, \cdots, m + 1$ one has
\[ k_D(y_i, y_{i+1}) \leq 4c^2 \] (3.8)
and
\[ l(\gamma(y_i, y_{i+1})) \leq 8c^2 d(y_i, \partial D). \] (3.9)

In fact, for $i = 1, 2, \cdots, m + 1$, take
\[ t = \frac{l(\gamma(y_i, y_{i+1}))}{d(y_i, \partial D)}. \] (3.10)

For any $x \in \gamma(y_i, y_{i+1})$, from (3.6) and (3.7) we have
\[ d(x, \partial D) \leq d(y_{i+1}, \partial D) \leq 2d(y_i, \partial D). \] (3.11)

Equation (3.10) and inequality (3.11) lead to
\[ t = \int_{\gamma(y_i, y_{i+1})} \frac{ds}{d(y_i, \partial D)} \leq 2k_D(y_i, y_{i+1}). \] (3.12)
From the definition of $y_i$ ($i = 1, 2, \cdots, m + 1$) in (3.7) we get

$$G_D(y_i, y_{i+1}) = \frac{1}{2} \log \left(1 + \frac{l(\gamma(y_i, y_{i+1}))}{d(y_i, \partial D)}\right) \left(1 + \frac{l(\gamma(y_i, y_{i+1}))}{d(y_{i+1}, \partial D)}\right)$$

$$\leq \log \left(1 + \frac{l(\gamma(y_i, y_{i+1}))}{d(y_i, \partial D)}\right) = \log(1 + t). \quad (3.13)$$

It follows from (1.7) and Lemma 2.2 together with (3.12) and (3.13) that

$$t \leq 2c(1 + t)^{\frac{1}{2}}. \quad (3.14)$$

If $t \geq 1$, then inequality (3.14) implies that

$$t \leq 2c(2t)^{\frac{1}{2}},$$

which leads to

$$t \leq 8c^2. \quad (3.15)$$

If $t < 1$, then we clearly see that (3.15) is also true.

From (1.7), Lemma 2.2, (3.10), (3.13) and (3.15) we have

$$k_D(y_i, y_{i+1}) \leq cG_D(y_i, y_{i+1}) \leq c\log(1 + t) \leq c(1 + t)^\frac{1}{2} \leq c(8c^2 + 8c^2)^\frac{1}{2} = 4c^2$$

and

$$l(\gamma(y_i, y_{i+1})) \leq 8c^2d(y_i, \partial D).$$

Therefore, both (3.8) and (3.9) are true.

Now, for any $x \in \gamma(y_i, y_{i+1})$, $i = 1, 2, \cdots, m + 1$, from [8, Lemma 2.1] and (3.8) we get

$$\log \frac{d(y_{i+1}, \partial D)}{d(x, \partial D)} \leq k_D(x, y_{i+1}) \leq k_D(y_i, y_{i+1}) \leq 4c^2,$$  

$$\quad (3.16)$$

which leads to

$$d(y_{i+1}, \partial D) \leq e^{4c^2}d(x, \partial D). \quad (3.17)$$

If $x \in \gamma(x_1, x_0)$, then there exists $i_0 \in \{1, 2, \cdots, m + 1\}$ such that $x \in$
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\[ \gamma(y_i, y_{i+1}), \text{ and (3.7), (3.9) and (3.17) imply that} \]
\[ \min_{j=1,2} l(\gamma(x_j, x)) \leq l(\gamma(x_1, x)) \leq \sum_{i=1}^{i_0} l(\gamma(y_i, y_{i+1})) \]
\[ \leq 8c^2 \sum_{i=1}^{i_0} d(y_i, \partial D) \]
\[ = 8c^2 (\sum_{i=1}^{i_0} 2^{i-1}) d(y_1, \partial D) \]
\[ < 8c^2 2^{i_0} d(y_1, \partial D) \]
\[ \leq 16c^2 d(y_{i_0+1}, \partial D) \]
\[ \leq 8c^2 e^{4c^2} d(x, \partial D). \quad (3.18) \]

If \( x \in \gamma(x_2, x_0) \), then making use of the same argument as above we get
\[ \min_{j=1,2} l(\gamma(x_j, x)) \leq 8c^2 e^{4c^2} d(x, \partial D). \quad (3.19) \]

Hence, for any \( x \in \gamma \) one has
\[ \min_{j=1,2} l(\gamma(x_j, x)) \leq 8c^2 e^{4c^2} d(x, \partial D). \quad (3.20) \]

Therefore, \( D \) is a \( b \)-John disk with \( b = 8c^2 e^{4c^2} \) follows from (3.2) and (3.20).

**Proof of Theorem 1.2.** Necessity. Suppose that \( D \) is a \( b \)-John disk and \( f \in \text{locLip}_G(D) \), then Theorem 1.1 implies that there exists a constant \( c \geq 1 \) which depends only on \( b \) such that (1.7) holds for all \( x_1, x_2 \in D \).

Next for any \( x_1, x_2 \in D \), let \( \gamma \subset D \) be the quasihyperbolic geodesic with endpoints \( x_1 \) and \( x_2 \), and \( \gamma(s) \) be the parameterization of \( \gamma \) with respect to arc length measure from \( x_1 \). Set \( y_1 = x_1 \), choose positive numbers \( r_i \) and \( l_i \), and points \( y_i \in \gamma \) as follows:
\[ r_1 = \frac{1}{2} d(y_1, \partial D), \quad l_1 = \max\{s : \gamma(s) \in \overline{B}^2(y_1, r_1)\}, \quad y_2 = \gamma(l_1); \]
\[ r_2 = \frac{1}{2} d(y_2, \partial D), \quad l_2 = \max\{s : \gamma(s) \in \overline{B}^2(y_2, r_2)\}, \quad y_3 = \gamma(l_3); \]
and so on. After a finite number of steps, \( N \), say, \( l_N = l(\gamma) \) and the process stops. Set \( y_{N+1} = x_2 \), then from inequality (1.7) and Lemma 2.4 together with
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$f \in \text{loc Lip}_G(D)$ we get

$$|f(x_1) - f(x_2)| \leq \sum_{i=1}^{N} |f(y_{i+1} - f(y_i)|$$

$$\leq \|f\|_{\text{loc}G} \sum_{i=1}^{N} G_D(y_i, y_{i+1})$$

$$\leq \|f\|_{\text{loc}G} \sum_{i=1}^{N} k_D(y_i, y_{i+1})$$

$$= \|f\|_{\text{loc}G} k_D(x_1, x_2)$$

$$\leq c \|f\|_{\text{loc}G} G_D(x_1, x_2). \tag{3.21}$$

Inequality (3.21) and the randomicity of $f \in \text{loc Lip}_G(D)$ lead to the conclusion that

$$\|f\|_G \leq c \|f\|_{\text{loc}G}$$

for all $f \in \text{loc Lip}_G(D)$. Therefore, $D$ is a $\text{Lip}_G$-extension domain.

Sufficiency. Suppose that $D$ is a $\text{Lip}_G$-extension domain, then there exists an absolute constant $a \geq 1$ such that inequality (1.9) holds for all $h \in \text{loc Lip}_G(D)$. Fixed any $x_0 \in D$, let

$$f(x) = k_D(x, x_0). \tag{3.22}$$

Then for any $x_1, x_2 \in B \subset D$, $B$ is an open disk, it follows from (3.22) and the triangle inequality that

$$|f(x_1) - f(x_2)| \leq k_D(x_1, x_2). \tag{3.23}$$

Let $\alpha \subset B$ be the segment of the circle through $x_1, x_2$ perpendicular to $\partial B$ with endpoints $x_1$ and $x_2$, then

$$l(\alpha) \leq \pi |x_1 - x_2| \tag{3.24}$$

and

$$\min_{j=1,2} l(\alpha(x, x_j)) \leq \pi d(x, \partial B) \leq \pi d(x, \partial D) \tag{3.25}$$

for all $x \in \gamma$. From Lemmas 2.1 and 2.3 together with inequalities (3.24) and (3.25) we clearly see that

$$k_D(x_1, x_2) \leq (3\pi + 2) \log \left(1 + \frac{l(\alpha)}{d(x_1, \partial D)}\right) \left(1 + \frac{l(\alpha)}{d(x_2, \partial D)}\right)$$

$$\leq (3\pi + 2) \pi \log \left(1 + \frac{|x_1 - x_2|}{d(x_1, \partial D)}\right) \left(1 + \frac{|x_1 - x_2|}{d(x_2, \partial D)}\right). \tag{3.26}$$
Let $\gamma$ be the quasihyperbolic geodesic in $D$ with endpoints $x_1$ and $x_2$, then inequality (3.26) leads to

$$k_D(x_1, x_2) \leq cG_D(x_1, x_2) \quad (3.27)$$

with $c = 2\pi(3\pi + 2)$.

Inequalities (3.23) and (3.27) lead to the conclusion that $f \in \text{locLip}_G(D)$ with $\|f\|_{\text{loc}G} \leq c = 2\pi(3\pi + 2)$. Then from (1.9) we get $f \in \text{Lip}_G(D)$ with $\|f\|_G \leq ac$, which implies that

$$k_D(x, x_0) = |f(x) - f(x_0)| \leq acG_D(x, x_0) \quad (3.28)$$

for all $x \in D$.

From (3.28) and the randomicity of $x_0$ we know that

$$k_D(x, y) \leq acG_D(x, y) \quad (3.29)$$

for all $x, y \in D$.

Therefore, $D$ is a John disk follows from (3.29) and Theorem 1.1.

References


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