Coefficient Inequalities for Classes of Uniformly Starlike and Convex Functions Defined by Generalized Ruscheweyh Operator

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Abstract. We introduce and study the classes $V_{\lambda}(\beta, b, \delta)$ and $VD_{\lambda}(\alpha, \beta, b, \delta)$ of analytic functions which are defined by making use of the generalized Ruscheweyh derivative operator. Coefficient inequalities conditions are investigated for these classes.

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Let $\mathcal{A}$ be the class of functions $f$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (1.1)$$

which are analytic in the open unit disk $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$.

In the present paper, we use a recent generalization of the Ruscheweyh derivative \cite{1} denoted by $D_{\lambda}^{\delta}$, defined as follows:

$$D_{\lambda}^{\delta} f(z) = \frac{z}{(1 - z)^{1+\delta}} \ast D_{\lambda} f, \quad f \in \mathcal{A}, \quad z \in \mathcal{U}$$

where $\ast$ stands for the convolution or Hadamard product of two power series. Further, we have,

$$D_{\lambda} f(z) = (1 - \lambda)f(z) + \lambda z f'(z), \delta > -1, \lambda \geq 0, \quad z \in \mathcal{U}.$$ 

We can easily see that $D_{\lambda}^{\delta} f(z)$ admits a representation of the form

$$D_{\lambda}^{\delta} f(z) = z + \sum_{n=2}^{\infty} [1 + (n-1)\lambda]B_n(\delta)a_n z^n$$

where,

$$B_n(\delta) = \frac{(\delta + 1)(\delta + 2)\ldots(\delta + n - 1)}{(n-1)!}.$$

We introduce the class $\mathcal{V}_{\lambda}(\beta, b, \delta)$ as the subclass of $\mathcal{A}$ consisting of functions $f$ obeying the condition

$$\Re \left\{ 1 - \frac{2}{b} + \frac{2}{b} D_{\lambda}^{\delta+1} f(z) \right\} > \beta, \quad (z \in \mathcal{U}) \quad (1.2)$$

where $b$ is a non-zero complex number, $0 \leq \beta < 1, \lambda \geq 0$ and $\delta > -1$.

This class generalizes the class $\mathcal{V}(\beta, b, \delta) \cite{4}$ and is of special interest for, it contains many well known as well as new classes of analytic functions. It provides a transition from starlike functions to convex functions more specifically $\mathcal{V}_0(\beta, 2, 0)$ is the family of starlike functions of order $\beta$ and $\mathcal{V}_0(\beta, 1, 1)$ is the class of convex functions of
order $\beta$. Shams, Kulkarni and Jahangiri [8] introduced the subclass $SD(\alpha, \beta)$ of $A$ consisting of functions $f$ satisfying

$$
\Re \left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha \left| \frac{zf'(z)}{f(z)} - 1 \right| + \beta, \quad (z \in U)
$$

for some $\alpha(\alpha \geq 0)$ and $\beta(0 \leq \beta < 1)$.

The class $KD(\alpha, \beta)$, another subclass of $A$, is defined as the set of all functions $f$ obeying

$$
\Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \alpha \left| \frac{zf''(z)}{f'(z)} \right| + \beta, \quad (z \in U)
$$

for some $\alpha(\alpha \geq 0)$ and $\beta(0 \leq \beta < 1)$.

We introduce the class $VD(\alpha, \beta, b, \delta)$ as the subclass of $A$ consisting of functions $f$ which satisfy

$$
\Re \left\{ 1 - \frac{2}{b} + \frac{2 D_\lambda^{\delta+1} f(z)}{b D_\lambda^\delta f(z)} \right\} > \alpha \left| \frac{2 D_\lambda^{\delta+1} f(z)}{b D_\lambda^\delta f(z)} - \frac{2}{b} \right| + \beta, \quad (z \in U)
$$

where $b$ is a non-zero complex number for some $\alpha \geq 0, 0 \leq \beta < 1, \lambda \geq 0$ and $\delta > -1$.

For the parametric values $\lambda = 0; \lambda = 0, b = 2, \delta = 0$ and $\lambda = 0, b = \delta = 1$ we obtain the classes $VD(\alpha, \beta, b, \delta), SD(\alpha, \beta)$ and $KD(\alpha, \beta)$ respectively.

2. **MAIN RESULTS**

We prove some coefficient inequalities for functions in the class $VD(\alpha, \beta, b, \delta)$.

**Theorem 2.1.** If $f \in VD(\alpha, \beta, b, \delta)$ with $0 \leq \alpha \leq \beta$, then $f \in V_\lambda(\frac{\beta-\alpha}{1-\alpha}, b, \delta)$.

**Proof.** Since $\Re(\omega) \leq |\omega|$ for any complex number $\omega, f \in VD(\alpha, \beta, b, \delta)$ implies that

$$
\Re \left\{ 1 - \frac{2}{b} + \frac{2 D_\lambda^{\delta+1} f(z)}{b D_\lambda^\delta f(z)} \right\} > \alpha \left| \frac{2 D_\lambda^{\delta+1} f(z)}{b D_\lambda^\delta f(z)} - \frac{2}{b} \right| + \beta, \quad (z \in U)
$$

Equivalently,

$$
\Re \left\{ 1 - \frac{2}{b} + \frac{2 D_\lambda^{\delta+1} f(z)}{b D_\lambda^\delta f(z)} \right\} > \frac{\beta - \alpha}{1 - \alpha}, \quad (z \in U).
$$

If $0 \leq \alpha \leq \beta$, then we have, $0 \leq \frac{\beta - \alpha}{1 - \alpha} < 1$. \qed
Corollary 2.2. For the parametric value $\lambda = 0$, we get Theorem 2.1 in [3] which reads:
If $f \in VD(\alpha, \beta, b, \delta)$ with $0 \leq \alpha \leq \beta$, then $f \in V(\frac{\beta-\alpha}{1-\alpha}, b, \delta)$.

Corollary 2.3. For the parametric values $b = 2, \delta = 0$ and $\lambda = 0$, we get Theorem 2.1 in [5] which reads:
If $f \in S(\alpha, \beta)$ with $0 \leq \alpha \leq \beta$, then $f \in S^*(\frac{\beta-\alpha}{1-\alpha})$.

Corollary 2.4. The parametric values $b = \delta = 1$ and $\lambda = 0$ yield Corollary 2.2 in [6] stated as:
If $f \in KD(\alpha, \beta)$ with $0 \leq \alpha \leq \beta$, then $f \in K(\frac{\beta-\alpha}{1-\alpha})$.

Theorem 2.5. If $f \in VD(\alpha, \beta, b, \delta)$, then
\begin{equation}
|a_2| \leq \frac{|b|(1-\beta)}{(1+\lambda)|1-\alpha|}.
\end{equation}

and
\begin{equation}
|a_n| \leq \frac{|b|(1-\beta)(\delta+1)}{(n-1)[1+(n-1)\lambda]|1-\alpha|B_n(\delta)} \prod_{j=1}^{n-2} \left(1 + \frac{b(\delta+1)(1-\beta)}{j|1-\alpha|}\right) \quad (n \geq 3).
\end{equation}

Proof. We note that for $f \in VD(\alpha, \beta, b, \delta)$
\[\Re \left\{1 - \frac{2}{b} + \frac{2 D^{\delta+1}_\lambda f(z)}{b D^\delta f(z)}\right\} > \frac{\beta-\alpha}{1-\alpha}, \quad (z \in U).\]

If we define the function $p$ by
\[p(z) = \frac{(1-\alpha) \left[1 - \frac{2}{b} + \frac{2 D^{\delta+1}_\lambda f(z)}{b D^\delta f(z)}\right] - (\beta-\alpha)}{(1-\beta)} \quad (z \in U)\]
then $p$ is analytic in $U$ with $p(0) = 1$ and $\Re(p(z)) > 0$, \quad $(z \in U)$.

Let $p(z) = 1 + p_1 z + p_2 z^2 + \ldots$, then we have,
\[1 - \frac{2}{b} + \frac{2 D^{\delta+1}_\lambda f(z)}{b D^\delta f(z)} = \frac{(1-\beta)p(z) + (\beta-\alpha)}{(1-\alpha)} = 1 + \left(\frac{1-\beta}{1-\alpha}\right) \sum_{n=1}^{\infty} p_n z^n.\]
That is,

\[(2.3) \quad 2 \left(D^{\delta+1}_\lambda f(z) - D^{\delta+1}_\lambda\right) = bD^{\delta+1}_\lambda \left[\left(\frac{1 - \beta}{1 - \alpha}\right) \sum_{n=1}^{\infty} p_n z^n\right].\]

Therefore, (2.3) implies that

\[
\frac{2 [1 + (n - 1)\lambda] B_n(\delta)(n - 1)a_n}{(\delta + 1)} = \frac{b(1 - \beta)}{(1 - \alpha)} [p_{n-1} + (1 + \lambda)B_2(\delta)a_2p_{n-2} + (1 + 2\lambda)B_3(\delta)a_3p_{n-3} + \ldots + [1 + (n - 2)\lambda]B_{n-1}(\delta)a_{n-1}p_1]
\]

Applying the coefficient estimates such that \(|p_n| \leq 2\ (n \geq 1)\) for Caratheodory functions [2], we obtain that

\[
|a_n| \leq \frac{|b|(1 - \beta)(\delta + 1)}{|1 - \alpha|(n - 1)[1 + (n - 1)\lambda]B_n(\delta)} \left[1 + (1 + \lambda)B_2(\delta)|a_2| + (1 + 2\lambda)B_3(\delta)|a_3| + \ldots + [1 + (n - 2)\lambda]B_{n-1}(\delta)|a_{n-1}| \right]
\]

For \(n = 2\), we have

\[
|a_2| \leq \frac{|b|(1 - \beta)}{|1 - \alpha|(1 + \lambda)}
\]

which proves (2.1). For \(n = 3\),

\[
|a_3| \leq \frac{|b|(1 - \beta)(\delta + 1)}{2(1 + 2\lambda)|1 - \alpha|B_3(\delta)} \left[1 + \frac{|b|(1 - \beta)(\delta + 1)}{|1 - \alpha|} \right]
\]
Therefore \( (2.2) \) holds for \( n = 3 \). Suppose that \( (2.2) \) is true for \( n = k \).

Consider,

\[
|a_{k+1}| \leq \frac{|b|(1 - \beta)(\delta + 1)}{k(1 + k\lambda)|1 - \alpha|B_{k+1}(\delta)} \left\{ \left( 1 + \frac{|b|(1 - \beta)(\delta + 1)}{|1 - \alpha|} \right) \right. \\
+ \frac{|b|(1 - \beta)(\delta + 1)}{2|1 - \alpha|} \left( 1 + \frac{|b|(1 - \beta)(\delta + 1)}{|1 - \alpha|} \right) \\
\left. + \ldots + \frac{|b|(1 - \beta)(\delta + 1)}{(k - 1)|1 - \alpha|} \prod_{j=1}^{k-2} \left( 1 + \frac{|b|(1 - \beta)(\delta + 1)}{j|1 - \alpha|} \right) \right\} \\
= \frac{|b|(1 - \beta)(\delta + 1)}{k(1 + k\lambda)|1 - \alpha|B_{k+1}(\delta)} \prod_{j=1}^{k-1} \left( 1 + \frac{|b|(1 - \beta)(\delta + 1)}{j|1 - \alpha|} \right).
\]

Therefore, the result is true for \( n = k + 1 \). Using mathematical induction, \( (2.2) \) holds true for all \( n \geq 3 \).

\[\square\]

**Corollary 2.6.** *The parametric value \( \lambda = 0 \) yields Theorem 2.4 in [3] which states that: If \( f \in VD(\alpha, \beta, \delta) \), then*

\[
|a_2| \leq \frac{|b|(1 - \beta)}{|1 - \alpha|}
\]

and

\[
|a_n| \leq \frac{|b|(1 - \beta)(\delta + 1)}{(n - 1)|1 + (n - 1)\lambda|1 - \alpha|B_n(\delta)} \prod_{j=1}^{n-2} \left( 1 + \frac{|b|(1 - \beta)(\delta + 1)}{j|1 - \alpha|} \right), \quad (n \geq 3).
\]

Since \( VD_\lambda(0, \beta, b, \delta) \equiv V_\lambda(\beta, b, \delta) \), we have the following Corollary.

**Corollary 2.7.** *If \( f \in V_\lambda(\beta, b, \delta) \), then*

\[
|a_2| \leq \frac{|b|(1 - \beta)}{(1 + \lambda)}
\]

and

\[
|a_n| \leq \frac{|b|(1 - \beta)(\delta + 1)}{(n - 1)|1 + (n - 1)\lambda|B_n(\delta)} \prod_{j=1}^{n-2} \left( 1 + \frac{|b|(\delta + 1)(1 - \beta)}{j} \right), \quad (n \geq 3).
\]
**Corollary 2.8.** For the parametric values \( b = 2, \delta = 0 \) and \( \lambda = 0 \), we get Theorem 2.3 in [5] which states that:

If \( f \in SD(\alpha, \beta) \), then

\[
|a_2| \leq \frac{2(1 - \beta)}{|1 - \alpha|}
\]

and

\[
|a_n| \leq \frac{2(1 - \beta)}{(n - 1)|1 - \alpha|} \prod_{j=1}^{n-2} \left( 1 + \frac{2(1 - \beta)}{j|1 - \alpha|} \right), \quad (n \geq 3).
\]

**Corollary 2.9.** Putting \( \alpha = 0 \) in Corollary 2.8, we get

\[
|a_n| \leq \frac{\prod_{j=2}^{n} (j - 2\beta)}{(n - 1)!}, \quad (n \geq 2).
\]

a result by Robertson [6].

**Corollary 2.10.** For the parametric values \( b = \delta = 1 \) and \( \lambda = 0 \), we obtain Corollary 2.5 in [5] given by:

If \( f \in KD(\alpha, \beta) \), then

\[
|a_2| \leq \frac{(1 - \beta)}{|1 - \alpha|}
\]

and

\[
|a_n| \leq \frac{2(1 - \beta)}{n(n - 1)|1 - \alpha|} \prod_{j=1}^{n-2} \left( 1 + \frac{2(1 - \beta)}{j|1 - \alpha|} \right), \quad (m \geq 3).
\]

**Corollary 2.11.** Putting \( \alpha = 0 \) in Corollary 2.10, we get the inequality by Robertson [6] given by:

\[
|a_n| \leq \frac{\prod_{j=2}^{n} (j - 2\beta)}{n!}, \quad (n \geq 2).
\]

**References**


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