Notes on Hilbert Lattices

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Abstract

Hilbert algebras started to be studied in the 50’s and they constitute the algebraic counterpart of the implicative fragment of the propositional implicational calculus ([13]). Later, Figallo, Ramón and Saad ([9]), studied distributive Hilbert algebras (or $dH$-algebras), these algebras are Hilbert algebras which are bounded distributive lattices with respect to their natural order.

In this work, we introduced the notion of Hilbert lattices (H-lattice) as $dH$-algebras dropping the distributivity condition and the existence of the zero element 0. We considered pure Hilbert lattices (pH-lattice) which are a special class of H-lattices. These algebras are a particular case of order algebras studied by Chajda ans Halaš [5]. We introduced the concept of ideal for pH-lattice; and we proved that every congruence of an arbitrary pH-lattice is a Rees congruence and that every pH-lattice is a Rees ideal algebra. Besides, we characterized some subdirectly irreducible and simple pH-lattices.

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1 Introduction and Preliminaries

Hilbert algebras have been widely studied since 1950. To see the development and properties of this algebras the reader may consult [7], [13], [2], [6] and [10, 11].

Recall that, according to [7], a Hilbert algebra is an algebra $\langle A, \rightarrow, 1 \rangle$ of type $(2,0)$ that satisfies the following equations:
(H1) \( x \rightarrow x = 1 \),
(H2) \( (x \rightarrow x) \rightarrow x = x \),
(H3) \( x \rightarrow (y \rightarrow z) = (x \rightarrow y) \rightarrow (x \rightarrow z) \),
(H4) \( (x \rightarrow y) \rightarrow ((y \rightarrow x) \rightarrow x) = (y \rightarrow x) \rightarrow ((x \rightarrow y) \rightarrow y) \).

In [7], it was remarked that it is possible to define more than one implication from a poset with last element, that is to say, that we can define more than one Hilbert algebra structure over a poset. Indeed, in the important work by J. Cîrulis ([6]), a method for obtaining all Hilbert algebras associated to a poset is exhibited, thus giving a positive answer to an old problem stated by A. Diego.

In [5], a special class of Hilbert algebras which were called order algebras was studied. It is well-known that for every ordered set with last element \( 1 \), \( \langle A, \leq, 1 \rangle \), the associated Hilbert algebra \( H(A) = \langle A, \rightarrow, 1 \rangle \) can be obtained, where \( \rightarrow \) is a binary operation defined by \( x \rightarrow y \neq 1 \) implies \( x \rightarrow y = y \).

On the other hand, in [1], order algebras under the name of pure Hilbert algebras were considered and the variety generated by these algebras was investigated. Besides, the authors proved that this variety is locally finite, finitely based and that the algebras of this variety have an equationally definable order relation. In [9], there were defined distributive Hilbert algebra (\( dH \)-algebra) as algebra \( \langle A, \rightarrow, \lor, \land, 0, 1 \rangle \) of type \( (2, 2, 2, 0, 0) \) where \( \langle A, \rightarrow, 1 \rangle \) is a Hilbert algebra and \( \langle A, \lor, \land, 0, 1 \rangle \) is a bounded distributive lattice where the following identities are satisfied:

\[
(HL_1) \quad x \land (x \rightarrow y) = x \land y,
(HL_2) \quad (x \rightarrow (y \land z)) \rightarrow ((x \rightarrow z) \land (x \rightarrow y)) = 1,
(HL_3) \quad (x \lor y) \rightarrow z = (x \rightarrow z) \land (y \rightarrow z).
\]

These algebras are Hilbert algebras which are bounded distributive lattices with respect to their natural order, which constitute the algebraic counterpart of a weaker calculus than the intuitionist one.

In a \( dH \)-algebra \( A \) the following properties hold for all \( x, y, z \in A \) ([9]):

1. \( x \leq y \iff x \rightarrow y = 1 \iff x = x \land y \iff y = x \lor y \),
2. \( (x \lor y) \rightarrow z = (x \rightarrow z) \land (y \rightarrow z) \),
3. \( x \rightarrow (y \land z) \leq (x \rightarrow z) \land (x \rightarrow y) \),
4. \( x \rightarrow (y \rightarrow z) \leq (x \land y) \rightarrow z \),
5. \( x \land (x \rightarrow y) = x \land y \).
On the other hand, in [9] the notion of absorbent deductive system was introduced in order to characterize the congruence lattice of $dH$-algebra. More precisely, a subset $D$ of a $dH$-algebra $A$ is an absorbent deductive system if it verifies the following conditions:

(D1) $1 \in D$,
(D2) $x, x \rightarrow y \in D$ imply $y \in D$,
(D3) $x \in D$ implies $z \rightarrow (z \land x) \in D$ for all $z \in A$.

In [12], equational axiomatizations for Heyting algebras and implicative semi-lattices are given respectively. Then, we consider what it was established in [11], we can present a new axiomatization for Heyting algebras consisting of all axioms for $dH$-algebra plus the axiom (HL$A_4$) $x \rightarrow (y \rightarrow (x \land y)) = 1$. It is not difficult to see that linear $dH$-algebra coincide with linear Heyting algebras.

Let us remark that the distributivity condition and the existence of the zero element are not necessary; since they are not used in any proof of the results obtained in [9]. Then, we are interested in abandon those requirements and consider Hilbert algebras which have an order given by an arbitrary lattice to which we are going to call Hilbert lattices (H-lattices). It is worth mentioning that, in other branches of algebraic logic, this same name has been used to design special orthomodular lattices.

## 2 Pure Hilbert lattices

In what follows, a special class of H-lattice is considered which is a particular case of order algebra from [5] and it can be defined in the following way:

**Definition 1** let $A$ be a H-lattice is an algebra $\langle A, \land, \lor, \rightarrow, 1 \rangle$ of type $(2,2,2,0)$ where $\langle A, \rightarrow, 1 \rangle$ and $\langle A, \land, \lor \rangle$ an lattices with $1$ verifies:

(HL$A_1$) $x \land (x \rightarrow y) = x \land y$,
(HL$A_2$) $(x \rightarrow (y \land z)) \rightarrow ((x \rightarrow z) \land (x \rightarrow y)) = 1$,
(HL$A_3$) $(x \lor y) \rightarrow z = (x \rightarrow z) \land (y \rightarrow z)$.

and a pure H-lattice (pH-lattice) is a H-lattice that satisfies the following axiom:

(pHL$A$) $x \rightarrow y \neq 1$ implies $x \rightarrow y = y$.

The class of pH-lattices do not constitute a quasivariety because it is not closed under direct products, as it happens with order algebras ([5]). In order to study the Rees congruence ([3]) of a pH-lattice $A$, we will consider certain sets that we named nodes of $A$, which will be useful for the development of this work.
Definition 2 Let $A$ be a pH-lattice. A nonempty subset $N$ of $A$ is a node if:

(N1) $n_1, n_2 \in N$ implies $n_1 \land n_2 \in N$ and (N2) $x \in A \setminus N$ implies $x \rightarrow n = 1$ for all $n \in N$.

Then, we can assert that a node is a filter, but the converse does not hold.

The next proposition establishes the relation between nodes and absorbent deductive systems.

Proposition 1 Let $A$ be a pH-lattice and $N \subseteq A$. Then, $N$ is a node if and only if $N$ is an absorbent deductive system.

Proof. Let $N$ be a node. It is clear that $1 \in N$. Let $x, x \rightarrow y \in N$, by (N1) $x \land (x \rightarrow y) \in N$. Then, by $(HL_1)$ we have that $x \land y \in N$, from what we have that $y \in N$. Let us suppose that $x \in A$ and $n \in N$, then we can have the following cases: (i) $x \in N$ and (ii) $x \notin N$, from (i) it results that $x \land n \in N$, and therefore $x \rightarrow (x \land n) \in N$. On the other hand, from (ii) and (N2) we have $x \rightarrow (x \land n) = 1 \in N$.

Conversely, if $N$ is an absorbent deductive system, (N1) holds. Let $x \in A \setminus N$ and $n \in N$, then two possible cases arise: (iii) $x \rightarrow (x \land n) = 1$ or (iv) $x \rightarrow (x \land n) = x \land n$. From (iii), it holds $x = x \land n$ and therefore $x \rightarrow n = 1$. If it is the case of (iv), we get a contradiction. □

It is worth mentioning that the above proposition is not true for H-lattices in general. Indeed,

Remark 1 Let us consider the algebra H-lattice $A$ which is described as follows:

\[
\begin{array}{c|ccc}
   & 0 & a & b \\
\hline
0 & 1 & 1 & 1 \\
a & b & 1 & 1 \\
b & a & 1 & 1 \\
1 & 0 & a & b \\
\end{array}
\]

It is clear that $F = \{b, 1\}$ is a filter which it is not a node.

Let $\mathcal{N}(A)$ be the set of all nodes of the pH-lattices $A$. Then,

Proposition 2 $\mathcal{N}(A)$ is a chain.

Proof. Let $N_1$ and $N_2$ be two nodes of $A$ and suppose that $N_1 \not\subseteq N_2$. Then, there exists $x$ such that $x \in N_1$ and $x \notin N_2$. Then, $x \rightarrow n = 1 \in N_1$ for all $n \in N_2$, from where we have that $n \in N_1$ for all $n \in N_2$. Therefore, $N_2 \subseteq N_1$. □

The lattice of all congruences of a given pH-lattices $A$ may be described as follows.
Theorem 1 Let $A$ be a pH-lattice Then,

(i) $\text{Con}(A) = \{ R(N) : N \in \mathcal{N}(A) \}$, where $R(N) = \{(x, y) \in A \times A : x, y \in N \text{ or } y = x \}$.

(ii) $\text{Con}(A)$ and $\mathcal{N}(A)$ are isomorphic lattices by the maps $\Theta \mapsto |1|_\Theta$ and $N \mapsto R(N)$.

Proof. In [9], the authors proved that the lattice of all congruences on a H-lattice $A$ is $\text{Con}(A) = \{ R(D) : D \in \mathcal{D}_a(A) \}$, where $R(D) = \{(x, y) \in A^2 : x \rightarrow y, y \rightarrow x \in D \}$ and that the set $\mathcal{D}_a(A)$ of all absorbent deductive systems of $A$ is isomorphic to $\text{Con}(A)$ by means of the correspondences $\Theta \mapsto [1]_\Theta$ and $D \mapsto R(D)$ where $[1]_\Theta$ stands for the equivalence class of 1 modulo $\Theta$. Then, by Proposition 1 we have $R(D) = \{(x, y) \in A^2 : x \rightarrow y, y \rightarrow x \in D, D \in \mathcal{D}_a(A) \} = \{(x, y) \in A^2 : x \rightarrow y, y \rightarrow x \in N, N \in \mathcal{N}(A) \} = \{(x, y) \in A \times A : x, y \in N \text{ or } y = x, N \in \mathcal{N}(A) \}$ from the axiom $(pHL)$. □

Next, let $A$ be a H-lattice, we shall show conditions for the notions of principal filters and absorbent deductive systems to coincide. Then,

Lemma 1 Let $A$ a H-lattice, $D \in \mathcal{D}_a(A)$, $D = [a]$ be a principal filter of $A$ and $S = \{ z \in A : z \rightarrow (x \rightarrow (x \land z)) = 1, \text{ for all } x \in A \}$. Then, $[a] \in \mathcal{D}_a(A)$ if only if $a \in S$.

Proof. Let $D = [a] \in \mathcal{D}_a(A)$. Then, by (D3) $x \rightarrow (x \land a) \in D$, for every $x \in A$, and $a \leq x \rightarrow (x \land a)$. Therefore, $1 = a \rightarrow (x \rightarrow (x \land a))$.

Conversely, if $a \in S$, $D = [a]$, then $a \rightarrow (x \rightarrow (x \land a)) = 1$ for all $x \in A$, its clear that $x \rightarrow (x \land a) \in D$, for every $x \in A$. Let $y \in D$, such that $a \leq y$, then $x \land a \leq x \land y$, by Hilbert algebras properties we have $x \rightarrow (x \land a) \leq x \rightarrow (x \land y)$. Therefore, $x \rightarrow (x \land y) \in D$, for every $x \in A$, then $D \in \mathcal{D}_a(A)$. □

In a pH-lattice $A$, we may consider node generated by the nonempty set $X$ as the intersection of all nodes of $A$ that contain $X$ and we will denote it with $N(X)$. If $X = \{a\}$ then we shall write $N(a)$ instead of $N(\{a\})$.

Let us remark that if $A$ is a denumerable H-lattice and $S = \{ z \in A : z \rightarrow (x \rightarrow (x \land z)) = 1, \text{ for all } x \in A \}$, then, if $a, b \in S$ it holds that $a \lor b \in S$, for all $w \in A$ there exists the greatest element of $\{ z \in S : z \leq w \}$.

Proposition 3 Let $A$ be a denumerable pH-lattice that all its filters are principal filters. Then, $N(a) = [d]$ where $d$ is the last element of the set $\{ z \in S : z \leq a \}$, $[d]$ is the principal filter generated by $\{d\}$, and $a \in A$.

Proof. Since $d \leq a$, then $a \in [d]$. On the other hand, from $d \in S$, and lemma 1, we conclude that $F(d)$ is a node of $A$. Let $N$ be a node of $A$ such that (1) $a \in N$. Then there exists $c \in A$ such that (2) $N = [c]$. From (1) and (2) we conclude that $c \leq a$ and therefore $c \in S$. Then, $c \leq d$ and $F(d) \subseteq N$. □
Remark 2 It can be stated an analogous result for H-lattices. More precisely, let $A$ be a denumerable H-lattice that all its filters are principal filters. If $a \in A$ then the absorbent deductive system generated by $a$ is $[d]$ where $d$ is the last element of the set $\{z \in S : z \leq a\}$.

3 Rees congruence and Rees ideal algebra

Next, we are going to exhibit some properties of the congruences.

Theorem 2 Let $A$ be a nontrivial pH-lattice. Then $A$ is congruence distributive, congruence permutable and weakly regular.

Proof. Distributivity is consequence of Theorem 1 and Proposition 2. Commutativity can be obtained showing that $\Theta_I \circ \Theta_J = \Theta_{I \cup J}$ for all $I, J \in \mathcal{N}(A)$. Indeed, let $(x, y) \in \Theta_I \circ \Theta_J$ then, there exists $z \in A$ such that $(x, z) \in \Theta_J$ and $(z, y) \in \Theta_I$. Then, (1) $x \rightarrow z \in J$, (2) $z \rightarrow y \in I$. On the other hand, from Proposition 2 we have that $I \cup J = I$ or $I \cup J = J$. Let us suppose that $I \cup J = J$, since $(x \rightarrow (z \rightarrow y)) \rightarrow ((x \rightarrow z) \rightarrow (x \rightarrow y)) \in J$, from (1) and (2) it results that $x \rightarrow y \in J$. Analogously, we prove that $y \rightarrow x \in J$, then $(x, y) \in \Theta_{I \cup J}$ and therefore, $\Theta_I \circ \Theta_J \subseteq \Theta_{I \cup J}$. The other inclusion is immediate. Then, we conclude that $\Theta_I \circ \Theta_J = \Theta_J \circ \Theta_I$. This last assertion implies that $A$ is congruence permutable.

Finally, from Theorem 1 we deduce that $A$ is weakly regular congruence.

Let us remark that in order algebras congruences are not permutable, in general. Indeed, consider the algebra of Remark 1, and the deductive systems $I = \{a, 1\}$ and $J = \{b, 1\}$. Then, it holds that $(b, a) \in \Theta_I \circ \Theta_J$ but $(b, a) \notin \Theta_J \circ \Theta_I$.

In [4], the concept of ideal in Hilbert algebras was studied and was defined as follow:

Definition 3 A non-void subset $I$ of a Hilbert algebras $(A, \rightarrow, 1)$ is called an ideal of $A$ if

1. $x, y \in A$, then $x \rightarrow y \in I$,
2. $x, y_1, y_2 \in A$, then $(y_2 \rightarrow (y_1 \rightarrow x)) \rightarrow x \in I$.

It is immediately clear that $1 \in I$ for every ideal $I$ of $A$. It was shown in [5] that the notions of order filter and ideal coincide. As expected, it was proved that ideals and deductive systems coincide (see [8]). Then, we may consider the notion of absorbent ideals adding the following condition:

3. $z \in I$, then $x \rightarrow (x \land z) \in I$. 

In order to study Rees ideal algebras, it is worth mentioning that absorbent ideals in pH-lattices satisfy conditions established in [3].

The concept of Rees congruence was introduced for semigroups by D. Rees and generalized by R. Tichy for universal algebra. In [5], definitions in [3] were adapted for order algebras and, taking into account that pH-lattices are a particular case of order algebras, we define:

Let $A = (A, F)$ be an algebra.

(i) We shall say if $\Theta \in Con(A)$ is a Rees congruence, if it has at most one class with more that one element.

(ii) If $A$ has a constant, denoted for instance by 1, with respect to which the concept of ideal is defined, we shall say that $A$ is a Rees ideal algebra if for all $\Theta \in Con(A)$

(a) $\{1\}_\Theta$ is an ideal and the only class that can have more that one element,

(b) for all ideals $I$ of $A$ we have that $(I \times I) \cup \omega_A \in Con(A)$, where $\omega_A$ is the identity relation on $A$.

**Theorem 3** Let $A$ be an pH-lattice and $\Theta \in Con(A)$. Then, $\Theta$ is a Rees congruence and $A$ is a Rees ideal algebra.

**Proof.** In an analogous way to that indicated in [5, Theorem 3] we can assert that $\Theta$ is a Rees congruence. On the other hand, from Proposition 1 and Theorem 1, since every absorbent ideal $I$ is an ideal in the sense of [5], we have that $(I \times I) \cup \omega_A \in Con(A)$ for all ideals $I$ of $A$.

In [9], it was proved that any $dH$-algebra with one co-atom is a subdirectly irreducible algebra. In order to study these algebras we are going to exhibit an important class of subdirectly irreducible H-lattice.

**Definition 4** Let $A$ be an pH-lattice. A node $N \neq A$ of $A$ is maximal if for any $M \in \mathcal{N}(A)$ with $N \subseteq M$, either $N = M$ or $M = A$.

**Theorem 4** Let $A$ be a non trivial pH-lattice. Then, $A$ is subdirectly irreducible if and only if $A$ has a maximal node.

**Proof.** It is consequence of Theorem 1.

From this last theorem and Proposition 2, we have that all denumerable pH-lattice $A$, has a denumerable family of nodes $\mathcal{N}(A)$ which is a chain. Then, it
has a penultimate element and, therefore, $A$ is subdirectly irreducible algebra. If we consider the pH-lattice $[0,1]$ where $[0,1]$ the real interval, it is easy to check that $N([0,1])$ do not have a maximal node and it is not a subdirectly irreducible algebra.

**Theorem 5** Let $A$ be an $H$-lattice for which every absorbent deductive system is a principal filter. The conditions that follow are equivalent:

1. $A$ is subdirectly irreducible,
2. $S = \{ z \in A : z \rightarrow (x \rightarrow (x \land z)) = 1, \text{for every } x \in A \} = \{ z \in A : z \rightarrow (x \land z) = z \rightarrow x, \text{for every } x \in A \}$, has a penultimate element.

**Proof.** By lemma 1, we have that all principal filters of $A$ generated by $a$ are elements of $S$. \qed

**Theorem 6** Let $A$ be an $H$-lattice for which every absorbent deductive system is a principal filter. The following conditions are equivalent:

1. $A$ is simple,
2. $S$ have two elements.

**Proof.** It is consequence of theorem 5. \qed

Then, we characterize important classes of subdirectly irreducible and simple $H$-lattices.

**References**


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