Common Fixed Point Theorem in Probabilistic Metric Spaces

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Abstract

In this paper we prove a common fixed point theorem for four mappings under weakly condition in probabilistic metric space. We point out that the continuity for the existence of fixed point is not required. Here we improve an earlier result.

Keywords: Probabilistic Metric spaces, weak commuting mapping, compatible mappings, common fixed point

1 Introduction

In 1942, Menger [5] was first who thought about distance distribution function in metric space and introduced the concept of probabilistic metric space. He replaced distance function $d(x, y)$, the distance between two point $x, y$ by distance distribution function $F_{x,y}(t)$ where the value of $F_{x,y}(t)$ is interpreted as probability that the distance between $x, y$ is less than $t$, $t > 0$. The study of fixed point theorem in probabilistic metrics space is useful in the study of existence of solution of operator equation in probabilistic metric space probabilistic functional analysis.

PM space has a nice topological properties. Many different topological structures may be defined on a PM space. The one that has received the most attention to date is the strong topology and it is the principle tool of this study. The convergence with respect to this topology is called strong
Schweizer and Sklar [1], developed the study of fixed point theory in probabilistic metric spaces. In 1966, Sehgal [12] initiated the study of contraction mapping theorem in probabilistic metric spaces. Several interesting and elegant result have been proved by various author in probabilistic metric spaces.

In 2005, Mihet [2] proved a fixed point theorem concerning probabilistic contractions satisfying an implicit relation. The purpose of the present paper is to prove a common fixed point theorem for six mappings via pointwise R-weakly commuting mappings in probabilistic metric spaces satisfying contractive type implicit relations. This generalizes several known results in the literature including those of Kumar and Pant [8], Kumar and Chugh [7] and others.

2 Preliminaries

Definition 2.1 A distribution function is a non-decreasing function \( F : R \to R^+ \) that satisfy \( \inf F(t) : t \in R = 0 \) and \( \sup F(t) : t \in R = 1 \) and is continuous.

\( \Delta_+ \) is the set of all distribution function defined on \([\infty, \infty]\). For \( a \geq 0 \), \( H_a \) is an element of \( \Delta_+ \) defined by

\[
H_a(t) = \begin{cases} 
0 & \text{if } t \leq a \\
1 & \text{if } t > a
\end{cases}
\]

If \( X \) is a non-empty set, a mapping \( F : X \times X \to \Delta_+ \) is called a probabilistic distance on \( X \) and the value of \( F \) at \( (x, y) \in X \times X \) is denoted by \( F_{x,y} \).

Definition 2.2 If \( F \) is a probabilistic distance on \( X \), the pair \( (X, F) \) is called probabilistic Metric space (briefly PM space) if the following condition are satisfied:

(PM1) \( F_{x,y}(t) = 1 \) iff \( x = y \),

(PM2) \( F_{x,y}(0) = 0 \),

(PM3) \( F_{x,y}(t) = F_{y,x}(t) \),

(PM4) If \( F_{x,y}(t) = 1 \) and \( F_{y,z}(S) = 1 \), then \( F_{x,z}(t + s) = 1 \) for all \( x, y, z \in X \) and \( s, t \geq 0 \).

every metric space can always realized as a PM space by considering \( F : X \times X \to \Delta_+ \) defined by \( F_{x,y}(t) = H(t - d(x, y)) \) for all \( x, y \in X \).

Definition 2.3 A triangle function is a binary operation \( \tau \) on \( \Delta_+ \) \((\tau : \Delta_+ \times \Delta_+ \to \Delta_+)\) which is commutative, associative non-decreasing at each place, and has \( H_0 \) as identity.
Definition 2.4 A triangle norm (briefly t-norm) is a binary operation $\Delta$ on $[0, 1]$ which is commutative, associative non-decreasing with $\Delta(a, 1) = a$ for all $a \in [0, 1]$. There are four basic t-norm as follows:

(i) The minimum t-norm, $\Delta_M$ is defined by $\Delta_M(x, y) = \min(x, y)$.

(ii) The product t-norm, $\Delta_p$ is defined by $\Delta_p(x, y) = x \cdot y$.

(iii) The Lukasiecz t-norm, $\Delta_L$ is defined by $\Delta_L(x, y) = \max(x + y - 1, 0)$

(iv) The weakest t-norm, the drastic product $\Delta_D$, is defined by

$$\Delta_D(x, y) = \begin{cases} 
\min(x, y) & \text{if } \max(x, y) = 1, \\
0 & \text{otherwise}
\end{cases}$$

Definition 2.5 A mengaer space is a tripled $(X, F, \Delta)$ where $(X, F)$ is a PM space and $\Delta$ is a t-norm such that the inequality $F_{x,z}(t+s) \geq \Delta(F_{x,y}(t), F_{y,z}(s))$ holds for all $x, y, z \in X$ and $t, s \geq 0$.

Definition 2.6 A sequence $x_n$ in a mengaer space $(X, F, \Delta)$ is said to be converges to a pont $x$ in $X$ iff for each $\epsilon > 0$ and $t \in (0, 1)$, there exist is an integer $M(\epsilon) \in \mathbb{N}$ such that $F_{x_n,x}(\epsilon) > 1 - t$ for all $n \geq M(\epsilon)$.

Definition 2.7 The sequence $\{x_n\}$ is said to be Cauchy sequence if for each $\epsilon > 0$ and $t \in (0, 1)$, there exist is an integer $M(\epsilon) \in \mathbb{N}$ such that $F_{x_n,x_m}(\epsilon) > 1 - t$ for all $n, m \geq M(\epsilon)$.

Definition 2.8 A mengaer space $(X, F, \Delta)$ is said to be complete, if every Cauchy sequence in $X$ converges to a point $x$ in $X$.

Definition 2.9 Two self mapping $f$ and $g$ of a mengaer space $(X, F, \Delta)$ are said to be weakly commuting if $F_{fgx,gfx}(t) \geq F_{fx,gx}(t)$ for each $x$ in $X$ and $t > 0$.

Definition 2.10 Two self mapping $f$ and $g$ of a mengaer space $(X, F, \Delta)$ are said to be pointwise $R$-weakly commuting if $F_{fgx,gfx}(t) \geq F_{fx,gx}(t/R)$ for each $x$ in $X$ and $t > 0$.

Definition 2.11 Two self mapping $f$ and $g$ of a mengaer space $(X, F, \Delta)$ are said to be compatible iff $F_{fgx_n,gfx_n}(t) \rightarrow 1$for all $t > 0$, whenever $\{x_n\}$ is a sequence in $X$ such that $fx_n, gx_n \rightarrow z \in X$ for some $z \in X$. 

Definition 2.12 Self mapping $f$ and $g$ of a menger space $(X, F, \Delta)$ are said to be weakly compatible (or coincidentally commuting) if they commute at their coincidence points, that is, if $fx = gx$ for some $x \in X$, then $fgx = gfx$.

Remark 2.1 If self mapping $f$ and $g$ of a menger space $(X, F, \Delta)$ are compatible then they are weakly compatible but the converse is not true as shown in the following example.

Example 2.1 Let $(X, d)$ be a metric space defined by $d(x, y) = |x - y|$ where $X = [0, 6]$ and $(X, F, \Delta)$ be the induced Menger space with

$$f_{x,y}(t) = f(x) = \begin{cases} \frac{t}{t+|x-y|} & \text{if } t > 0 \\ 0 & \text{if } t = 0 \end{cases}$$

Then $(X, T)$ is a probabilistic metric space. Let $A$ and $B$ are self mapping of $X$ defined as

$$Ax = \begin{cases} 6 - x & \text{if } 0 \leq x < 3 \\ 6 & \text{if } 3 \leq x \leq 6 \end{cases}$$

$$Bx = \begin{cases} x & \text{if } 0 \leq x < 3 \\ 6 & \text{if } 3 \leq x \leq 6 \end{cases}$$

Taking $x_1 = 3 - 1/n$. We get $Ax_1 = 3 + 1/n$, $Bx_1 = 3 - 1/n$. Thus, $Ax_n \to 3, Ax \to 3$. Hence $x = 3$. Further $ABx_1 = 3 + 1/n, BAx_1 = 6$. Now, $\lim_{n \to \infty} F_{ABx_1, BAx_1}(t) = \lim_{n \to \infty} F_{3+1/n,6}(t) = \frac{t}{t+3} < 1$, for all $t > 0$. Hence $(A, B)$ is not compatible.

Definition 2.13 Two self mapping $f$ and $g$ of a menger space $(X, F, \Delta)$ are said to be reciprocally continuous if $fgx_n \to z$ and $gfx_n \to z$, whenever $\{x_n\}$ is a sequence in $X$ such that $fx_n, gx_n \to z$ for some $z \in X$.

If $f$ and $g$ are both continuous, then they are obviously reciprocally continuous but converse is not true.

Lemma 2.1 Let $\{x_n\}$ is a Cauchy sequence in a menger space $(X, F, \Delta)$. If there exist a constant $k \in (0, 1)$ such that $F_{x_n, x_{n+1}}(kt) \geq F_{x_{n-1}, x_n}(t), n = 1, 2, 3, \ldots$, then $\{x_n\}$ is a Cauchy sequence in $X$.

3 Common Fixed Point Theorem

Theorem 3.1 Let $A, B, P$ and $Q$ are self maps on a complete probabilistic metric space $(X, F)$ satisfying:

(a) $P(X) \subset B(X), Q(X) \subset A(X)$;
(b) \( F_{Pw,Qx}(kt) \geq \max \{ F_{Ax,By}(t), \frac{F_{Pw,Ax}(t) + F_{Qw,Bx}(kt)}{2} \} \); for all \( x, y \in X, t > 0, k \in (0, 1) \)

(c) If one of \( P(X), B(X), Q(X), A(X) \) is complete subset of \( X \) then

(i) \( P \) and \( A \) have a coincidence point and

(ii) \( Q \) and \( B \) have a coincidence point and

if the pair \( (P, A) \) and \( (Q, B) \) are weakly compatible, then \( A, B, P \) and \( Q \) have a unique common fixed point in \( X \).

Proof: Since \( P(X) \subset B(X) \) and \( Q(X) \subset A(X) \) so we can define sequences \( (x_n) \) and \( (y_n) \) in \( X \) such that for all \( n = 0, 1, 2, 3, ... \)

\[ y_{2n+1} = P x_{2n} = B x_{2n+1}, y_{2n+2} = Q x_{2n+1} = A x_{2n+2} \]

Now we have,

\[ F_{P2n,Q2n+1}(kt) \geq \max \{ F_{Ax,By}(t), \frac{F_{P2n,Ax}(t) + F_{Q2n,Bx}(kt)}{2} \} \]

\[ F_{y_{2n+1},y_{2n+2}}(kt) \geq \max \{ F_{y_{2n+1},y_{2n+1}}(t), \frac{F_{y_{2n+1},y_{2n+1}}(t) + F_{y_{2n+1},y_{2n+1}}(kt)}{2} \} \]

\[ F_{y_{2n+1},y_{2n+2}}(kt) \geq F_{y_{2n+1},y_{2n+1}}(t) \]

Similarly,

\[ F_{y_{2n+1},y_{2n+2}}(kt) \geq F_{y_{2n+1},y_{2n+2}}(t) \]

In general for any \( n \) and \( t \), we have \( F_{y_{n+1},y_{n+1}}(kt) \geq F_{y_{n-1},y_{n}}(t) \). Hence \( Y_n \) is a Cauchy sequence in \( X \). By completeness, \( y_n \to z \in X \). Thus the subsequence \( y_{2n}, y_{2n+1} \) and \( y_{2n+2} \) also converges to \( z \). Therefore \( B x_{2n+1}, P x_{2n}, Q x_{2n+1} \) and \( A x_{2n} \) also converges to \( z \). Now suppose \( A(X) \) is complete. Note that the subsequence \( y_{2n+2} \) contained in \( A(X) \) and has a limit in \( A(X) \) say \( z \). Let \( w \in (A)^1 - 1(z) \). Then \( Aw = z \).

Now consider

\[ F_{Pw,Qx_{2n+1}}(kt) \geq \max \{ F_{Aw,Bx_{2n+1}}(t), \frac{F_{Pw,Aw}(t) + F_{Qw,Bw}(kt)}{2} \} \]

\[ F_{Pw,y_{2n+2}}(kt) \geq \max \{ F_{Aw,y_{2n+1}}(t), \frac{F_{Pw,Aw}(t) + F_{Qw,Bw}(kt)}{2} \} \]

Taking limit \( n \to \infty \), we have
\[ F_{Pw,z}(kt) \geq \max\{F_{z,z}(t), \frac{F_{Pw,z}(t) + F_{Qw,Bw}(kt)}{2}\} \]

\[ F_{Pw,z}(kt) \geq F_{z,z}(t) = 1 \Rightarrow F_{Pw,z}(kt) = 1 \Rightarrow Pw = z. \text{ Since } Aw = z \text{ so } w \text{ is a coincidence point of } P \text{ and } A. \]

Since \( P(X) \subset B(X) \) ans \( Pw = z \) implies that \( z \in B(X) \). Let \( v \in (B)^{-1}z \), then \( Bv = z \).

now consider

\[ F_{P_{x2n+1}Qv}(kt) \geq \max\{F_{A_{x2n}Bv}(t), \frac{F_{P_{x2n}A_{x2n}}(t) + F_{Q_{x2n}B_{x2n}}(kt)}{2}\} \]

\[ F_{y2n+1,Qv}(kt) \geq \max\{F_{y2n,Bv}(t), \frac{F_{y2n+1,y2n}(t) + F_{y2n+1,y2n}(kt)}{2}\} \]

Taking limit \( n \to \infty \), we have

\[ F_{z,Qv}(kt) \geq \max\{F_{z,z}(t), \frac{F_{z,z}(t) + F_{z,z}(kt)}{2}\} \]

\[ F_{z,Qv}(kt) \geq F_{z,z}(t) = 1 \Rightarrow F_{z,Qv}(kt) = 1 \Rightarrow Qv = z \]

Since \( Bv = z \) so \( v \) is a coincidence point of \( Q \) and \( B \).

Since the pair \( (P,A) \) is weakly compatible therefore \( P \) and \( A \) commute at their coincidence point that is \( PAw = APw \) or \( Pz = Az \) and the pair \( (Q,B) \) is weakly compatible therefore \( Q \) and \( B \) commute at their coincidence point that is \( QBv = BQv \) or \( Qz = Bz \).

Now we will prove that \( Pz = z \). By (b), we have

\[ F_{Pz,Q_{x2n+1}}(kt) \geq \max\{F_{A_{z},B_{x2n+1}}(t), \frac{F_{Pz,Az}(t) + F_{Qz,Bz}(kt)}{2}\} \]

\[ F_{Pz,y2n+2}(kt) \geq \max\{F_{Az,y2n+1}(t), \frac{F_{Pz,Az}(t) + F_{Qz,Bz}(kt)}{2}\} \]

Taking limit \( n \to \infty \), we have

\[ F_{Pz,z}(kt) \geq \max\{F_{Az,z}(t), \frac{F_{Pz,Az}(t) + F_{Qz,Bz}(kt)}{2}\} \]
Common fixed point theorem

\[ F_{P,z}(kt) \geq \max\{F_{A,z}(t), 1\} \text{ since } Az = Pz \text{ and } Qz = Bz \]
\[ F_{P,z}(kt) = 1 \text{ then } Pz = z. \]

Similarly we will prove that \( Qz = z \). By (b), we have
\[ F_{P_{x_{2n}},Qz}(kt) \geq \max\{F_{A_{x_{2n}},Bz}(t), \frac{F_{P_{x_{2n}},A_{x_{2n}}}(t) + F_{Q_{x_{2n}},B_{x_{2n}}}(kt)}{2} \} \]
\[ F_{y_{2n+1},Qz}(kt) \geq \max\{F_{y_{2n},Bz}(t), \frac{F_{y_{2n+1},y_{2n}}(t) + F_{y_{2n+1},y_{2n}}(kt)}{2} \} \]

Taking limit \( n \to \infty \), we have
\[ F_{z,Qz}(kt) \geq \max\{F_{z,Bz}(t), \frac{F_{z,z}(t) + F_{z,z}(kt)}{2} \} \]
\[ F_{z,Qz}(kt) \geq \max\{F_{z,Bz}(t), 1\} \]
\[ F_{z,Qz}(kt) = 1 \text{ then } Qz = z. \]

Hence \( z \) is a common fixed point of \( A, B, P \) and \( Q \).

**Uniqueness** Let \( w \) is an another common fixed point of \( A, B, P \) and \( Q \). then we have
\[ F_{Pw,Qz}(kt) \geq \max\{F_{Aw,Bw}(t), \frac{F_{Pw,Aw}(t) + F_{Qw,Bw}(kt)}{2} \} \]
\[ F_{w,z}(kt) \geq \max\{F_{w,z}(t), \frac{F_{w,w}(t) + F_{w,w}(kt)}{2} \} \]

\[ F_{w,z}(kt) \geq max F_{w,z}(t), 1 \Rightarrow F_{w,z}(kt) = 1 \Rightarrow w = z \]

Hence \( z \) is unique common fixwd point of \( A, B, P \) and \( Q \).

**Corollary 3.2** (3, Theorem 3.1)

Let \( A, B \) and \( P \) are self maps on a complete probabilistic metric space \((X, F)\) satisfying:

**d** \( P(X) \subset B(X), P(X) \subset A(X); \)

**e** \( F_{P_x,P_y}(kt) \geq \max\{F_{A_x,B_y}(t), \frac{F_{P_x,A_x}(t) + F_{P_y,B_y}(kt)}{2} \}; \) for all \( x, y \in X, t > 0, k \in (0, 1) \)

**f** If one of \( P(X), B(X), A(X) \) is compalete subset of \( X \) then
(iii) $P$ and $A$ have a coincidence point and
(iv) $P$ and $B$ have a coincidence point and

if the pair $(P, A)$ and $(P, B)$ are weakly compatible, then $A, B$ and $P$ have a unique common fixed point in $X$.

**Proof:** Substitute $P = Q$, in the theorem 3.1.

**Corollary 3.3 (3, Theorem 3.1)**

Let $P$ and $Q$ are self maps on a complete probabilistic metric space $(X, F)$ satisfying

(a) $F_{P x, Q y}(kt) \geq \max \{F_{x, y}(t), \frac{F_{P x, x}(t)+F_{Q y, y}(kt)}{2}\}$; for all $x, y \in X, t > 0, k \in (0, 1)$

if one of $P(X), Q(X)$ is a complete subset of $X$, then $P$ and $Q$ have a unique common fixed point in $X$.

**Proof:** Substitute $A = B = I$, the identity mapping, in the theorem 3.1.

**Corollary 3.4 (3, Corollary 3.3)**

Let $P$ be a self maps on a complete probabilistic metric space $(X, F)$ satisfying

(a) $F_{P x, y y}(kt) \geq \max \{F_{x, y}(t), \frac{F_{P x, x}(t)+F_{P y, y}(kt)}{2}\}$; for all $x, y \in X, t > 0, k \in (0, 1)$

if $P(X)$ is a complete subset of $X$, then $P$ has a unique common fixed point in $X$.

**Proof:** Substitute $P = Q$, in the corollary 3.3.

**References**


Common fixed point theorem


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