Fuzzy Ideals of KU - Algebras

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Abstract

In this paper, we consider KU - ideals of KU - algebras and some fundamental properties to KU - algebra are discussed. The notion of fuzzy KU - ideals in KU - algebras are introduced, several appropriate examples are provided and their some properties are investigated. The image and the inverse image of fuzzy KU - ideals in KU - algebras are defined and how the image and the inverse image of fuzzy KU - ideals in KU - algebras become fuzzy KU - ideals are studied. Moreover, the cartesian product of fuzzy KU - ideals in cartesian product KU – algebras are given.

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Keywords: KU - algebra, homomorphisms of KU - algebra, fuzzy KU - subalgebra, KU - ideals, fuzzy KU - ideal, image and preimage of fuzzy KU - ideals
1. Introduction

BCK-algebras form an important class of logical algebras introduced by K.Iseki and were extensively investigated by several researchers. The class of all BCK-algebras is quasi variety. K.Iseki posed an interesting problem (solved in [21]) whether the class of all BCK-algebras is a variety. In connection with this problem, Y.Komori introduced in [14] a notion of BCC algebras and W.A.Dudek (cf.[2],[3]) redefined the notion of BCC-algebras by using a dual form of the ordinary definition in the sense of Y. Komori.

In [18], C.Prabpayak and U.Leerawat studied ideals and congruences of BCC-algebras ([8], [9]) and introduced a new algebraic structure which is called KU-algebra. They gave the concept of homomorphisms of KU-algebras and investigated some related properties.

L.A. Zadeh [23] introduced the notion of fuzzy sets. At present this concept has been applied to many mathematical branches, such as group, functional analysis, probability theory, topology, and so on. In 1991, O. G. Xi [22] applied this concept to BCK-algebras, and he introduced the notion of fuzzy sub-algebras (ideals) of the BCK-algebras with respect to minimum, and since then Y.B. Jun et al studied fuzzy ideals (cf.[10], [11], [12], [13], [17]), and moreover several fuzzy structures in BCC-algebras are considered (cf.[5], [6], [8], [9]).

In this paper, we introduce the notion of fuzzy KU-ideals of KU-algebras and then we investigate several basic properties which are related to fuzzy KU-ideals. We describe how to deal with the homomorphic image and inverse image of fuzzy KU-ideals. We have also proved that the cartesian product of fuzzy KU-ideals in cartesian product of fuzzy KU-algebras are fuzzy KU-ideals.

2. Preliminaries

By an KU-algebra we mean an algebra \((X, *, 0)\) of type \((2, 0)\) with a single binary operation \(*\) that satisfies the following identities: for any \(x, y, z \in X\),
\[
\begin{align*}
(ku_1) : & \quad (x * y) * [(y * z) * (x * z)] = 0, \\
(ku_2) : & \quad x * 0 = 0, \\
(ku_3) : & \quad 0 * x = x, \\
(ku_4) : & \quad x * y = 0 = y * x \text{ implies } x = y.
\end{align*}
\]

In what follows, let \((X, *, 0)\) denote an KU-algebra unless otherwise specified. For brevity we also call \(X\) a KU-algebra. In \(X\) we can define a binary relation \(\leq\) by:
\[
x \leq y \text{ if and only if } y * x = 0.
\]

Then \((X, *, 0)\) is a KU-algebra if and only if it satisfies that:
\[
\begin{align*}
k'u_1 : & \quad (y * z) * (x * z) \leq (x * y), \\
k'u_2 : & \quad 0 \leq x.
\end{align*}
\]
Fuzzy ideals of KU-algebras

(k'u3) : \( x \leq y \), \( y \leq x \) implies \( x = y \),
(k'u4) : \( x \leq y \) if and only if \( y \ast x = 0 \).

In an KU-algebra, the following identities are true: If we put in (k'u1) \( y = x = 0 \) we get \((0 \ast 0) \ast [ (0 \ast z) \ast (0 \ast z) ] = 0 \), and it follows that : (KU3) \( z \ast z = 0 \), and if we put \( y = 0 \) in (k'u1), we get (p1) \( z \ast (x \ast z) = 0 \).

Example 2.1. Let \( X = \{0, 1, 2, 3, 4\} \) in which \( \ast \) is defined by the following table:

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It is easy to show that \( X \) is KU-algebra.

Definition 2.2 [19]. A subset \( S \) of KU-algebra \( X \) is called sub-algebra of \( X \) if \( x \ast y \in S \), whenever \( x, y \in S \).

Definition 2.3 [19]. A non-empty subset \( A \) of a KU-algebra \( X \) is called an KU-ideal of \( X \) if it satisfies the following conditions:

1) \( 0 \in A \),
2) \( x \ast (y \ast z) \in A \), \( y \in A \) implies \( x \ast z \in A \), for all \( x, y, z \in X \).

Example 2.4. Let \( X = \{0, a, b, c, d, e\} \) in which \( \ast \) is defined by the following table:

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Then \( (X, \ast, 0) \) is KU-algebra. It is easy to show that \( A_1 = \{0, a\} \), and \( A_2 = \{0, a, b, c, d\} \) are KU-ideals of \( X \).

Lemma 2.5. In a KU-algebra \((X, \ast, 0)\), the following hold:

\( x \leq y \) imply \( y \ast z \leq x \ast z \).
Proof. Since \( x \leq y \) implies \( y \cdot x = 0 \), by \( ku_1 \), we obtain \((y \cdot x) \cdot ((x \cdot z) \cdot (y \cdot z)) = 0 \). But \( y \cdot x = 0 \), then \( 0 \cdot ((x \cdot z) \cdot (y \cdot z)) = 0 \), by \((ku_2, ku_3)\), we get \((x \cdot z) \cdot (y \cdot z) = 0 \).

i.e. \( y \cdot z \leq x \cdot z \).

**Lemma 2.6.** In KU-algebra \( X \), we have

\[
z \cdot (y \cdot x) = y \cdot (z \cdot x)
\]

for all \( x, y, z \in X \).

**Proof.** From \((ku_1)\) we get \((0 \cdot z) \cdot ((z \cdot x) \cdot (0 \cdot x)) = 0 \), this implies \( z \cdot ((z \cdot x) \cdot x) = 0 \).

i.e. \( z \cdot x \leq z \) \hspace{1cm} (a)

Making use of (a) and \((ku_1)\), we get \( z \cdot (y \cdot x) \leq ((z \cdot x) \cdot x) \cdot (y \cdot x) \leq y \cdot (z \cdot x) \) since \( x, y, z \) are arbitrary, interchanging \( y \) and \( z \) in the above inequality, we obtain \( y \cdot (z \cdot x) \leq z \cdot (y \cdot x) \). By \((ku_4)\), we get \( z \cdot (y \cdot x) = y \cdot (z \cdot x) \).

**Lemma 2.7.** If \( X \) is KU-algebra, then

\[
y \cdot [(y \cdot x) \cdot x] = 0.
\]

**Proof.** Using lemma (2.6), then \((y \cdot x) \cdot (y \cdot x) = 0 \).

**Definition 2.8 [19].** Let \((X, \ast, 0)\) and \((X', \ast', 0')\) be KU-algebras, a homomorphism is a map \( f: X \rightarrow X' \) satisfying

\[
f(x \cdot y) = f(x) \cdot' f(y)
\]

for all \( x, y \in X \).

**Theorem 2.9 [19].** Let \( f \) be a homomorphism of a KU-algebra \( X \) into a KU-algebra \( X' \), then

(i) If 0 is the identity in \( X \), then \( f(0) \) is the identity in \( X' \).

(ii) If \( S \) is a KU-subalgebra of \( S \), then \( f(S) \) is a KU-subalgebra of \( X' \).

(iii) If \( I \) is an KU-ideal of \( X \), then \( f(I) \) is an KU-ideal in \( f(X) \).

(iv) If \( S \) is a KU-subalgebra of \( f(X') \), then \( f^{-1}(S) \) is a KU-algebra of \( X \).

(v) If \( B \) is an KU-ideal in \( f(X) \), then \( f^{-1}(B) \) is an KU-ideal in \( X \).

(vi) If \( f \) is a homomorphism from KU-algebra \( X \) to a KU-algebra \( X' \), then \( f \) is one to one if and only if \( \ker f = \{0\} \).

**Proposition 2.10.** Suppose \( f: X \rightarrow X' \) is a homomorphism of KU-algebras, then

1. \( f(0) = 0' \).
2. If \( x \leq y \) implies \( f(x) \leq f(y) \).

**Proof.** Since \( x \leq y \) then \( y \cdot x = 0 \), then \( f(y \cdot x) = f(y) \cdot' f(x) = f(0) \).

i.e. \( f(x) \leq f(y) \).

**Proposition 2.11.** Let \((X, \ast, 0)\) and \((X', \ast', 0')\) be KU-algebras and \( f: X \rightarrow X' \) be a homomorphism, then \( ker f \) is KU-ideal of \( X \).

**Proof.** \( 0 \in ker f \), since \( f(0) = 0' \). Let \( x \cdot (y \cdot z) \in ker f \), \( y \in ker f \), then \( f(x \cdot (y \cdot z)) = 0' \), \( f(y) = 0' \), since \( 0' = f(x \cdot (y \cdot z)) = f(x) \cdot f(y \cdot z) = 0' = f(x) \cdot (f(y) \cdot f(z)) = f(y) \cdot f(x) \cdot f(z) \).

(by lemma 2.6) together with \( f(y) = 0' \), we get \( 0' = f(x) \cdot f(z) \), this implies \( f(x) \cdot f(z) = f(x \cdot z) = 0' \).

i.e. \( x \cdot z \in ker f \), then \( ker f \) is an KU-ideal of \( X \).
In this section, we will discuss and investigate a new notion called fuzzy KU-ideals of KU-algebras and study several basic properties which related to fuzzy KU-ideals.

**Definition 3.1** [23]. Let $X$ be a set, a fuzzy set $\mu$ in $X$ is a function $\mu : X \rightarrow [0,1]$.

**Definition 3.2**. Let $X$ be a KU-algebra, a fuzzy set $\mu$ in $X$ is called fuzzy sub-algebra if it satisfies:

1. $\mu(0) \geq \mu(x)$,
2. $\mu(x) \geq \{\mu(x \cdot y), \mu(y)\}$ for all $x, y \in X$.

**Definition 3.3**. Let $X$ be a KU-algebra, a fuzzy set $\mu$ in $X$ is called a fuzzy KU-ideal of $X$ if it satisfies the following conditions:

1. $\mu(0) \geq \mu(x)$,
2. $\mu(x \cdot z) \geq \min\{\mu(x \cdot (y \cdot z)), \mu(y)\}$.

**Example 3.4**. Let $X = \{0,1,2,3,4\}$ in which $\ast$ is defined by the following table:

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Then $(X, \ast, 0)$ is KU-algebra. Define a fuzzy set $\mu : X \rightarrow [0,1]$ by $\mu(0) = t_0, \mu(1) = \mu(2) = t_1, \mu(3) = \mu(4) = t_2$, where $t_0, t_1, t_2 \in [0,1]$ with $t_0 > t_1 > t_2$.

Routine calculation gives that $\mu$ is a fuzzy KU-ideal of KU-algebras $X$.

**Lemma 3.5**. Let $\mu$ be a fuzzy KU-ideal of KU-algebra $X$, if the inequality $x \ast y \leq z$ holds in $X$, then $\mu(y) \geq \min\{\mu(x), \mu(z)\}$.

**Proof**. Assume that the inequality $x \ast y \leq z$ holds in $X$, then $z \ast (x \ast y) = 0$ and by $(F_2)\mu(z \ast y) \geq \min\{\mu(z \ast (x \ast y)), \mu(x)\}$, if we put $z=0$ then $\mu(0 \ast y) = \mu(y) \geq \min\{\mu(x \ast y), \mu(x)\}$ (i).

But $\mu(x \ast y) \geq \min\{\mu(x \ast (z \ast y)), \mu(z)\}$

$= \min\{\mu(z \ast (x \ast y)), \mu(z)\}$

$= \min\{\mu(0), \mu(z)\} = \mu(z)$ (ii).

From (i), (ii), we get $\mu(y) \geq \min\{\mu(z), \mu(x)\}$, this completes the proof.
Lemma 3.6. If $\mu$ is a fuzzy KU-ideal of KU-algebra $X$ and if $x \leq y$, then $\mu(x) \geq \mu(y)$.

Proof. If $x \leq y$, then $y \ast x = 0$, this together with $0 \ast x = x$ and $\mu(0) \geq \mu(y)$, we get $\mu(0 \ast x) = \mu(x) \geq \min \{\mu(0 \ast (y \ast x)), \mu(y)\} = \min \{\mu(0), \mu(y)\} = \mu(y)$.

Proposition 3.7. The intersection of any set of fuzzy KU-ideals of KU-algebra $X$ is also fuzzy ideal.

Proof. Let $\{\mu_i\}$ be a family of fuzzy KU-ideals of KU-algebra $X$, then for any $x, y, z \in X$,

$(\bigcap \mu_i)(0) = \inf(\mu_i(0)) \geq \inf(\mu_i(x)) = (\bigcap \mu_i)(x)$ and $(\bigcap \mu_i)(x \ast z) = \inf(\mu_i(x \ast z)) \geq \inf(\min \{\mu_i(x \ast (y \ast z)), \mu_i(y)\}) = \min \{\inf(\mu_i(x \ast (y \ast z))), \inf(\mu_i(y))\} = \min \{(\bigcap \mu_i)(x \ast (y \ast z)), (\bigcap \mu_i(y))\}$.

This completes the proof.

Theorem 3.8. Let $\mu$ be a fuzzy set in $X$ then $\mu$ is a fuzzy KU-ideal of $X$ if and only if it satisfies:

For all $\alpha \in [0,1]$, $U(\mu, \alpha) \neq \emptyset$ implies $U(\mu, \alpha)$ is KU-ideal of $X$-----(A)

where $U(\mu, \alpha) = \{x \in X / \mu(x) \geq \alpha\}$.

Proof. Assume that $\mu$ is a fuzzy ideal of $X$, let $\alpha \in [0, 1]$ be such that $U(\mu, \alpha) \neq \emptyset$, and let $x, y \in X$ be such that $x \in U(\mu, \alpha)$, then $\mu(x) \geq \alpha$ and so by (F2),

$\mu(y \ast 0) = \mu(0) \geq \min \{\mu(y \ast (x \ast 0)), \mu(x)\} = \min \{\mu(0), \mu(x)\} = \alpha$.

thus $0 \in U(\mu, \alpha)$.

Let $x \ast (y \ast z) \in U(\mu, \alpha)$, $y \in U(\mu, \alpha)$, It follows from(F2) that $\mu(x \ast z) \geq \min \{\mu(x \ast (y \ast z)), \mu(y)\} = \alpha$, so that $x \ast z \in U(\mu, \alpha)$. Hence $U(\mu, \alpha)$ is KU-ideal of $X$.

Conversely, suppose that $\mu$ satisfies (A), let $x, y, z \in X$ be such that $\mu(x \ast z) < \min \{\mu(x \ast (y \ast z)), \mu(y)\}$, taking $\beta_0 = 1/2 \{\mu(x \ast z) + \min \{\mu(x \ast (y \ast z)), \mu(y)\}\}$, we have $\beta_0 \in [0,1]$ and $\mu(x \ast z) < \beta_0 < \min \{\mu(x \ast (y \ast z)), \mu(y)\}$

it follows that $x \ast (y \ast z) \in U(\mu, \beta_0)$ and $x \ast z \notin U(\mu, \beta_0)$, this is a contradiction and therefore $\mu$ is a fuzzy KU-ideal of $X$.

Proposition 3.9. If $\mu$ is a fuzzy KU-ideal of $X$, then

$\mu(x \ast (x \ast y)) \geq \mu(y)$

proof. Taking $z = x \ast y$ in (F2) and using (ku2) and (F1), we get

$\mu(x \ast (x \ast y)) \geq \min \{\mu(x \ast (y \ast (x \ast y))), \mu(y)\}$

$= \min \{\mu(x \ast (y \ast 0)), \mu(y)\} = \min \{\mu(0), \mu(y)\} = \mu(y)$. 

4. Characterization of fuzzy KU - ideal by their level KU - ideals

**Theorem 4.1.** A fuzzy subset \( \mu \) of KU - algebra \( X \) is a fuzzy KU - ideal of \( X \) if and only if , for every \( t \in [0, 1] \), \( \mu_t \) is either empty or an KU - ideal of \( X \).

**Proof.** Assume that \( \mu \) is a fuzzy KU - ideal of \( X \), by (F1), we have \( \mu(0) \geq \mu(x) \) for all \( x \in X \) therefore \( \mu(0) \geq \mu(x) \geq t \) for \( x \in \mu_t \) and so \( 0 \in \mu_t \). Let \( x \ast (y \ast z) \in \mu_t \) and \( y \in \mu_t \), then \( \mu(x \ast (y \ast z)) \geq t \) and \( \mu(y) \geq t \), since \( \mu \) is a fuzzy KU - ideal it follows that \( \mu(x \ast z) \geq \min \{\mu(x \ast (y \ast z)), \mu(y)\} \geq t \) and that \( x \ast z \in \mu_t \). Hence \( \mu_t \) is an KU - ideal of \( X \).

Conversely, we only need to show that (F1) and (F2) are true. If (F1) is false , then there exist \( x' \in X \) such that \( \mu(0) < \mu(x') \). If we take \( t' = (\mu(x') + \mu(0))/2 \), then \( \mu(0) < t' \) and \( 0 \leq t' < \mu(x') \leq 1 \), thus \( x' \in \mu \) and \( \mu \neq \phi \). As \( \mu \) is an KU-ideal of \( X \), we have \( 0 \in \mu_t \), and so \( \mu(0) \geq t' \). This is a contradiction. Now , assume (F2) is not true ,then there exist \( x' , y' \) and \( z' \) such that ,

\[
\mu(x', z') < \min \{\mu(x' \ast (y' \ast z')) , \mu(y')\}.
\]

Putting \( t' = (\mu(x') + \mu(0))/2 \), then \( \mu(0) < t' \) and \( 0 \leq t' < \min \{\mu(x' \ast (y' \ast z')) , \mu(y')\} \leq 1 \), hence \( \mu(x' \ast (y' \ast z')) > t' \) and \( \mu(y') > t' \), which imply that \( x' \ast (y' \ast z') \in \mu(t') \) and \( y' \in \mu_t \), since \( \mu_t \) is an KU - ideal ,it follows that \( x' \ast z' \in \mu_t \) and that \( \mu(x' \ast z') \geq t' \), this is also a contradiction. Hence \( \mu \) is a fuzzy KU – ideal of \( X \).

**Corollary 4.2.** If a fuzzy subset \( \mu \) of KU - algebra \( X \) is a fuzzy KU - ideal , then for every \( t \in \text{Im}(\mu) \), \( \mu_t \) is an KU - ideal of \( X \).

**Definition 4.3.** let \( \mu \) be a fuzzy KU - ideal of KU - algebra \( X \), the KU - ideals \( \mu_t \) \( t \in [0, 1] \) are called level KU - ideal of \( \mu \).

**Corollary 4.4.** let \( I \) be an KU - ideal of KU - algebra \( X \), then for any fixed number \( t \) in an open interval \((0, 1)\) , there exist a fuzzy KU - ideal \( \mu \) of \( X \) such that \( \mu_t = I \).

**Definition 4.5.**

Let \( f \) be a mapping from the set \( X \) to a set \( Y \). If \( \mu \) is a fuzzy subset of \( X \), then the fuzzy subset \( B \) of \( Y \) defined by

\[
f(\mu)(y) = B(y) = \begin{cases} \sup_{x \in f^{-1}(y)} \mu(x), & \text{if } f^{-1}(y) = \{x \in X, f(x) = y\} \neq \emptyset \\ 0 & \text{otherwise} \end{cases}
\]

is said to be the image of \( \mu \) under \( f \).
Similarly if \( \beta \) is a fuzzy subset of \( Y \), then the fuzzy subset \( \mu = \beta \circ f \) in \( X \) (i.e. the fuzzy subset defined by \( \mu (x) = \beta (f(x)) \) for all \( x \in X \)) is called the primage of \( \beta \) under \( f \).

**Theorem 4.6.** An onto homomorphic preimage of a fuzzy KU-ideal is also a fuzzy KU-ideal.

**Proof.** Let \( f : X \to X' \) be an into homomorphism of KU-algebras, \( \beta \) a fuzzy KU-ideal of \( X' \) and \( \mu \) the preimage of \( \beta \) under \( f \), then \( \beta (f(x)) = \mu (x) \), for all \( x \in X \). Let \( x \in X \), then \( \mu (0) = \beta (f(0)) \geq \beta (f(x)) = \mu (x) \). Now let \( x, y, z \in X \) then \( \mu (x * z) = \beta (f(x) * f(z)) \geq \min \{ \beta (f(x) * (f(y) * f(z))), \beta (f(y)) \} = \min \{ \beta (f((x * (y * z))), \beta (f(y))) \} = \min \{ \mu(x * (y * z)), \mu (y) \} \), and the proof is completed.

**Definition 4.7 [16].** A fuzzy subset \( \mu \) of \( X \) has sup property if for any subset \( T \) of \( X \), there exist \( t_0 \in T \) such that \( \mu (t_0) = \text{SUP} \mu (t) \).

**Theorem 4.8.** Let \( X \to Y \) be a homomorphism between KU-algebras \( X \) and \( Y \). For every fuzzy KU-ideal \( \mu \) in \( X \), \( f(\mu) \) is a fuzzy KU-ideal of \( Y \).

**Proof.** By definition \( B(y') = f(\mu)(y') = \sup_{x \in f^{-1}(y')} \mu(x) \) for all \( y' \in Y \) and \( \sup \phi = 0 \).

We have to prove that \( B(x' * z') \geq \min \{ B(x' * (y' * z')), B(y') \} \), \( \forall x', y', z' \in Y \).

Let \( f : X \to Y \) be an onto homomorphism of KU-algebras, \( \mu \) a fuzzy KU-ideal of \( X \) with sup property and \( \beta \) the image of \( \mu \) under \( f \), since \( \mu \) is a fuzzy KU-ideal of \( X \), we have \( \mu(0) \geq \mu(x) \) for all \( x \in X \). Note that \( 0 \in f^{-1}(0') \), where \( 0, 0' \) are the zero of \( X \) and \( Y \) respectively.

Thus, \( B(0') = \sup_{x \in f^{-1}(0')} \mu(t) = \mu(0) \geq \mu(x) \), for all \( x \in X \), which implies that

\[
B(0') \geq \sup_{x \in f^{-1}(x')} \mu(t) = B(x'), \text{ for any } x' \in Y.
\]

For any \( x', y', z' \in Y \), let \( x_0 \in f^{-1}(x') \), \( y_0 \in f^{-1}(y') \), \( z_0 \in f^{-1}(z') \) be such that

\[
\mu(x_0 * z_0) = \sup_{x \in f^{-1}(x')} \mu(t), \quad \mu(y_0) = \sup_{y \in f^{-1}(y')} \mu(t)
\]

and

\[
\mu(x_0 * (y_0 * z_0)) = B[f(x_0 * (y_0 * z_0))] = B((x' * (y' * z')))
\]

\[
= \sup_{(x_0 * (y_0 * z_0) \in f^{-1}(x' * (y' * z'))} \mu((x_0 * (y_0 * z_0))) = \sup_{t \in f^{-1}(y')} \mu(t).
\]

Then \( B(x' * z') = \sup_{t \in f^{-1}(y')} \mu(t) = \mu(x_0 * z_0) \geq \min \{ \mu(x_0 * (y_0 * z_0), \mu(y_0) \} = \min \{ B(x' * (y' * z')) \}

Hence \( B \) is a fuzzy KU-ideal of \( Y \).
5. Cartesian product of fuzzy KU-ideal

**Definition 5.1[1]**. A fuzzy $\mu$ is called a fuzzy relation on any set $S$, if $\mu$ is a fuzzy subset $\mu : S \times S \rightarrow [0,1]$.

**Definition 5.2 [1]**. If $\mu$ is a fuzzy relation a set $S$ and $\beta$ is a fuzzy subset of $S$, then $\mu$ is a fuzzy relation on $\beta$ if $\mu(x, y) \leq \min \{\beta(x), \beta(y)\}, \forall x, y \in S$.

**Definition 5.3 [1]**. Let $\mu$ and $\beta$ be fuzzy subset of a set $S$, the cartesian product of $\mu$ and $\beta$ is define by $(\mu \times \beta)(x, y) = \min \{\mu(x), \beta(y)\}, \forall x, y \in S$.

**Lemma 5.4[1]**. Let $\mu$ and $\beta$ be fuzzy subset of a set $S$ then,

(i) $\mu \times \beta$ is a fuzzy relation on $S$.

(ii) $(\mu \times \beta)^t = \mu^t \times \beta^t$ for all $t \in [0,1]$.

**Definition 5.5 [1]**. If $\beta$ is a fuzzy subset of a set $S$, the strongest fuzzy relation on $S$, that is, a fuzzy relation on $\beta$ is $\mu_\beta$ given by $\mu_\beta(x, y) = \min \{\beta(x), \beta(y)\}, \forall x, y \in S$.

**Lemma 5.6**. For a given fuzzy subset $S$, let $\mu_\beta$ be the strongest fuzzy relation on $S$ then for $t \in [0,1]$, we have $(\mu_\beta)^t = \beta_\beta^t$.

**Proposition 5.7**. For a given fuzzy subset $\beta$ of KU-algebra $X$, let $\mu_\beta$ be the strongest fuzzy relation on $X$. If $\mu_\beta$ is a fuzzy KU-ideal of $X \times X$, then $\beta(x) \leq \beta(0)$ for all $x \in X$.

**Proof**. Since $\mu_\beta$ is a fuzzy KU-ideal of $X \times X$, it follows from (F1) that $\mu_\beta(x, x) = \min \{\beta(x), \beta(x)\} \leq (0, 0) = \min \{\beta(0), \beta(0)\}$, where $(0, 0) \in X \times X$ then $\beta(x) \leq \beta(0)$.

**Remark 5.8**. Let $X$ and $Y$ be KU-algebras, we define $*$ on $X \times Y$ by:

For every $(x, y), (u, v) \in X \times Y$, $(x, y) * (u, v) = (x * u, y * v)$, then clearly $(x * y, *, (0, 0))$ is a KU-algebra.

**Theorem 5.9**. Let $\mu$ and $\beta$ be a fuzzy subset of KU-algebra $X$, such that $\mu \times \beta$ is a fuzzy KU-ideal of $X \times X$, then $\beta(x) \leq \beta(0)$ for all $x \in X$.

**Proof**. For any $(x, y) \in X \times X$, we have,

$$\mu \times \beta)(0, 0) = \min \{\mu(0), \beta(0)\} \geq \min \{\mu(x), \beta(x)\} = (\mu \times \beta)(x, y).$$

Now let $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in X \times X$, then,

$$(\mu \times \beta)(x_1 \ast z_1, x_2 \ast z_2) = \min \{\mu(x_1, z_1), \beta(x_2, z_2)\}$$

$$\geq \min \{\min \{\mu(x_1 \ast (y_1 \ast z_1)), \mu(y_1)\}, \min \{\beta(x_2 \ast (y_2 \ast z_2)), \beta(y_2)\}\}$$

$$= \min \{\min \{\mu(x_1 \ast (y_1 \ast z_1)), \mu(x_2 \ast (y_2 \ast z_2))\}, \min \{\mu(y_1), \beta(y_2)\}\}$$

$$= \min \{\min \{\mu \times \beta)(x_1 \ast (y_1 \ast z_1), x_2 \ast (y_2 \ast z_2)\}, (\mu \times \beta)(y_1, y_2)\}.$$  

Hence $\mu \times \beta$ is a fuzzy KU-ideal of $X \times X$.

Analogous to theorem 3.2 [15], we have a similar results for KU-ideal, which can be proved in similar manner, we state the results without proof.

**Theorem 5.10**. Let $\mu$ and $\beta$ be a fuzzy subset of KU-algebra $X$, such that $\mu \times \beta$ is a fuzzy KU-ideal of $X \times X$, then
(i) either $\mu(x) \leq \mu(0)$ or $\beta(x) \leq \beta(0)$ for all $x \in X$,
(ii) if $\mu(x) \leq \mu(0)$ for all $x \in X$, then either $\mu(x) \leq \beta(0)$ or $\beta(x) \leq \beta(0)$,
(iii) if $\beta(x) \leq \beta(0)$ for all $x \in X$, then either $\mu(x) \leq \mu(0)$ or $\beta(x) \leq \mu(0)$,
(iv) either $\mu$ or $\beta$ is a fuzzy KU- ideal of $X$.

**Theorem 5.11.** let $\beta$ be a fuzzy subset of KU-algebra $X$ and let $\mu_\beta$ be the strongest fuzzy relation on $X$, then $\beta$ is a fuzzy KU - ideal of $X$ if and only if $\mu_\beta$ is a fuzzy KU- ideal of $X \times X$.

**proof.** Assume that $\beta$ is a fuzzy KU- ideal $X$, we note from (F1) that:

$\mu_\beta(0, 0) = \min \{\beta(0), \beta(0)\} \geq \min \{\beta(x), \beta(y)\} = \mu_\beta(x, y)$.

Now, for any $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in X \times X$, we have from (F2):

$\mu_\beta(x_1 \ast z_1, x_2 \ast z_2) = \min \{\beta(x_1 \ast z_1), \beta(x_2 \ast z_2)\}$

$\geq \min \{\min \{\beta(x_1 \ast (y_1 \ast z_1)), \beta(x_2 \ast (y_2 \ast z_2))\}, \min \{\beta(x_1 \ast (y_1 \ast z_1)), \beta(x_2 \ast (y_2 \ast z_2))\}\}$

$= \min \{\min \{\beta(x_1 \ast (y_1 \ast z_1)), \beta(x_2 \ast (y_2 \ast z_2))\}, \min \{\beta(y_1), \beta(y_2)\}\}$

$= \min \{\mu_\beta(x_1 \ast (y_1 \ast z_1), x_2 \ast (y_2 \ast z_2)), \mu_\beta(y_1, y_2)\}$. 

Hence $\mu_\beta$ is a fuzzy KU - ideal of $X \times X$. 

Conversely : for all $(x, y) \in X \times X$, we have

$\min \{\beta(0), \beta(0)\} = \mu_\beta(x, y) = \min \{\beta(x), \beta(y)\}$

It follows that $\beta(0) \geq \beta(x)$ for all $x \in X$, which prove (F1).

Now, let $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in X \times X$, then

$\min \{\beta(x_1 \ast z_1), \beta(x_2 \ast z_2)\} = \mu_\beta(x_1 \ast z_1, x_2 \ast z_2)$

$\geq \min \{\mu_\beta(x_1, x_2) \ast ((y_1 \ast y_2) \ast (z_1, z_2)), \mu_\beta(y_1, y_2)\}$

$= \min \{\mu_\beta(x_1 \ast (y_1 \ast z_1)), \mu_\beta(x_2 \ast (y_2 \ast z_2)), \mu_\beta(y_1, y_2)\}$

$= \min \{\min \{\beta(x_1 \ast (y_1 \ast z_1)), \beta(x_2 \ast (y_2 \ast z_2))\}, \min \{\beta(y_1), \beta(y_2)\}\}$

$= \min \{\min \{\beta(x_1 \ast (y_1 \ast z_1)), \beta(y_1)\}, \min \{\beta(x_2 \ast (y_2 \ast z_2)), \beta(y_2)\}\}$

In particular, if we take $x_2 = y_2 = z_2 = 0$, then,

$\beta(x_1 \ast z_1) \geq \min \{\beta(x_1 \ast (y_1 \ast z_1)), \beta(y_1)\}$

This prove (F1) and completes the proof.

**References**

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