Three Notes About Linear Stochastic Oscillator Equations

Reza Habibi
Department of Statistics
Central Bank of Iran, Tehran, Iran
habibi1356@gmail.com

Hamed Habibi
Department of Mechanics
University of Tehran, Tehran, Iran

Hamid Habibi
Department of Airworthiness
Civil Aviation Organization, Tehran, Iran

Abstract

This paper considers three notes about linear stochastic oscillator equations. First, we derive the exact and numerical approximations for the expectation of quadratic forms of solution of linear stochastic oscillator equations. This extends Strømøen Melbø and Higham’s result (2004). The second note considers the same problem under measurement error assumptions and in the third one, we add a damping term to stochastic differential equations.

Keywords: Damping term; Ito lemma; Linear stochastic oscillator; Measurement error; Numerical approximation; Quadratic form

1 Introduction. Stochastic differential equations are strong tools for analyzing the continuous stochastic processes. They are used in modeling investment finance, population dynamics, biological waste treatment, neuronal models. They have been applied for modeling phenomena in some fields of engineering, for example, to model the fatigue damage of materials or fluid mechanical turbulence or modeling movement of a stochastic oscillator. There
are many strong approaches to derive the analytical solution of a SDE. However, an important objective in this field is to achieve reasonable approximations to the solution of a SDE. There are many good approximation methods, for example the Euler method, the Milstein procedure and the Runge Kutta approach. These methods and their error estimates have been received considerable attentions in the literatures, see Kloeden and Platen (1999), Oksendal (2000) and Henderson and Plaschko (2006) among the others.

Strømmen Melbø and Higham (2004) (hereafter SMH) studied the expectation of

\[ g_t = x_t^2 + y_t^2, \]

and its numerical approximations where \( z_t = (x_t, y_t)^T \) satisfies the following two-dimensional stochastic differential equation

\[
\begin{aligned}
    dx_t &= y_t dt \\
    dy_t &= -x_t dt + \sigma dw_t,
\end{aligned}
\]

at which \( \sigma > 0 \) and \( w_t \) is standard Wiener process. Let \( x_0 = 1, y_0 = 0 \). Notation \( T \) stands for the transpose of a vector or matrix. This system of differential equations is referred to stochastic oscillator equation. In this note, we first study the exact and numerical approximation of the expectation of a quadratic form

\[ f_t = z_t^T A z_t, \]

where \( A \) is a symmetry, positive definite matrix. Then, we obtain the expectation of \( g_t \) under measurement error problems. Finally, we add a damping term to the stochastic differential equations. We first solve SMH’s problem by our approach. The multivariate Ito lemma implies that

\[
dg_t = g_x dx + g_y dy + (1/2)(g_{xx}(dx)^2 + g_{yy}(dy)^2) + 2g_{xy}dx dy,
\]

where \( g_x \) is the partial derivative of \( g \) with respect to \( x_t \). The \( g_y, g_{xx}, g_{yy} \) and \( g_{xy} \) are defined analogously. One can see that \( (dx)^2 = dx dy = 0, (dy)^2 = \sigma^2 dt \) and \( g_x = 2x, g_y = 2y \) and \( g_{yy} = 2 \). Therefore, we have

\[
dg_t = 2(x_t dx_t + y_t dy_t) + \sigma^2 dt.
\]

It is easy to see that

\[ x_t dx_t + y_t dy_t = \sigma y_t dw_t, \]
and then \( dg_t = 2\sigma y_t dw_t + \sigma^2 dt \). Then,
\[
g_t = g_0 + \sigma^2 t + 2\sigma \int_0^t y_s dw_s.
\]
Since \( g_0 = x_0^2 + y_0^2 = 1 + 0 = 1 \) then
\[
g_t = 1 + \sigma^2 t + 2\sigma \int_0^t y_s dw_s.
\]
Once can see that
\[
E(g_t) = 1 + \sigma^2 t.
\]
Theorem 1 in SMH (2004) states the same result. They solved the equations and derived the expectation, instead we used the Ito lemma. This approach helps us to extend this problem to other cases, as follows. A natural extension to \( g_t \) is \( f_t \), the quadratic form of \( z_t \). Here, we derive the expectation of \( f_t \) under SMH’s equations. Suppose that
\[
A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}.
\]
Then,
\[
f_x = 2(ax + by), \quad f_y = 2(cy + bx),
\]
and \( f_{xx} = 2a, \; f_{yy} = 2c \) and \( f_{xy} = 2b \). Therefore, using Ito lemma and integrating with respect to \( x \), we have
\[
f_t = a + c\sigma^2 t + 2\int_0^t h_1(x_s, y_s) ds + 2\sigma \int_0^t h_2(x_s, y_s) dw_s,
\]
where
\[
h_1(x_s, y_s) = (a - c)x_s y_s + b(y_s^2 - x_s^2) \quad \text{and} \quad h_2(x_s, y_s) = bx_s + cy_s.
\]
It is seen that
\[
E(f_t) = a + c\sigma^2 t + 2E(\int_0^t h_1(x_s, y_s) ds),
\]
where
\[
E(\int_0^t h_1(x_s, y_s) ds) = (a - c) \int_0^t E(x_s y_s) ds + b \int_0^t E(y_s^2 - x_s^2) ds.
\]
By letting $a = c = 1$ and $b = 0$, the previous result is obtained. Twice use of the Ito lemma proves that

$$E(x_s y_s) = -\cos(s) \sin(s) + \frac{\sigma^2}{2}(1 - \cos^2(s)),$$

and $E(y_s^2 - x_s^2) = -\cos(2s) - \frac{\sigma^2}{2}\sin(2s)$. However, note that calculating the $E(x_s y_s)$ and $E(y_s^2 - x_s^2)$ is straightforward. To see them, note that

$$E(x_s y_s) = E[(\cos(s) + \sigma \int_0^s \sin(s-t)dw_t) \times (-\sin(s) + \sigma \int_0^s \cos(s-t)dw_t)]
= -\cos(s) \sin(s) + \sigma^2 \int_0^s \sin(s-t) \cos(s-t) dt
= -\cos(s) \sin(s) + \frac{\sigma^2}{2}(1 - \cos^2(s)).$$

Also, one can see that

$$E(y_s^2 - x_s^2) = -\cos(2s) - \frac{\sigma^2}{2}\sin(2s).$$

**Remark.** One can see that $f_t = ax_t^2 + 2bx_t y_t + cy_t^2$. Therefore, the direct calculation of $E(f_t)$ seems to be more easier than the above mentioned method. However, it is worth noting that this method uses only the multivariate version of Ito lemma. Authors believe that this approach may be useful for the other problems in stochastic differential equation field.

2 Extensions. In this section, we propose some other extensions to problem considered by SMH (2004). They include Measurement error and equations with damping terms cases. In each case, the exact value of expectations and their numerical approximations are given.

2.1 Measurement error. As another extension, we consider the measurement error case. Here, we assume that data process $dx_t$ is observed with errors, that is $dx_t$, itself, obeys another stochastic differential equation, i.e.,

$$\begin{align*}
\left\{ \begin{array}{ll}
dx_t &= y_t dt + \delta dw_{1t} \\
dy_t &= -x_t dt + \sigma dw_{2t},
\end{array} \right.
\end{align*}$$

at which $w_1$ and $w_2$ are two correlated Brownian motions such that $dw_1 \cdot dw_2 = \rho dt$. This problem has been studied by many authors, for example see Baltazar-Larios and Sørensen (2009) and Habibi et al. (2010). The Ito lemma implies that
Linear stochastic oscillator equations

\[ dg = g_x dx + g_y dy + 1/2(g_{xx}(dx)^2 + g_{yy}(dy)^2 + 2g_{xy}dxdy). \]

One can see that \((dx)^2 = \delta^2 dt, (dy)^2 = \sigma^2 dt, dxdy = \delta \sigma dt\) and \(g_{xx} = g_{yy} = 2\) and \(g_{xy} = 0\). Beside this,

\[ g_x dx + g_y dy = 2(\delta x_i dw_{1t} + \sigma y_i dw_{2t}), \]

then, we have

\[ g_t = 1 + (\delta^2 + \sigma^2)t + 2\delta \int_0^t x_s dw_{1s} + 2\sigma \int_0^t y_s dw_{2s}. \]

It is seen that

\[ E(g_t) = 1 + (\delta^2 + \sigma^2)t. \]

For numerical approximation, let \(\Delta t\) be integral step-size required in Euler-Maruyama (EM) procedure and \(t_n = n\Delta t, n = 1, 2, \ldots\). Here, the EM approximation is

\[
\begin{align*}
x_{n+1} &= x_n + \Delta t y_n + \delta \Delta w_{1n} \\
y_{n+1} &= y_n - \Delta t x_n + \sigma \Delta w_{2n},
\end{align*}
\]

where \(x_n = x(t_n), y_n = y(t_n)\) and \(\Delta w_{in} = w_i(t_{n+1}) - w_i(t_n), i = 1, 2\). Following SMH (2004), since \(\Delta w_{in}\) and \(x_n\) and \(y_n\) (for each \(i\)) are independent, then

\[ E(g_{n+1}) = (1 + (\Delta t)^2)E(g_n) + (\delta^2 + \sigma^2)\Delta t \]

\[ \geq (1 + (\Delta t)^2)E(g_n). \]

Hence,

\[ E(g_n) \geq (1 + (\Delta t)^2)^n \geq e^{(\Delta t/2)t_n}. \]

This shows that for small \(\Delta t\), the squared length of \(z_n\), the EM solution of stochastic differential equation, will grow exponentially with \(t_n\). The Backward EM (BEM) procedure gives the following solutions

\[
\begin{align*}
x_{n+1} &= x_n + \Delta t y_{n+1} + \delta \Delta w_{1n} \\
y_{n+1} &= y_n - \Delta t x_{n+1} + \sigma \Delta w_{2n},
\end{align*}
\]

One can see that

\[ E(g_{n+1}) = \frac{1}{1 + (\Delta t)^2} \{ E(g_n) + (\delta^2 + \sigma^2)\Delta t \}. \]
This fact implies that $E(g_n)/t_n \to 0$ as $n$ goes to infinity.

2.2 Equations with damping term. SMH (2004) ignored the damping term in second order stochastic differential equations. Here, we consider the following equations at which the damping term is added. They are given by

$$\begin{cases}
    dx_t = y_t dt \\
    dy_t = (-x_t + \beta y_t) dt + \sigma dw_t.
\end{cases}$$

Some algebra manipulations show that

$$g_t = 1 + \sigma^2 t + 2\beta \int_0^t y_s^2 ds + 2\sigma \int_0^t y_s dw_s.$$  

It is seen that

$$E(g_t) = 1 + \sigma^2 t + 2\beta \int_0^t E(y_s^2) ds.$$  

Suppose that $g_n$ is the EM solution of above mentioned stochastic differential equation. Following SMH (2004), we can show that

$$E(g_n) \geq e^{(\Delta t/2)t_n},$$

and for BEM solution again it can be shown that $E(g_n)/t_n \to 0$ as $n$ goes to infinity.

References


**Received: June, 2011**