Fuzzy Convex Invariants and Product Spaces

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Abstract

In abstract convexity theory, the classical convex invariants namely Helly number, Caratheodory number, Radon number and Exchange number play a central role. In [3], some basic relations between these invariants in a fuzzy convex structure are studied. The behaviour of these invariants under the formation of FCP images are also studied. In this paper, the fuzzy convex invariants of product spaces are discussed and certain relations are established.

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1 Introduction

Caratheodory and Helly numbers of convex product structures are defined and studied by G. Sierksma in 1975. In the crisp case [6], it was shown that the Helly number of a convex product structure is the supremum of the Helly numbers of the factor spaces. In this paper, it is proved in the fuzzy context. The Caratheodory number of the convex product space with two factors is also determined in the fuzzy context.
2 Basic definitions and properties

Definition 2.1 [1] A fuzzy subset $A$ on a nonempty set $X$ is a function from $X$ to $I = [0, 1]$. The set of all fuzzy subsets of $X$ is denoted by $I^X$. A fuzzy point $a_\alpha$ is a fuzzy subset defined as

$$a_\alpha(y) = \begin{cases} \alpha, & \text{if } y = a \\ 0, & \text{otherwise} \end{cases}$$

Definition 2.2 [1] Support of a fuzzy subset $A$ is $\text{Supp}(A) = \{x \in X | A(x) > 0\}$. A fuzzy subset is said to be finite if its support is finite.

Definition 2.3 [1] Given two fuzzy sets $A, B \in I^X, A \subseteq B$ (or $A \leq B$) if $A(x) \leq B(x), \forall x \in X$. Also, $A \setminus B = A \land B'$.

Definition 2.4 [4] A family $C$ of fuzzy subsets of $X$ is called a fuzzy convexity if

(i) $\emptyset, 1 \in C$

(ii) If $F \subseteq C$, then $\land F \in C$

(iii) If $F \subseteq C$ is non-empty and totally ordered by inclusion, then $\lor F \in C$.

The pair $(X, C)$ is called a fuzzy convex structure or a fuzzy convexity space. Members of $C$ are called fuzzy convex sets.

Note. $a$ denotes a constant function whose value at $x$ is $a, \forall x \in X$

Definition 2.5 [3] If $F$ is any fuzzy subset, then convex hull of $F$, is defined as

$$\text{Co}(F) = \bigwedge \{A \in C | F \subseteq A \}.$$ 

Definition 2.6 [2] Assume $X$ is the universe of discourse, $I$ is the unit interval $[0, 1]$ and $N$ is the set of nonnegative integers. Then a fuzzy bag $A$ is a mapping from the product $X \times I$ to $N$ characterized by

$$FC_A(u/x) : X \times I \rightarrow N.$$ 

A fuzzy bag is also denoted as $A(u/x) = [FC_A(u/x) / (u/x)]$ where $FC_A(u/x)$ is the count of $x$ with $u$ as its membership grade.

Definition 2.7 [3] Let $(X, C)$ be a fuzzy convex structure. A nonzero finite fuzzy subset $F$ of $X$ is Helly dependent (or $H$-dependent) if $\land_{\alpha \in F} \text{Co}(F \setminus a_\alpha) \neq 0$ where $F \setminus a_\alpha = F \land a_\alpha$. Otherwise, it is $H$-independent.

The Helly number of $X$ is the smallest $'n'$ such that for each nonzero finite fuzzy subset $F$ of $X$, with cardinality of its support at least $'n+1'$, is Helly dependent. It is denoted by $h(X)$ or by $h$. 
Definition 2.8 [3] Let \((X, C)\) be a fuzzy convex structure. A nonzero finite fuzzy subset \(F\) of \(X\) is Caratheodory dependent (or \(C\)-dependent) provided
\[
\text{Co}(F) \leq \bigvee_{\alpha_a \in F} \text{Co}(F\setminus a) 
\]
Otherwise, it is \(C\)-independent.

The Caratheodory number of \(X\) is the smallest \(n\) such that for each nonzero finite fuzzy subset \(F\) of \(X\), with cardinality of its support at least \(n+1\), is Caratheodory dependent. It is denoted by \(c(X)\) or by \(c\).

Definition 2.9 [3] Let \((X, C)\) be a fuzzy convex structure. A nonzero finite fuzzy subset \(F\) of \(X\) is Exchange dependent (or \(E\)-dependent) if for each \(p\alpha \in F, \text{Co}(F\setminus p) \leq \bigvee (\text{Co}(F\setminus a\beta); a\beta \in F\setminus p; a \neq p))\). Otherwise, it is \(E\)-independent.

The Exchange number of \(X\) is the smallest \(n\) such that for each nonzero finite fuzzy subset \(F\) of \(X\), with cardinality of its support at least \(n+1\), is Exchange dependent. It is denoted by \(e(X)\) or by \(e\).

Definition 2.10 [5] Let \((X, C_1), (Y, C_2)\) be fuzzy convex structures. Let \(f : X \to Y\). Then \(f\) is said to be a fuzzy convexity preserving function (FCP) if for each fuzzy convex set \(K\) in \(Y\), \(f^{-1}(K)\) is a fuzzy convex set in \(X\).

Definition 2.11 [5] Let \((X, C_1), (Y, C_2)\) be fuzzy convex structures. Let \(f : X \to Y\). Then \(f\) is said to be a fuzzy convex to convex function (FCC) if for each fuzzy convex set \(K\) in \(X\), \(f(K)\) is a fuzzy convex set in \(Y\).

Proposition 2.12 [5] Let \(\{(X_i, C_i); i \in I\}\) be a family of fuzzy convex structures. Let \(X = \prod_{i \in I} X_i\) be the product and let \(\pi_i : X \to X_i\) be the projection map. Then \(X\) can be equipped with the fuzzy convexity \(C\) generated by the fuzzy convex sets of the form
\[
\{\pi_i^{-1}(C_i); C_i \in C_i; i \in I\}.
\]
Then \(C\) is called the product fuzzy convexity on \(X\) and \((X, C)\) is called the product fuzzy convexity space.

Proposition 2.13 [5] In a product space, the polytopes are of the product type i.e., for each finite fuzzy subset \(F\) of the product, \(\text{Co}(F) = \prod_{i \in I} \text{Co}(\pi_i(F))\).

Proposition 2.14 [5] The projection map \(\pi_i\) of a product to its factors is both FCP and FCC.

Theorem 2.15 [3] Let \((X, C)\) be a fuzzy convex structure and let \(n < \infty\). If each finite collection of fuzzy convex sets in \(X\) meeting \(n\) by \(n\) has a nonzero intersection then \(h \leq n\).
Theorem 2.16 [3] Let \((X,C)\) be a fuzzy convex structure and let \(n < \infty\). Then \(c \leq n\) iff for each nonzero fuzzy subset \(A \subseteq X\) and \(p_\alpha \in \text{Co}(A)\) there is a fuzzy subset \(F\) of \(A\) with cardinality of its support at most \(n\) and having \(p_\alpha \in \text{Co}(F)\).

Theorem 2.17 [3] Let \((X,C_1),(Y,C_2)\) be fuzzy convex structures. Let \(f : X \rightarrow Y\) be a surjective FCP function. Then \(h(X) \geq h(Y)\).

3 Relationships between fuzzy convex invariants

In this section, we establish some relationships between fuzzy convex invariants in a fuzzy convex product space.

Theorem 3.1 Let \(\{X_i, i \in I\}\) be a family of nonzero fuzzy convex structures and let \(X = \prod_{i \in I} X_i\) be the fuzzy convex product space. Then the Helly number \(h\) of \(X\) is the supremum of the Helly numbers of the factor spaces. i.e., \(h = \text{Sup}\{h_i; i \in I\}\) where \(h_i\)'s are the Helly numbers of the factor spaces.

Proof

Let \(\{(X_i,C_i) i \in I\}\) be a family of nonzero fuzzy convex structures and let \(X = \prod_{i \in I} X_i\) be the product space. Let \(\pi_i : X \rightarrow X_i\) be the projection map which is a surjective FCP function. By Theorem 2.17, a surjective FCP function does not increase the Helly number. Hence \(h \geq h_i; \forall i \in I\). So,

\[
h \geq \text{Sup}\{h_i; i \in I\}.
\]

Next suppose \(n = \text{Sup}\{h_i; i \in I\}\) is finite. Let \(F\) be a fuzzy bag on \(X \times I\) with support\(\{x_1, x_2, ..., x_m\}; m > n\). Let \(F_k = F\backslash\{x_{k_i}\}\). By the product formula for polytopes we have

\[
\text{Co}(F_k) = \prod_{i \in I} \text{Co}(\pi_i(F_k)) = \prod_{i \in I} (\pi_i\text{Co}(F_k))
\]

since the projection map \(\pi_i\) of a product to its factors is both FCP and FCC. Now \(m > n \Rightarrow m > h_i, \forall i\). Hence \(\pi_i(F)\) is Helly dependent. Then by definition, \(\bigwedge_{k=1}^{m} \pi_i\text{Co}(F_k) \neq \emptyset\).

Consider

\[
\bigwedge_{k=1}^{m} \text{Co}(F_k) = \bigwedge_{k=1}^{m} \left\{ \prod_{i \in I} (\pi_i\text{Co}(F_k)) \right\} = \prod_{i \in I} \left\{ \bigwedge_{k=1}^{m} (\pi_i\text{Co}(F_k)) \right\} \neq \emptyset
\]
Then by definition, $F$ is Helly dependent. So $h \leq n$.

i.e., $h \leq \sup \{h_i; i \in I\}$.

From (1) and (2),

$$h = \sup \{h_i; i \in I\}$$

Theorem 3.2 Let $X_1, X_2$ be two fuzzy convex structures and $X = X_1 \times X_2$ be the fuzzy convex product space. If $c_1, c_2$ and $c$ are the Caratheodory numbers of $X_1, X_2$ and $X$ respectively, then $c \leq c_1 + c_2$.

Proof

Let $A$ be a fuzzy subset of $X$ and $p_\alpha \in Co(A)$.

Let $\pi_i : X \to X_i; i = 1, 2$ denote the $i$th projection. Then $\pi_1(A)$ and $\pi_2(A)$ are the projections of $A$ on $X_1$ and $X_2$ respectively. Given that $c_i$ is the Caratheodory number of $X_i$ for $i = 1, 2$. Hence by Theorem 2.16, for each $\pi_i(A) \subseteq X_i$ and $\pi_i(p_\alpha) \in Co(\pi_i(A))$, there is a fuzzy subset $F_i$ of $A$ with cardinality of its support at most $c_i$ such that $\pi_i(p_\alpha) \in Co(\pi_i(F_i))$ for $i = 1, 2$.

But

\[ p_\alpha \in Co(A) \Rightarrow p_\alpha \in Co(\pi_1F_1) \times Co(\pi_2F_2) \text{ where} \]
\[ Co(\pi_1F_1) \times Co(\pi_2F_2) \subseteq Co(\pi_1(F_1 \vee F_2)) \times Co(\pi_2(F_1 \vee F_2)) \]
\[ = Co(F_1 \vee F_2) \]

Thus

$$p_\alpha \in Co(F_1 \vee F_2).$$

Clearly,

$$\#Supp(F_1 \vee F_2) \leq c_1 + c_2$$

i.e., for each fuzzy subset $A \subseteq X$ and $p_\alpha \in Co(A)$, there is a fuzzy subset $F_1 \vee F_2$ of $A$ with cardinality of its support at most $c_1 + c_2$ and having $p_\alpha \in Co(F_1 \vee F_2)$.

Let $c_i \geq e_i$ for at least one of $i = 1, 2$. Then by Theorem 2.16,

$$c \leq c_1 + c_2 - 1.$$
c_i and having \( \pi_i(p_a) \in Co(\pi_i(F_i)) \); \( i = 1, 2 \). Also \( p_a \in Co(F_1 \lor F_2) \).
If \( \#Supp(F_i) < c_i \) for some \( i \) or if \( F_1 \land F_2 \neq \emptyset \) then
\[
\#Supp(F_1 \lor F_2) < c_1 + c_2 \leq c_1 + c_2 - 1.
\]
If \( \#Supp(F_i) = c_i \) for some \( i \) and if \( F_1 \land F_2 = \emptyset \) then take a fuzzy point \( a_\beta \in F_1 \). Since \( \#Supp(F_2 \lor a_\beta) > c_2 \geq c_2 \), the fuzzy subset \( F_2 \lor a_\beta \) is E-dependent. Then by definition,
\[
Co(F_2) \leq \bigvee \{Co((F_2 \lor a_\beta) \setminus b_\gamma) ; b_\gamma \in F_2\}
\]
i.e., \( Co(F_2) \leq \bigvee \{Co(F_2^*)\} \) where \( F_2^* = (F_2 \lor a_\beta) \setminus b_\gamma ; b_\gamma \in F_2 \).
Since \( \pi_i(p_a) \in Co(\pi_i(F_i)) \); \( i = 1, 2 \)
\[
\pi_2(p_a) \in Co(\pi_2(F_2)) \Rightarrow \pi_2(p_a) \in Co(\pi_2(F_2^*))
\]
Hence, \( p_a \in Co(F_1 \lor F_2^*) \) where cardinality of support of \( F_1 \lor F_2^* \) is at most \( c_1 + c_2 - 1 \). Thus,
\[
c \leq c_1 + c_2 - 1.
\]

**Theorem 3.4** Let \( X_1, X_2 \) be two fuzzy convex structures and \( X = X_1 \times X_2 \), the fuzzy convex product space. Suppose \( F_i \subseteq X_i \) be a fuzzy set with \( \#Supp(F_i) > 1 \) for \( i = 1, 2 \). Let \( p_i \in Supp(F_i) \) such that
\[
Co(F_i) \not\leq \bigvee \{Co(F_i \setminus q_a) ; q_a \in F_i ; q \neq p_i\} \text{ for } i = 1, 2.
\]
Then the fuzzy subset \( F = [(p_1) \times (F_2 \setminus p_2)] \cup [(F_1 \setminus p_1) \times \{p_2\}] \) of \( X \) is C-independent.

**Proof**
For \( i = 1, 2 \), consider a fuzzy point \( x_{i\lambda_i} \) in \( Co(F_i) \) but not in the sets
\[
Co(F_i \setminus q_a) ; q_a \in F_i ; q \neq p_i.
\]
i.e., \( x_{i\lambda_i} \in Co(F_i) \setminus \bigvee \{Co(F_i \setminus q_a) ; q_a \in F_i ; q \neq p_i\} \text{ for } i = 1, 2. \) (3)

Let \( x_\lambda = (x_{1\lambda_1}, x_{2\lambda_2}) \). Since \( \pi_i(F) = F_i, i = 1, 2 \) we get \( x_\lambda \in Co(F) \).
Let \( a_\beta \) be a fuzzy point in \( F \). Then either \( a_\beta = (p_1, q_\beta) \) with \( q \neq p_2 \) or \( a_\beta = (q_\beta, p_2) \) with \( q \neq p_1 \). If \( a_\beta = (p_1, q_\beta) \) then \( \pi_2(F \setminus a_\beta) = F_2 \setminus q_\beta \).
But by (3),
\[
x_{2\lambda_2} \notin Co(F_2 \setminus q_\alpha) ; q_\alpha \in F_2 ; q \neq p_2.i.e., x_{2\lambda_2} \notin Co(\pi_2(F \setminus a_\beta)).
\]
Similarly, if \( a_\beta = (q_\beta, p_2) \) with \( q \neq p_1 \), we can show that \( x_{1\lambda_1} \notin Co(\pi_1(F \setminus a_\beta)) \).
Hence in either case,
\[
x_\lambda \notin \bigvee \{Co(F \setminus a_\beta) ; a_\beta \in F\} ; \text{ where } x_\lambda \in Co(F).
\]
Therefore,
\[
Co(F) \not\leq \bigvee \{Co(F \setminus a_\beta) ; a_\beta \in F\}.
\]
So by definition, \( F \) is C-independent.
Theorem 3.5 Let $X_1, X_2$ be two fuzzy convex structures and $X = X_1 \times X_2$ be the fuzzy convex product space. Let $c_i$ and $e_i$ be the Caratheodory number and Exchange number of $X_i$ for $i = 1, 2$. Then the Caratheodory number $c$ of the product space is determined as follows.
i) If $c_i < e_i$ for $i = 1, 2$ then $c = c_1 + c_2$.
ii) If $c_i < e_i$ for exactly one of $i = 1, 2$ then $c = c_1 + c_2 - 1$.

Proof
i) In Theorem 3.2, it is proved that
\[ c \leq c_1 + c_2. \] (4)
Since $e_i > c_i$ for $i = 1, 2$ for each $i$, there is a nonexchangeable fuzzy set $F_i \subseteq X_i$ with cardinality of its support $> c_i$. In other words, $F_i$ for $i = 1, 2$ is $E$-independent. Then, by definition, for each $i$, there is a point $p_i \in Supp(F_i)$ such that
\[ Co(F_i \setminus p_i) \not\subseteq \bigvee \{Co(F_i \setminus q_\alpha); q_\alpha \in F_i; q \neq p_i\}. \]

Hence
\[ Co(F_i) \not\subseteq \bigvee \{Co(F_i \setminus q_\alpha); q_\alpha \in F_i; q \neq p_i\} \] for $i = 1, 2$.

Then by Theorem 3.4, the fuzzy subset
\[ F = [(p_1) \times (F_2 \setminus p_2)] \bigvee [(F_1 \setminus p_1) \times \{p_2\}] \]
of $X$ is $C$-independent. Since the cardinality of support of $F$ is at least $c_1 + c_2$, we have
\[ c \geq c_1 + c_2. \] (5)
Combining (4) and (5), $c = c_1 + c_2$.

ii) Suppose $c_1 < e_1$ and $c_2 \geq e_2$.
As $c_2 \geq e_2$, by Theorem 3.3,
\[ c \leq c_1 + c_2 - 1. \] (6)
Since $e_1 > c_1$, there is a nonexchangeable fuzzy set $F_1 \subseteq X_1$ with cardinality of its support at least $c_1 + 1$. i.e., $F_1$ is $E$-independent.
So, there is a point $p_1 \in Supp(F_1)$ such that
\[ Co(F_1 \setminus p_1) \not\subseteq \bigvee \{Co(F_1 \setminus q_\alpha); q_\alpha \in F_1; q \neq p_1\}. \]
Hence
\[ Co(F_1) \not\subseteq \bigvee \{Co(F_1 \setminus q_\alpha); q_\alpha \in F_1; q \neq p_1\}. \]
Since $c_2 \geq e_2$, there is a $C$-independent fuzzy set $F_2 \subseteq X_2$, with cardinality of its support as $c_2$ and having a point $p_2 \in Supp(F_2)$ such that
\[ Co(F_2) \not\subseteq \bigvee \{Co(F_2 \setminus q_\alpha); q_\alpha \in F_2; q \neq p_2\}. \]
Then by Theorem 3.4, the fuzzy subset
\[ F = [(p_1 \times (F_2 \setminus p_2))] \cup [(F_1 \setminus p_1) \times \{p_2\}] \]
of \(X\) is \(C\)-independent. As the cardinality of support of \(F\) is at least \(c_1 + c_2 - 1\),
\[ c \geq c_1 + c_2 - 1. \tag{7} \]
Combining (6) and (7),
\[ c = c_1 + c_2 - 1. \]

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**References**


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