**g-(M, μ)-Frames in Hilbert Spaces**

Xiyan Yao

Department of Applied Mathematics, Yuncheng College
Yuncheng 044000, P. R. China
yaoxiyan63@163.com

**Abstract**

Let \((M, \mu)\) be a measure space, \(U\) and \(V\) be two Hilbert Spaces. In this paper, we introduce and discuss \(g-(M, \mu)\)-frames \(\{\Lambda_m\}_{m \in M}\) for a Hilbert Space \(U\) with respect to a family of subspaces \(\{V_m\}_{m \in M}\) of \(V\) and then we characterize the properties of \(g-(M, \mu)\)-frames and dual of \(g-(M, \mu)\)-frames. Finally, we generalize the perturbations of \((M, \mu)\)-frames and \(g\)-frames to \(g-(M, \mu)\)-frames and derive some meaningful results.

**Mathematics Subject Classification:** 41A58, 42C99, 42C15

**Keywords:** \((M, \mu)\)-frame, \(g\)-frame, \(g-(M, \mu)\)-frame, dual \(g-(M, \mu)\)-frame, perturbation

**1 Introduction**

Frames for Hilbert spaces were first introduced by Duffin and Schaeffer [1] in 1952 to study some deep problems in nonharmonic Fourier analysis. They were reintroduced and developed in 1986 by Daubechies, Grassmann and Meyer [2], and popularized from them on, later generalized frame theory or \((M, \mu)\)-frame theory was introduced by Kaiser [3] as a natural generalization of frame theory in Hilbert spaces. In [4], Jean-Pierre Gabardo and Deguang Han studied the dilation and perturbation properties of \((M, \mu)\)-frames. Let \((M, \mu)\)be a measure space and \(U\) be a Hilbert space. Recall that mapping \(F : M \rightarrow U\) is called a \((M, \mu)\)-frame if it is weakly-measurable and if there exist two constants \(A, B > 0\) such that

\[
A \| f \| ^2 \leq \int_\mu \| \langle f, F(m) \rangle \| ^2 d\mu(m) \leq B \| f \| ^2 \quad (1.1)
\]

1 Supported by the emphasis Subject Science Foundation of Shanxi Province of China (Grant No.20091028) and the Science Foundation of Yuncheng University of China(Grant No.2009003)
holds for all \( f \in U \). The constants \( A \) and \( B \) are called the lower and upper frame bounds, respectively. A frame \( F \) is called tight if \( A = B \) and Parseval if \( A = B = 1 \). The mapping \( F \) is called \((M, \mu)\)-Bessel if the second inequality in (1.1) holds.

Recently, the theory of frames for Hilbert spaces has been generalized in several directions (see [5-7]) and applied to signal processing, image processing, and sampling theory and so on (see [8,9]). In [10], Wenchang Sun introduced g-frames for Hilbert spaces, which generalize the concepts of frames [11], Pseudoframes [12], oblique frame [13], frames of subspace [14]. Let \( U, V \) be two Hilbert spaces and let \( I \) be finite (or countable) index set. \( \{V_i\}_{i \in I} \) is a family of subspaces of \( V_i \), and \( L(U, V) \) is the collection of all bounded linear operators from \( U \) into \( V_i \). A family \( \{\Lambda_i : i \in I\} \) in \( L(U, V) \) is called a g-frame for \( U \) with respect to \( \{V_i : i \in I\} \), if there exist \( 0 < C \leq D < \infty \) such that for all \( f \in U \),

\[
C \| f \|^2 \leq \sum_{i \in I} \| \Lambda_i f \|^2 \leq D \| f \|^2, \quad \forall f \in U.
\]

The number \( C \) and \( D \) are called the lower and upper frame bounds. A g-frame \( \{\Lambda_i : i \in I\} \) is called tight if \( C = D \) and Parseval if \( C = D = 1 \). If only the right-hand inequality of (1.2) is satisfied, we call \( \{\Lambda_i : i \in I\} \) the g-Bessel sequence for \( U \) with respect to \( \{V_i : i \in I\} \) with Bessel bound \( D \).

Let \( \{\Lambda_i \in L(U, V_i) : i \in I\} \) be a g-frame. Then the g-frame operator

\[
S(f) = \sum_{i \in I} \Lambda_i^* \Lambda_i f, \quad f \in U
\]

associated with \( \{\Lambda_i : i \in I\} \) is a bounded, invertible and positive operator mapping \( U \) onto itself.

In the paper, we will introduce the concept of g-\((M, \mu)\)-frame for a Hilbert space \( U \) with respect to a family \( \{V_m\} \) of closed subspaces of \( V \). If \( M \) is finite (or countable) index set, then a g-\((M, \mu)\) frame \( \{\Lambda_m \}_{m \in M} \) for a Hilbert space \( U \) with respect to a family \( \{V_m\}_{m \in M} \) of subspaces of a Hilbert space \( V \) to be essentially a g-frame for \( U \) with respect to \( \{V_m\}_{m \in M} \). Hence, the concept of g-\((M, \mu)\)-frame for Hilbert spaces generalizes almost all of the concepts of frames.

The paper is organized as follows. In Section 2, we give some notations and some basic properties of g-\((M, \mu)\)-frames which we will use in later sections. In Section 3, we characterize dual of g-\((M, \mu)\)-frames. In Section 4, we extend the perturbations of \((M, \mu)\)-frames and g-frames to g-\((M, \mu)\)-frames in Hilbert spaces.
2 \( g_-(M, \mu) \)-frames for Hilbert Spaces

We first give some notations. Throughout this paper, let \((M, \mu)\) be a measure space, \(U\) and \(V\) be two Hilbert spaces and \(\{V_m\}_{m \in M}\) is a family of subspaces of \(V\), and \(L(U, V_m)\) is the collection of all bounded linear operators from \(U\) into \(V_m\). Put

\[
L^2(V_m, M) = \{\{f_m\}_{m \in M}, f_m : M \rightarrow U \mid \int_M \| f_m \|^2 d\mu(m) < \infty\},
\]

with the inner product given by

\[
\langle\{f_m\}_{m \in M}, \{g_m\}_{m \in M}\rangle = \int_M \langle f_m, g_m \rangle d\mu(m), \forall \{f_m\}, \{g_m\} \in L^2(V_m, M).
\]

It is clear that \(L^2(V_m, M)\) is a Hilbert space with respect to the operation.

**Definition 2.1** We call a family \(\{\Lambda_m \in L(U, V_m) : m \in M\}\) a generalized \((M, \mu)\)-frame, or simply a \(g_-(M, \mu)\)-frame, for \(U\) with respect to \(\{V_m\}_{m \in M}\). If for each \(f \in U\), \(\Lambda_m f \in L^2(V_m, M)\), and if there are two positive constants \(A\) and \(B\), such that

\[
A \| f \|^2 \leq \int_M \| \Lambda_m f \|^2 d\mu(m) \leq B \| f \|^2, \forall f \in U.
\]

We call \(A\) and \(B\) the lower and upper \(g_-(M, \mu)\)-frame bounds, respectively.

We call \(\{\Lambda_m : m \in M\}\) a tight \(g_-(M, \mu)\)-frame if \(A = B\) and a Parseval \(g_-(M, \mu)\)-frame if \(A = B = 1\).

We call this family \(\{\Lambda_m : m \in M\}\) a \(g_-(M, \mu)\)-frame for \(U\) with respect to \(V\) whenever \(V_m = V, \forall m \in M\).

**Example 2.2** Let \(U\) be a Hilbert space and \(\{f_m\}_{m \in M}\) be a \((M, \mu)\)-frame for \(U\). Let \(\Lambda f_m\) be the function induced by \(f_m\), i.e., \((\Lambda f_m)f = \langle f, f_m \rangle, f \in U\).

It is easy to check that \(\{\Lambda f_m\}_{m \in M}\) is a \(g_-(M, \mu)\)-frame for \(U\) with respect to \(\mathbb{C}\).

**Remark 2.3** For any \(m \in M\), if we let \(V_m = V = \mathbb{C}\) and \((\Lambda f_m)f = \langle f, f_m \rangle, f \in U\), and \(\{f_m\}_{m \in M}\) is weekly measure, in this case the \(g_-(M, \mu)\)-frame \(\{\Lambda f_m\}_{m \in M}\) is just a \((M, \mu)\)-frame for \(U\).

**Definition 2.4** Let \(\Lambda = \{\Lambda_m\}_{m \in M}\) be a \(g_-(M, \mu)\)-frame for \(U\) with respect to \(\{V_m\}_{m \in M}\). The synthesis operator for \(\{\Lambda_m\}_{m \in M}\) is the linear operator,

\[
T_\Lambda : L^2(V_m, M) \rightarrow U, T_\Lambda(\{f_m\}_{m \in M}) = \int_M \Lambda_m^*(f_m)d\mu(m).
\]

We call the adjoint \(T_\Lambda^*\) of the synthesis operator, the analysis operator.

\[
T_\Lambda^* : U \rightarrow L^2(V_m, M), T_\Lambda^*(f) = \{\Lambda_m f : m \in M\}.
\]
Proposition 2.5 Let \( \{ \Lambda_m \in L(U, V_m) : m \in M \} \) be a family of linear operators and for each \( f \in U \), \( \Lambda_m f \in L^2(V_m, M) \). The following conditions are equivalent.

1. \( \{ \Lambda_m : m \in M \} \) be a g-\((M, \mu)\)-frame for \( U \) with respect to \( \{ V_m \}_{m \in M} \).
2. The synthesis operator \( T_\Lambda \) is bounded, linear and onto.
3. The analysis operator \( T_\Lambda^* \) is bounded and one-to-one (\( \sqrt{A} \leq \| T_\Lambda^* \| \leq \sqrt{B} \)).

Proof Since \( \| T_\Lambda^* f \|^2 = \int_M \| \Lambda_m f \|^2 \, d\mu(m) \), (1) and (3) are equivalent. (2) and (3) are equivalent for each operator on a Hilbert space \( U \).

Given a g-\((M, \mu)\)-frame with the frame bounds \( A \) and \( B \), composing \( T_\Lambda \) with the adjoint operator \( T_\Lambda^* \), we get the g-\((M, \mu)\)-frame operator \( S : U \to U, Sf = T_\Lambda T_\Lambda^* f = \int_M \Lambda_m^* \Lambda_m f \, d\mu(m), f \in U \). Clearly, \( S \) is a linear operator on \( U \) and we have

\[
A \| f \|^2 \leq \langle Sf, f \rangle = \int_M \| \Lambda_m f \|^2 \, d\mu(m) \leq B \| f \|^2, \forall f \in U.
\]

Proposition 2.6 Let \( \{ \Lambda_m \in L(U, V_m) : m \in M \} \) be a family of operators and for each \( f \in U \), \( \Lambda_m f \in L^2(V_m, M) \). We define an operator \( S : U \to U \),

\[
Sf = \int_M \Lambda_m^* \Lambda_m f \, d\mu(m), f \in U
\]

(the formula (2.2) is well-defined in the weak sense in \( U \)). Then the family \( \{ \Lambda_m \in L(U, V_m) : m \in M \} \) is a g-\((M, \mu)\)-frame if and only if \( S \) is bounded, positive and invertible operator.

Proof Let \( \{ \Lambda_m \in L(U, V_m) : m \in M \} \) be a g-\((M, \mu)\)-frame for \( U \) with respect to \( \{ V_m \}_{m \in M} \). By (2.2), it is clear that \( S \) is a positive operator. We only need to prove that \( S \) is bounded invertible operator. It is easy to check that for any \( f_1, f_2 \in U \),

\[
\langle Sf_1, f_2 \rangle = \int_M \langle \Lambda_m^* \Lambda_m f_1, f_2 \rangle \, d\mu(m) = \int_M \langle f_1, \Lambda_m^* \Lambda_m f_2 \rangle \, d\mu(m) = \langle f_1, Sf_2 \rangle.
\]

and therefore

\[
\| S \| = \sup_{\| f \|=1} \langle Sf, f \rangle = \sup_{\| f \|=1} \int_M \| \Lambda_m f \|^2 \, d\mu(m) \leq B.
\]

Hence \( S \) is a bounded operator. Since \( A \| f \|^2 \leq \langle Sf, f \rangle \leq \| Sf \| \cdot \| f \| \), \( \forall f \in U \), we have \( \| Sf \| \geq A \| f \| \), which implies that \( S \) is injective and \( SU \) is closed in \( U \). Let \( f_2 \in U \) be such that \( \langle Sf_1, f_2 \rangle = 0 \) for every \( f_1 \in U \). Then we have \( \langle f_1, Sf_2 \rangle = 0, \forall f_1 \in U \). This implies that \( Sf_2 = 0 \) and therefore \( f_2 = 0 \). Hence \( SU = U \). Consequently, \( S \) is invertible and \( \| S^{-1} \| \leq \frac{1}{A} \).
Conversely, we assume that $S$ define by (2.2) is a bounded, positive and invertible operator, then for all $f \in U$, we have that
\[
\Delta(S) \cdot \| f \|^2 \leq \langle Sf, f \rangle \leq \| S \| \cdot \| f \|^2.
\]
Hence, by definition 2.1, the family of operators $\{\Lambda_m\}_{m \in M}$ is a $g$-$(M,\mu)$-frame with the frame bounds $\Delta(S)$ and $\| f \|$. Where $\Delta(S) = \inf\{\| Sf \| : f \in U, \| f \| = 1\}$ denote the minimal module of $S$.

3 Dual of $g$-$(M,\mu)$-frames

Definition 3.1 Let $\{\Lambda_m\}_{m \in M}$ and $\{\Theta_m\}_{m \in M}$ be two $g$-$(M,\mu)$-frames for a Hilbert space $U$ with respect to $\{V_m\}_{m \in M}$ such that
\[
f = \int_M \Lambda^*_m \Theta_m f d\mu(m), \forall f \in U,
\]
then $\{\Theta_m\}_{m \in M}$ is called an alternate dual of $\{\Lambda_m\}_{m \in M}$.

We have the following situation which shows that if $\{\Theta_m\}_{m \in M}$ is an alternate dual of $\{\Lambda_m\}_{m \in M}$, then $\{\Lambda_m\}_{m \in M}$ is an alternate dual of $\{\Theta_m\}_{m \in M}$.

Proposition 3.2 Let $\Lambda = \{\Lambda_m\}_{m \in M}$ and $\Theta = \{\Theta_m\}_{m \in M}$ be $g$-$(M,\mu)$-frames for a Hilbert space $U$ with respect to $\{V_m\}_{m \in M}$ such that $f = \int_M \Lambda^*_m \Theta_m f d\mu(m), \forall f \in U$, then for each $f \in U, f = \int_M \Theta^*_m \Lambda_m f d\mu(m)$.

Proof Let us define $T : U \rightarrow U$ by $Tf = \int_M \Theta^*_m \Lambda_m f d\mu(m)$. If the upper $g$-$(M,\mu)$-frame bounds of $\{\Lambda_m\}_{m \in M}$ and $\{\Theta_m\}_{m \in M}$ are $B$ and $D$, respectively, then
\[
\| T \| = \sup_{\| f \| = 1} \| (Tf, f) \|
\leq \sup_{\| f \| = 1} \left( \int_M \| \Lambda_m f \|^2 d\mu(m) \right)^{\frac{1}{2}} \left( \int_M \| \Theta_m f \|^2 d\mu(m) \right)^{\frac{1}{2}} \leq (BD)^{\frac{1}{2}}.
\]

So $T \in L(U, U)$. For $f, g \in U$, we have
\[
\langle Tf, g \rangle = \int_M \langle \Theta^*_m \Lambda_m f, g \rangle d\mu(m) = \int_M \langle \Lambda_m f, \Theta_m g \rangle d\mu(m),
\]
and
\[
\langle f, g \rangle = \int_M \Theta^*_m \Lambda_m g d\mu(m) = \int_M \langle \Theta_m f, \Lambda_m g \rangle d\mu(m).
\]
So $\langle Tf, g \rangle = \langle f, g \rangle$ for all $f, g \in U$, which implies that $T = I$.

Proposition 3.3 Let $\Lambda = \{\Lambda_m\}_{m \in M}$ be a $g$-$(M,\mu)$-frame for $U$ with respect to $\{V_m\}_{m \in M}$ with frame bounds $A$ and $B$, $S$ is the $g$-$(M,\mu)$-frame operator of $\Lambda$. Put $\Theta = \{\Theta_m\}_{m \in M} = \{\Lambda_m S^{-1}\}_{m \in M}$, then $\Theta$ is a $g$-$(M,\mu)$-frame for $U$ and
$g$-$(M, \mu)$-frame bounds $\frac{1}{B}, \frac{1}{A}$.

Proof. For any $f \in U$, we have

\[
\int_M \| \Theta_m f \|^2 \, d\mu(m) = \int_M \| \Lambda_m S^{-1} f \|^2 \, d\mu(m) \\
= \int_M \langle \Lambda_m S^{-1} f, \Lambda_m S^{-1} f \rangle \, d\mu(m) \\
= \int_M \langle \Lambda_m^{*} \Lambda_m S^{-1} f, S^{-1} f \rangle \, d\mu(m) \\
= \langle SS^{-1} f, S^{-1} f \rangle = \langle f, S^{-1} f \rangle \leq \frac{1}{A} \| f \|^2.
\]

On the other hand, since

\[
\| f \|^2 = \int_M \langle \Theta_m^{*} \Lambda_m f, f \rangle \, d\mu(m) \\
= \int_M \langle \Lambda_m f, \Theta_m f \rangle \, d\mu(m) \\
\leq (\int_M \| \Lambda_m f \|^2 \, d\mu(m))^\frac{1}{2} (\int_M \| \Theta_m f \|^2 \, d\mu(m))^\frac{1}{2} \\
\leq B^\frac{1}{2} \| f \| (\int_M \| \Theta_m f \|^2 \, d\mu(m))^\frac{1}{2},
\]

we have

\[
(\int_M \| \Theta_m f \|^2 \, d\mu(m))^\frac{1}{2} \geq \frac{1}{B} \| f \|.
\]

Hence, $\Theta = \{ \Theta_m \}_{m \in M}$ is a $g$-$(M, \mu)$-frame for $U$ with frame bounds $\frac{1}{B}$ and $\frac{1}{A}$. We call it the canonical dual $g$-$(\Omega, \mu)$-frame of $\{ \Lambda_m \}_{m \in M}$.

If $\{ \Lambda_m \}_{m \in M}$ is a $g$-$(M, \mu)$-frame for $U$ with respect to $\{ V_m \}_{m \in M}$, then any vector $f \in U$ can be represented as:

\[
f = \int_M \Lambda_m^{*} \Lambda_m S^{-1} f \, d\mu(m) = \int_M S^{-1} \Lambda_m^{*} \Lambda_m f \, d\mu(m), \forall f \in U. \tag{2.3}
\]

Where $S^{-1}$ is the inverse of the $g$-$(M, \mu)$-frame operator $S$ on $U$.

Let $\{ \Lambda_m \}_{m \in M}$ be a $g$-$(M, \mu)$-frame for $U$ with respect to $\{ V_m \}_{m \in M}$ and $S$ be the $g$-$(M, \mu)$-frame operator of $\{ \Lambda_m \}_{m \in M}$. By (2.3), $\{ \Lambda_m S^{-1} \}_{m \in M}$ is an alternate dual of $\{ \Lambda_m \}_{m \in M}$, $\{ \Lambda_m \}_{m \in M}$ is an alternate dual of $\{ \Lambda_m S^{-1} \}_{m \in M}$.

4 Perturbation of $g$-$(M, \mu)$-frames

The following theorems characterize perturbations of $g$-$(M, \mu)$-frames. It generalized some results of $(M, \mu)$-frames and g-frames.
Theorem 4.1 Let \( \{ \Lambda_m \}_{m \in M} \) be a \( g-(M, \mu) \)-frame for \( U \) with respect to \( \{ V_m \}_{m \in M} \) with the frame bounds \( A \) and \( B \). If any family of operators \( \{ \Theta_m \in L(U, V_m) : m \in M \} \) satisfied.

(1) For every \( f \in U \), \( \Theta_m f \in L^2(V_m, M), \forall m \in M \).

(2) \( R = \int_M \| \Lambda_m - \Theta_m \|^2 d\mu(m) < A \).

Then \( \{ \Theta_m \}_{m \in M} \) is a \( g-(M, \mu) \)-frame for \( U \) with respect to \( \{ V_m \}_{m \in M} \) with the frame bounds \( (A^{\frac{1}{2}} - R^{\frac{1}{2}})^2 \) and \( (B^{\frac{1}{2}} + R^{\frac{1}{2}})^2 \).

Proof For a family of operators \( \{ \Theta_m \}_{m \in M} \), the operator \( S \) defined by

\[
S f = \int_M \Theta_m^* \Theta_m f d\mu(m), \forall f \in U.
\]

It follows from (1) that the operator \( S \) is well-defined in the week sense in \( U \). By Proposition 2.6, to prove that \( \{ \Theta_m \}_{m \in M} \) is a \( g-(M, \mu) \)-frame, it is sufficient to show that \( S \) is a bounded, positive and invertible operator. Clearly, \( S \) is a positive operator. Next we firstly prove that \( S \) is bounded below. In fact, for all \( f \in U \),

\[
\langle Sf, f \rangle^{\frac{1}{2}} = \left( \int_M \| \Theta_m f \|^2 d\mu(m) \right)^{\frac{1}{2}} \\
\geq \left( \frac{1}{2} \int_M \| \Lambda_m f \|^2 d\mu(m) \right)^{\frac{1}{2}} - \left( \int_M \| \Theta_m f \|^2 d\mu(m) \right)^{\frac{1}{2}} \\
\geq A^{\frac{1}{2}} \| f \| \geq \left( \frac{1}{2} \int_M \| \Lambda_m f \|^2 d\mu(m) \right)^{\frac{1}{2}} \| f \| \\
= (A^{\frac{1}{2}} - R^{\frac{1}{2}}) \| f \|,
\]

by (2), we have get \( A^{\frac{1}{2}} - R^{\frac{1}{2}} > 0 \), so

\[
\langle Sf, f \rangle \geq (A^{\frac{1}{2}} - R^{\frac{1}{2}})^2 \| f \|^2, \forall f \in U. \tag{4.1}
\]

Secondly, we estimate a upper bound of \( S \), for all \( f \in U \), we have that

\[
\langle Sf, f \rangle^{\frac{1}{2}} = \left( \int_M \| \Theta_m f \|^2 d\mu(m) \right)^{\frac{1}{2}} \\
\leq \left( \frac{1}{2} \int_M \| \Lambda_m f \|^2 d\mu(m) \right)^{\frac{1}{2}} + \left( \int_M \| \Theta_m f \|^2 d\mu(m) \right)^{\frac{1}{2}} \\
\leq \left( \frac{1}{2} \int_M \| \Lambda_m f \|^2 d\mu(m) \right)^{\frac{1}{2}} + \left( \int_M \| \Theta_m f \|^2 d\mu(m) \right)^{\frac{1}{2}} \\
\leq B^{\frac{1}{2}} \| f \| \geq \left( \frac{1}{2} \int_M \| \Lambda_m f \|^2 d\mu(m) \right)^{\frac{1}{2}} \| f \| \\
= (B^{\frac{1}{2}} + R^{\frac{1}{2}}) \| f \|.
\]
Then \( \langle Sf, f \rangle \leq (B^2 + R^2)^2 \| f \|^2, \forall f \in U. \) \hfill (4.2)

Combining (4.1) and (4.2), we get that
\[
(A^2 - R^2)^2 \| f \|^2 \leq \langle Sf, f \rangle \leq (B^2 + R^2)^2 \| f \|^2, \forall f \in U.
\]

**Theorem 4.2** Let \( \{\Lambda_m\}_{m \in M} \) be a \( g-(M, \mu) \)-frame for \( U \) with respect to \( \{V_m\}_{m \in M} \) with the frame bounds \( A \) and \( B \). If any family of operators \( \{\Theta_m \in \mathcal{L}(U, V_m) : m \in M\} \) such that.

1. For every \( f \in U \), \( \Theta_m(f) \in L^2(V_m, M), \forall m \in M \).
2. There exists a constant \( K > 0 \), such that for any \( f \in U \), we have
\[
\| \{\Lambda_m f\}_{m \in M} \|^2 - \| \{\Theta_m f\}_{m \in M} \|^2 \leq K \| f \|^2, \forall f \in U.
\]

Then \( \{\Theta_m\}_{m \in M} \) is a \( g-(M, \mu) \)-frame for \( U \) with respect to \( \{V_m\}_{m \in M} \) with the frame bounds \( A - K \) and \( B + K \).

**Proof** By the formula (4.3), we get that
\[
\| \{\Lambda_m f\}_{m \in M} \|^2 - \| \{\Theta_m f\}_{m \in M} \|^2 \leq K \| f \|^2, \forall f \in U.
\]
Thus
\[
\| \{\Theta_m f\}_{m \in M} \|^2 \geq \| \{\Lambda_m f\}_{m \in M} \|^2 - K \| f \|^2 = (A - K) \| f \|^2.
\]

Similarly
\[
\| \{\Theta_m f\}_{m \in M} \|^2 - \| \{\Lambda_m f\}_{m \in M} \|^2 \leq K \| f \|^2, \forall f \in U.
\]
So
\[
\| \{\Theta_m f\}_{m \in M} \|^2 \leq K \| f \|^2 + \| \{\Lambda_m f\}_{m \in M} \|^2 \leq (B + K) \| f \|^2, \forall f \in U.
\]

**Theorem 4.3** Let \( \{\Lambda_m\}_{m \in M} \) be a \( g-(M, \mu) \)-frame for \( U \) with respect to \( \{V_m\}_{m \in M} \) with the frame bounds \( A \) and \( B \). Let \( \Theta_m \in \mathcal{L}(U, V_m) \) for any \( m \in M \), and for every \( f \in U \), \( \Theta_m(f) \in L^2(V_m, M) \). Then the following statements are equivalent.

1. \( \{\Theta_m\}_{m \in M} \) is a \( g-(M, \mu) \)-frame for \( U \) with respect to \( \{V_m\}_{m \in M} \).
2. There exists a constant \( H > 0 \), such that for any \( f \in U \), we have
\[
\| \{(\Lambda_m - \Theta_m)f\}_{m \in M} \|^2 \leq H \min(\| \Lambda_m f \|^2, \| \Theta_m f \|^2). \]
Proof First we let $\{\Theta_m\}_{m \in M}$ be a $g-(M, \mu)$-frame with frame bounds $C$ and $D > 0$. Then for any $f \in U$, we have
\[
\| \{ (\Lambda_m - \Theta_m) f \}_{m \in M} \|^2 \leq \| \{ \Lambda_m f \}_{m \in M} \|^2 + \| \{ \Theta_m f \}_{m \in M} \|^2 \\
\leq \| \{ \Lambda_m f \}_{m \in M} \|^2 + D \| f \|^2 \\
\leq \| \{ \Lambda_m f \}_{m \in M} \|^2 + \frac{D}{A} \| \{ \Lambda_m f \}_{m \in M} \|^2 \\
= \left(1 + \frac{D}{A}\right) \| \{ \Lambda_m f \}_{m \in M} \|^2 .
\]
Similarly, we can obtain
\[
\| \{ (\Lambda_m - \Theta_m) f \}_{m \in M} \|^2 \leq (1 + \frac{B}{C}) \| \{ \Theta_m f \}_{m \in M} \|^2 .
\]
Let putting $H = \min(1 + \frac{D}{A}, 1 + \frac{B}{C})$, the (4.4) holds. Next we suppose that (4.4) holds, for any $f \in U$, we have
\[
A \| f \|^2 \leq \| \{ (\Lambda_m - \Theta_m) f \}_{m \in M} \|^2 + \| \{ \Theta_m f \}_{m \in M} \|^2 \\
\leq H \| \{ \Theta_m f \}_{m \in M} \|^2 + \| \{ \Theta_m f \}_{m \in M} \|^2 \\
= \left(H + 1\right) \| \{ \Theta_m f \}_{m \in M} \|^2 .
\]
Also we obtain
\[
\| \{ \Theta_m f \}_{m \in M} \|^2 \leq \| \{ (\Lambda_m - \Theta_m) f \}_{m \in M} \|^2 + \| \{ \Lambda_m f \}_{m \in M} \|^2 \\
\leq H \| \{ \Lambda_m f \}_{m \in M} \|^2 + \| \{ \Lambda_m f \}_{m \in M} \|^2 \\
= \left(1 + H\right) \| \{ \Lambda_m f \}_{m \in M} \|^2 \\
\leq B(1 + H) \| f \|^2 .
\]
Combining above two formulas, we have $\{\Theta_m\}_{m \in M}$ is a $g-(M, \mu)$-frame for $U$ with respect to $\{V_m\}_{m \in M}$ with $g-(M, \mu)$-frame bounds $\frac{A}{H+1}$ and $B(H + 1)$.

References


Received: April, 2011