On Homogeneous Quasi-Translations 
in Dimension Five

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Abstract

Let $F = x + H$ be a homogeneous quasi-translation of degree $d < 15$ in dimension 5. We show that the components of $H$ are linearly dependent, or equivalently a homogeneous nice derivation of degree $d < 15$ in dimension 5 contains a linear coordinate in its kernel.

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1 Introduction

A polynomial map $F = x + H : \mathbb{C}^n \to \mathbb{C}^n$ is called a quasi-translation if $F$ is invertible and $F^{-1} = x - H$, and $F$ is called homogeneous of degree $d$, if each $H_i$ is homogeneous of degree $d$. A quasi-translation can be seen as a special type of locally nilpotent derivation—the so-called nice derivation. A derivation $D$ on $\mathbb{C}[x_1, \ldots, x_n]$ is called nice if $D^2(x_i) = 0, 1 \leq i \leq n$, see [6, Section 3.2.1]. (Note that this definition of nice derivation is not the same as the original one in [4, Section 7.3].) The following proposition describes the relation between quasi-translations and nice derivations.

**Proposition 1.1** [1, Proposition 1.1] Let $F = x + H : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial map and $D_H = H_1\partial_1 + \cdots + H_n\partial_n$, where $\partial_i = \frac{\partial}{\partial x_i}$. Then

1. $F$ is a (homogeneous) quasi-translation if and only if $JH \cdot H = 0$, if and only if $D_H$ is a (homogeneous) nice derivation;

2. The components of $H$ are linearly dependent over $\mathbb{C}$ if and only if $\ker D_H$ (the kernel of $D_H$) contains a linear coordinate. (A coordinate means a polynomial $f$ with $\mathbb{C}[f, f_2, \ldots, f_n] = \mathbb{C}[x_1, \ldots, x_n]$ for some $f_2, \ldots, f_n$.)
The studies of quasi-translations date back to 1876. At the time, P. Gordan and M. Nöther [7] studied them in order to understand singular Hessians better. Among other things, they showed that the components of $H$ are linearly dependent for any homogeneous quasi-translation $x + H$ in dimension 4. The interests in quasi-translations and singular Hessians arose again in the past few years, since in 2005, de Bondt and van den Essen [3] showed that it suffices to study the famous Jacobian conjecture for all homogeneous polynomial maps $x + H$ with $JH$ a nilpotent Hessian. And in 2006, de Bondt [1] found that, for every dimension $n \geq 6$, there exists a homogeneous quasi-translation $x + H$ such that the components of $H$ are linearly independent. This gave counterexamples to the Homogeneous Dependence Problem, which arose in the research of the Jacobian conjecture. However the following problem is still open.

**Problem 1.2** Let $F = x + H$ be a homogeneous quasi-translation in dimension 5, say of degree $d$. Are the components of $H$ linear dependent?

A description of the structure of (possible) counterexamples to Problem 1.2 was given by de Bondt, and he gave an affirmative answer to this problem for $d \leq 8$, see [2, Theorem 3.5.6]. The difficulty grows rapidly when the degree $d$ increases. A bottle of Joustra Beerenburg (Frisian spirit) will be offered by de Bondt to the one who first solves Problem 1.2.

In this paper, we show that Problem 1.2 has an affirmative answer for $d < 15$, or equivalently, in dimension 5, every homogeneous nice derivation of degree $d < 15$ contains a linear coordinate in its kernel. In contrast to our result, it is well-known that every locally nilpotent derivation on $\mathbb{C}[x_1, x_2]$ contains a coordinate in its kernel (due to the work of Rentschler [8]). Freudenburg [5] showed that there exist homogeneous locally nilpotent derivations on $\mathbb{C}[x_1, x_2, x_3]$ which contain no coordinates in its kernel, and Sun [9] showed that, in dimension no more than 9, a quadratic homogeneous nice derivation contains a linear coordinate in its kernel.

## 2 Quasi-translations in dimension 5

A quasi-translation $F = x + G$ is called irreducible if $\gcd\{G_1, \ldots, G_n\} = 1$. If $F = x + H$ is a quasi-translation, then $x + g^{-1}H$ is an irreducible quasi-translation, where $g = \gcd\{H_1, \ldots, H_n\}$ (cf. [2, Proposition 3.2.3]). In addition, if $F' = x + H'$ is conjugate to a quasi-translation $F = x + H$, i.e., $F' = T^{-1} \circ F \circ T$ for some invertible linear map $T \in \text{Gl}_n(\mathbb{C})$, then $F'$ is a quasi-translation as well, where $\circ$ means the composition.

In Section 3.5 of his Ph.D. thesis, de Bondt [2] investigated homogeneous quasi-translations in dimension 5, and he showed the following result.
Theorem 2.1 [2, Theorem 3.5.6] Let $F = x + H$ be an irreducible homogeneous quasi-translation in dimension 5. If the components of $H$ are linearly independent, then up to linear conjugation, $H$ is of the form

$$H = (h_1(p, q), h_2(p, q), h_3(p, q), h_4(p, q), r),$$

(2.1)

with the condition:

$h_i \in \mathbb{C}[y_1, y_2], i = 1, \ldots, 4$, are homogeneous of the same degree $s \geq 3$;

$p, q \in \mathbb{C}[x]$ are homogeneous of the same degree $t \geq 3$; $p, q, r$ are algebraically independent; $0 < \deg x_5 p < t - 1$, $0 < \deg x_5 q < t - 1$.

Theorem 2.1 ensures that Problem 1.2 has an affirmative answer for $d \leq 8$, and to study Problem 1.2 one only need to consider quasi-translations of the form (2.1) in Theorem 2.1.

In what follows, unless otherwise mentioned, $F = x + H$ is always an irreducible homogeneous quasi-translation in dimension 5 such that $H$ is of the form (2.1) and satisfies the condition (*).

Let $D_H = \sum_{i=1}^{5} H_i \partial_i$. By Proposition 1.1, $D_H(x_i) = D_H(H_i) = 0, 1 \leq i \leq 5$. Since $D_H$ is locally nilpotent, $\ker D_H$ is factorial closed, i.e., if $fg \in \ker D_H$ where $f \neq 0, g \neq 0$, then $f \in \ker D_H$ or $g \in \ker D_H$ (cf. [4, Proposition 1.3.32]). Up to linear conjugation, one may assume that $pq|H_i$ for some $1 \leq i \leq 4$, whence $p, q \in \ker D_H$. Hence

$$D_H(p) = H_1 \partial_1 p + H_2 \partial_2 p + H_3 \partial_3 p + H_4 \partial_4 p + r \partial_5 p = 0;$$

(2.2)

$$D_H(q) = H_1 \partial_1 q + H_2 \partial_2 q + H_3 \partial_3 q + H_4 \partial_4 q + r \partial_5 q = 0;$$

(2.3)

$$D_H(r) = H_1 \partial_1 r + H_2 \partial_2 r + H_3 \partial_3 r + H_4 \partial_4 r + r \partial_5 r = 0.$$  

(2.4)

And we obtain by (2.2) and (2.3) that

$$(\partial_5 p \partial_1 q - \partial_5 q \partial_1 p)H_1 + (\partial_5 p \partial_2 q - \partial_5 q \partial_2 p)H_2 + (\partial_5 p \partial_3 q - \partial_5 q \partial_3 p)H_3$$

$$+ (\partial_5 p \partial_4 q - \partial_5 q \partial_4 p)H_4 = 0.$$  

(2.5)

Proposition 2.2 We may assume that $\deg x_5 p < \deg x_5 q$.

Proof. Assume that $\deg x_5 (H_1, H_2, H_3, H_4) = k$. Let $G_i$ be the part of $H_i$ that has degree $k$ with respect to $x_5$, for $1 \leq i \leq 4$, and $G_5$ be the part of $H_5$ that has degree $k + 1$ with respect to $x_5$. Since $\text{tr} JH = 0$, either $G_5 = 0$ or $G_5$ is the leading part of $H_5$ with respect to $x_5$.

Looking at the coefficient of $x_5^{2k}$ of $JH_i \cdot H = 0$ for $1 \leq i \leq 4$, and the coefficient of $x_5^{2k+1}$ of $JH_5 \cdot H = 0$, we see that $JG \cdot G = 0$, whence $x + G$ is a quasi-translation. Let $D_G = \sum_{i=1}^{5} G_i \partial_i$. Then $D_G$ is a nice derivation, i.e., $D_G^n(x_i) = D_G^n(G_i) = 0, 1 \leq i \leq 5$, in particular, $G_5 \in \ker D_G$. Noticing that $\ker D_G$ is factorial closed and $x_5|G_5$, we have $G_5 = D_G(x_5) = 0$ and

$$(x_1 + x_5^{-k}G_1, x_2 + x_5^{-k}G_2, x_3 + x_5^{-k}G_3, x_4 + x_5^{-k}G_4, x_5 + 0)$$
is a quasi-translation as well. By removing its last component we obtain a quasi-translation in dimension 4. Thus \(G_1, G_2, G_3, G_4\) are linearly dependent over \(\mathbb{C}\) by a result of P. Gordan and M. Nöther[7].

Now let \((P, Q)\) be the leading homogeneous part of \((p, q)\) with respect to \(x_5\). Then \(h_i(P, Q) = G_i, 1 \leq i \leq 4\). Noticing that \(h_i, 1 \leq i \leq 4\), are linearly independent and \(h_i(P, Q), 1 \leq i \leq 4\), are linearly dependent, we have \(P\) and \(Q\) are algebraically dependent. But \(P\) and \(Q\) are homogeneous of the same degree, thus \(P\) and \(Q\) must be linearly dependent, say that \(\lambda P + \mu Q = 0\) with \(\lambda, \mu \in \mathbb{C}\) not all zero (say \(\lambda \neq 0\)). Then \(\deg x_5(\lambda p + \mu q) < \deg x_5 q\). Replacing \(p\) by \(p' = \lambda p + \mu q\), we obtain the desired result.

\[\text{Theorem 2.3} \quad \text{If } t = 3, \text{ i.e., } \deg p = \deg q = 3, \text{ then the components of } H \text{ are linearly dependent.} \]

\[\text{Proof.} \quad \text{We may assume that } \deg x_5 p < \deg x_5 q \text{ due to Proposition 2.2, and that } \deg x_5 p < 2 \text{ and } \deg x_5 q < 2 \text{ due to Theorem 2.1. Then } \deg x_5 p = 0, \text{ which implies that the components of } H \text{ are linearly dependent.} \]

The following lemma will be needed later.

\[\text{Lemma 2.4} \quad [1, \text{Proposition 1.2}] \quad \text{Let } F = x + H \text{ be a quasi-translation and } g \text{ a homogeneous polynomial of degree } m. \text{ Then } D_H^m(g) = 0 \text{ if and only if } g(H) = 0. \]

\[\text{Theorem 2.5} \quad \text{If } t = 4, \text{ i.e., } \deg p = \deg q = 4, \text{ then the components of } H \text{ are linearly dependent.} \]

\[\text{Proof.} \quad \text{By Proposition 2.2 and Theorem 2.1, we may assume that } \deg x_5 p < \deg x_5 q < 3. \text{ Notice that we are done when } \deg x_5 p = 0 \text{ or } \deg x_5 q = 0. \text{ Hence } \deg x_5 p = 1 \text{ and } \deg x_5 q = 2. \text{ Let} \]

\[
\begin{aligned}
\begin{cases}
p = p_4 + p_3 x_5; \\
q = q_4 + q_3 x_5 + q_2 x_5^2,
\end{cases}
\end{aligned}
\]

where \(p_i\) and \(q_j\) are of degree \(i\) and \(j\) respectively, and contain no \(x_5\). Noticing that \(p, q \in \ker D_H\), we have \(p(H) = q(H) = 0\) due to Lemma 2.4, and thus \(p_i(H) = q_i(H) = 0\), since \(p, q, r\) are algebraic independent.

If \(p\) is reducible, then \(p\) contains a factor \(w\) such that \(w\) is linear or \(w = l x_5 + w'\), where \(l\) is linear and \(\deg x_5 w' = 0\). Since \(p \in \ker D_H\) and \(D_H\) is factorial closed, we have \(w \in \ker D_H\), and thus \(w(H) = 0\) due to Lemma 2.4.

It follows that the components of \(H\) are linearly dependent. Hence we assume that \(p\) is irreducible, and similarly we may assume that so is \(q\).

If \(H_i \equiv 0 \pmod{p}\) for \(i = 1, \ldots, 4\), then we obtain by (2.2) that \(p|r \partial_5 p\). Since \(F = X + H\) is irreducible, we have \(p \nmid r\), and thus \(\partial_5 p = 0\). Similarly,
when \( H_i \equiv 0 (\text{mod } q) \) for \( i = 1, \ldots, 4 \), we have also \( \partial_5 p = 0 \). Therefore, up to linear conjugation, it suffices to consider the following two cases:

**Case 1.** \( H_1 \equiv 0, H_2 \equiv p^s + q^s, H_i \equiv 0, 3 \leq i \leq 4 \) (mod \( pq \));

**Case 2.** \( H_1 \equiv p^s, H_2 \equiv q^s, H_i \equiv 0, 3 \leq i \leq 4 \) (mod \( pq \)).

Let \( R = \mathbb{C}[x_1, x_2, x_3, x_4] \) and let \( I_0 \) be the ideal of \( R \) of relations between \( H_1, H_3, H_4 \). Then \( \dim R/I_0 = \text{trdeg}_\mathbb{C} \mathbb{C}(H_1, H_3, H_4) = 2 \), whence \( \text{height} I_0 = 1 \), and thus \( I_0 \) is a (prime) ideal generated by an irreducible polynomial.

By the proof of Proposition 2.2, we see that \( \deg x_5 \leq \deg x_5 (H_1, \ldots, H_4) = 2s \), say \( r = r_{4s} + r_{4s-1}x_5 + \cdots + r_2x_5^2s \). Looking at the coefficients of \( x_5^{2s+1} \) in (2.2), \( x_5^{2s+2} \) in (2.3) and \( x_5^{4s} \) in (2.4), we have \( q_2^s \frac{\partial p}{\partial x_2} \equiv 0 \), \( q_2^s \frac{\partial q}{\partial x_2} = 0 \) and thus \( \frac{\partial p}{\partial x_2} = \frac{\partial q}{\partial x_2} = \frac{\partial r}{\partial x_2} = 0 \). Then \( q_2, p_3, r_2 \in I_0 \). Since \( q_2(H) = 0 \), we may assume that \( q_2 \) is irreducible. It follows that \( I_0 = \langle q_2 \rangle \) and \( p_3 = q_2 L \) for some linear polynomial \( L \in R \).

**Claim.** We are done when \( q_2 \frac{\partial q}{\partial x_2} \).

In fact, if \( q_2 \frac{\partial q}{\partial x_2} \), then noticing that \( p = p_4 + (q_2 L)x_5 \) and \( q_2 \in R = \mathbb{C}[x_1, x_2, x_3, x_4] \), we have \( p = w + q_2 w' \) for some \( w, w' \), where \( w \in R \). Since \( w(H) = p(H) - q_2(H)w'(H) = 0 \), we have \( w \not\in I_0 = \langle q_2 \rangle \) and thus \( q_2|p \), contradicting that \( p \) is irreducible. Thus the Claim has been proved.

**Case 1.** By (2.5), \( pq|\langle p^s + q^s \rangle (\partial_5 p \partial_5 q - \partial_5 q \partial_5 p) \), and thus \( \partial_5 p \partial_5 q - \partial_5 q \partial_5 p = 0 \), since both \( p \) and \( q \) are irreducible and \( p, q \) are relatively prime.

Suppose that \( \partial_5 q = q_3 + 2q_2x_5 \) is reducible. Then \( q_2|q_3 \), say \( q_3 = q_2 L' \). Noticing that

\[
\frac{\partial}{\partial x_5} (\partial_5 p \partial_5 q - \partial_5 q \partial_5 p) = \partial_5 p \frac{\partial^2 q}{\partial x_2 \partial x_5} - \partial_5 q \frac{\partial^2 p}{\partial x_2 \partial x_5} - \partial_5 q \partial_5 q \frac{\partial^2 p}{\partial x_2 \partial x_5 - x_5} = p_3 \frac{\partial q}{\partial x_2} - 2q_2 \frac{\partial p}{\partial x_2} = q_2 L \frac{\partial(q_2 L')}{\partial x_2} - 2q_2 \frac{\partial p}{\partial x_2} = \frac{q_2 (L q_2 \frac{\partial L'}{\partial x_2} - 2 \frac{\partial q}{\partial x_2})}{\partial x_2} = 0,
\]

we have \( q_2 \frac{\partial q}{\partial x_2} \). So it suffices to consider the case that \( \partial_5 q \) is irreducible.

Since \( \partial_5 p \partial_5 q - \partial_5 q \partial_5 p = 0 \) and \( \partial_5 q \not\mid \partial_5 p \), we have \( \partial_5 q = c \partial_5 q \) for some \( c \in \mathbb{C} \). Replacing \( F = X + H \) by \( T^{-1} \circ F \circ T \), where \( T = (x_1, x_2, x_3, x_4, x_5 - cx_2) \), we may assume that \( \partial_5 q = 0 \), whence \( \partial_5 (q_j) = 0 \) for each \( j \). It follows that \( q_j \in I_0 = \langle q_2 \rangle \) for each \( j \), and thus \( q_2|q \), contradicting that \( q \) is irreducible.

**Case 2.** Recall that \( H_1 \equiv p^s, H_2 \equiv q^s, H_i \equiv 0, 3 \leq i \leq 4 \) (mod \( pq \)). If \( pq^s-1 \) appears in \( H_3 \) or \( H_4 \), then we may assume that it doesn’t appear in the other \( H_i \). So it suffices to consider the following two subcases:

**Case 2.1** \( H_1 \equiv p^s, H_2 \equiv q^s, H_3 \equiv 0, H_4 \equiv pq^s-1 \) (mod \( p^2 q \));

**Case 2.2** \( H_1 \equiv p^s + c_1 pq^s-1, H_2 \equiv q^s + c_2 pq^s-1, H_i \equiv 0, i = 3, 4 \) (mod \( p^2 q \)).
And we may assume that $c_1 = 1$. (In fact, if $c_1 = 0$, then by (2.5), we have $p^2(q^* + c_2 pq^{* - 1})(\partial_5 p \partial_2 q - \partial_5 q \partial_2 p)$, and thus $\partial_5 p \partial_2 q - \partial_5 q \partial_2 p = 0$. We are done by the proof of Case 1.)

In both Case 2.1 and Case 2.2, looking at the coefficients of $x^2_{s+1}$ in (2.2), $x^2_{s+1}$ in (2.3), and $x^2_{s-1}$ in tr$JH = 0$, we have

$$q_2^2 \frac{\partial p}{\partial x_2} + p_3 q_2^{-1} \frac{\partial q_2}{\partial x_j} + r_2 s p_3 = 0; \quad (2.6)$$

$$q_2^2 \frac{\partial q_2}{\partial x_2} + p_3 q_2^{-1} \frac{\partial q_2}{\partial x_j} + 2 r_2 q_2 = 0; \quad (2.7)$$

$$s q_2^{-1} \frac{\partial q_2}{\partial x_2} + q_2^{-1} \frac{\partial q_2}{\partial x_j} + (s - 1) p_3 q_2^{-2} \frac{\partial q_2}{\partial x_j} + 2 s r_2 s = 0, \quad (2.8)$$

where $j = 4$ for Case 2.1 and $j = 1$ for Case 2.2, and hereinafter.

Since $p_3 = q_2 L$, we obtain by (2.6) that $L | \frac{\partial q_2}{\partial x_2}$, say $\frac{\partial q_2}{\partial x_2} = Lu$. We may assume that $u \neq 0$ for otherwise $L | \frac{\partial q_2}{\partial x_2}$. Noticing that $q_2, p_3, r_2 s \in R = \mathbb{C}[x_1, x_3, x_4]$, we have $\frac{\partial q_2}{\partial x_2} \in R$ due to (2.6) and (2.7).

Eliminating $r_2 s$ in (2.6) and (2.7), we obtain that $\frac{\partial q_2}{\partial x_2} = \frac{\partial q_2}{\partial x_j}$. And then eliminating $r_2 s$ in (2.6) and (2.7), we have $\frac{\partial q_2}{\partial x_2} = 2 u + L | \frac{\partial q_2}{\partial x_j}$.

Now by (2.5), $p | q'(\partial_3 p \partial_2 q - \partial_5 q \partial_2 p)$, and thus $p | q'(\partial_3 p \partial_2 q - \partial_5 q \partial_2 p)$, say $p u_2 = \partial_5 p \partial_2 q - \partial_5 q \partial_2 p$. Noticing that $p = p_4 + p_3 x_5$ and $\partial_5 p \partial_2 q - \partial_5 q \partial_2 p = (p_3 \frac{\partial q_2}{\partial x_2} - 2 \frac{\partial q_2}{\partial x_2} q_2) x_5 + (p_3 \frac{\partial q_2}{\partial x_2} - q_2 \frac{\partial q_2}{\partial x_2})$, we have

$$p_3 u_2 = p_3 \frac{\partial q_2}{\partial x_2} - 2 \frac{\partial q_2}{\partial x_2} q_2; \quad (2.9)$$

$$p_4 u_2 = p_3 \frac{\partial q_2}{\partial x_2} - q_3 \frac{\partial q_2}{\partial x_2}. \quad (2.10)$$

By (2.9), $u_2 = \frac{\partial q_2}{\partial x_2} - 2 u = L | \frac{\partial q_2}{\partial x_2}$. Taking partial derivative with respect to $x_2$ in (2.10), we obtain that $\frac{\partial q_2}{\partial x_2} u_2 = p_3 \frac{\partial q_2}{\partial x_2} - \frac{\partial q_2}{\partial x_2} \frac{\partial q_2}{\partial x_2}$, i.e., $\frac{\partial q_2}{\partial x_2} (u_2 + \frac{\partial q_2}{\partial x_2}) = p_3 \frac{\partial^2 q_2}{\partial x_2}$. Since $u_2 = L | \frac{\partial q_2}{\partial x_2}$ and $\frac{\partial q_2}{\partial x_2} = 2 u + L | \frac{\partial q_2}{\partial x_2}$, we have $2 \frac{\partial q_2}{\partial x_2} (u + L | \frac{\partial q_2}{\partial x_2}) = p_3 \frac{\partial^2 q_2}{\partial x_2}$, i.e., $2 L u (u + L | \frac{\partial q_2}{\partial x_2}) = L q_2 \frac{\partial^2 q_2}{\partial x_2}$, and thus

$$2 u (u + L | \frac{\partial q_2}{\partial x_2}) = q_2 \frac{\partial^2 q_2}{\partial x_2}. \quad (2.11)$$

We may assume that $q_2 \not| u$, for otherwise $q_2 | \frac{\partial q_2}{\partial x_2}$. Hence $q_2 | u + L | \frac{\partial q_2}{\partial x_2}$. It follows that $u + L | \frac{\partial q_2}{\partial x_2} = c q_2$ for some $c \in \mathbb{C}$, and $\frac{\partial^2 q_2}{\partial x_2} = 2 c u$.

In conclusion,

$$p_3 = q_2 L; \quad \frac{\partial q_2}{\partial x_2} = \frac{\partial q_2}{\partial x_2} L; \quad \frac{\partial q_2}{\partial x_2} = L u; \quad \frac{\partial q_2}{\partial x_2} = 2 u + L | \frac{\partial q_2}{\partial x_2}; \quad u + \frac{\partial q_2}{\partial x_2} = c q_2; \quad \frac{\partial^2 q_2}{\partial x_2} = 2 c u. \quad (2.11)$$

Also by (2.5), $q | p' (\partial_5 p \partial_1 q - \partial_5 q \partial_1 p)$ and thus $q | p' (\partial_5 p \partial_1 q - \partial_5 q \partial_1 p)$, say $q v_2 = \partial_5 p \partial_1 q - \partial_5 q \partial_1 p$. Noticing that $q = q_4 + q_3 x_5 + q_2 x_5^2$ and

$$\partial_5 p \partial_1 q - \partial_5 q \partial_1 p = (\frac{\partial q_2}{\partial x_1} - 2 \frac{\partial q_2}{\partial x_1} q_2) x_5^2 + (p_3 \frac{\partial q_2}{\partial x_1} - q_3 \frac{\partial q_2}{\partial x_1} - 2 q_2 \frac{\partial q_2}{\partial x_1}) x_5 + (p_3 \frac{\partial q_2}{\partial x_1} - q_3 \frac{\partial q_2}{\partial x_1}),$$
we have
\[ q_2 v_2 = \frac{\partial L}{\partial x_1} p_3 - 2 \frac{\partial p}{\partial x_1} q_2; \tag{2.12} \]
\[ q_3 v_2 = p_3 \frac{\partial q_2}{\partial x_1} - q_3 \frac{\partial p}{\partial x_1} - 2 q_2 \frac{\partial p}{\partial x_1}; \tag{2.13} \]
\[ q_4 v_2 = p_3 \frac{\partial q_2}{\partial x_1} - q_3 \frac{\partial p}{\partial x_1}. \tag{2.14} \]

By (2.12), \( v_2 = \frac{\partial q_2}{\partial x_1} L - 2 \frac{\partial p}{\partial x_1} = \frac{\partial q_2}{\partial x_1} L - 2 \partial q_2 \frac{\partial L}{\partial x_1}, \) and then by (2.14), \( q_4(\frac{\partial q_2}{\partial x_1} L - 2 \frac{\partial p}{\partial x_1}) = p_3 \frac{\partial q_2}{\partial x_1} - q_3 \frac{\partial p}{\partial x_1}. \) It follows that
\[ \frac{1}{2} \frac{\partial^2 q_2}{\partial x_1^2} \left( \frac{\partial q_2}{\partial x_1} L - 2 \frac{\partial p}{\partial x_1} \right) = \frac{1}{2} p_3 \frac{\partial L}{\partial x_1} \left( \frac{\partial^2 q_2}{\partial x_2^2} \right) - \frac{\partial q_2}{\partial x_2} \frac{\partial p}{\partial x_1}, \]
and by (2.11) we have \( cu(- \frac{\partial q_2}{\partial x_1} L - 2 \frac{\partial p}{\partial x_1}) = cp_3 \frac{\partial q_2}{\partial x_1} - q_3 \frac{\partial p}{\partial x_1} (\frac{\partial q_2}{\partial x_1} u + L \frac{\partial u}{\partial x_1}), \) i.e.,
\[ u \left( - c \frac{\partial q_2}{\partial x_1} L - 2 cp_2 \frac{\partial L}{\partial x_1} \right) = \frac{\partial u}{\partial x_1} (-Lu), \]
whence
\[ c \frac{\partial q_2}{\partial x_1} + \frac{\partial q_2}{\partial x_1} \frac{\partial L}{\partial x_1} = \frac{\partial u}{\partial x_1}. \]

Then \( u \left( - c \frac{\partial q_2}{\partial x_1} L - 2 cp_2 \frac{\partial L}{\partial x_1} \right) = \frac{\partial q_2}{\partial x_1} \left( -Lu \right), \)
we have
\[ 2 \frac{\partial q_2}{\partial x_1} \frac{\partial L}{\partial x_1} \frac{\partial q_2}{\partial x_1} + L \frac{\partial^2 q_2}{\partial x_1^2} = 0. \tag{2.15} \]

**Case 2.1** Recall that \( H_1 \equiv p^s, H_2 \equiv q^s, H_3 \equiv 0, H_4 \equiv \text{mod } p^2 q \) and \( j = 4 \) in this case. Taking partial derivative with respect to \( x_1 \) in (2.15), we have
\[ \frac{\partial q_2}{\partial x_1} = 0. \] Noticing that \( p, q \) are algebraically independent and \( q_2(H) = 0 \), we have \( q_2 \) contains no \( x_1^2 \) and \( x_4^2 \). But \( q_2 \) is irreducible, so \( \frac{\partial^2 q_2}{\partial x_1 \partial x_4} \neq 0 \), and then \( \frac{\partial L}{\partial x_1} = 0. \) Hence by (2.15), \( L \frac{\partial^2 q_2}{\partial x_1^2} = 0, \) and thus \( L = 0. \) It follows that \( \frac{\partial p}{\partial x_2} = Lu = 0. \) We are done by the Claim.

**Case 2.2** Recall that \( H_1 \equiv p^s + pq^{s-1}, H_2 \equiv q^s + c_2pq^{s-1}, H_i \equiv 0, i = 3, 4 \) (mod \( p^2 q \)) and \( j = 1 \) in this case. We may assume that \( \text{deg}_p H_3 > \text{deg}_p H_4. \) Since \( q_2(H) = 0 \) and \( p, q \) are algebraically independent, we have \( q_2 \) contains no terms \( x_1^2 \) and \( x_1 x_3, \) and thus \( q_2 \) is of the form \( k_0 x_3^2 + k_1 x_2^2 + k_2 x_1 x_4 + k_3 x_3 x_4, \) where \( k_0 \neq 0. \) Up to linear conjugation, we may assume that \( q_2 = x_3^2 - x_1 x_4. \)

By (2.11),
\[ p_3 = q_2 L; \quad \frac{\partial p_3}{\partial x_1} = \frac{\partial q_2}{\partial x_1} L; \quad \frac{\partial p_3}{\partial x_2} = Lu; \quad u = c \frac{\partial q_2}{\partial x_1} L; \quad \frac{\partial q_3}{\partial x_2} = 2u + \frac{\partial p_3}{\partial x_1} L; \quad \frac{\partial q_3}{\partial x_2} = 2cu. \]
Since $p_3 = q_2L$ and $\frac{\partial p_3}{\partial x_1} = \frac{\partial q_2}{\partial x_1}L$, we see that $\frac{\partial L}{\partial x_1} = 0$, say $L = a_3x_3 + a_4x_4$, where $a_3, a_4 \in \mathbb{C}$. And we may choose an appropriate linear conjugation such that $L = a_3x_3$ or $L = a_4x_4$, and that $q_2$ is still equal to $x_3^2 - x_1x_4$.

By (2.12), $v_2 = \frac{\partial q_2}{\partial x_1}L - 2\frac{\partial q_2}{\partial x_1} - 2\frac{\partial (q_2L)}{\partial x_1} = -\frac{\partial q_2}{\partial x_1}$. Then we obtain by (2.13) that $p_3\frac{\partial q_2}{\partial x_1} - 2q_2\frac{\partial q_2}{\partial x_1} = q_3(v_2 + \frac{\partial q_2}{\partial x_1}) = 0$, and thus $L\frac{\partial q_2}{\partial x_1} = 2\frac{\partial q_2}{\partial x_1}$. By (2.14), we see that $q_4(-\frac{\partial q_2}{\partial x_1}) = q_2L\frac{\partial q_2}{\partial x_1} - q_2\frac{\partial q_2}{\partial x_1}$, and then

$$q_4(-\frac{\partial q_2}{\partial x_1}) + q_3\frac{1}{2}\frac{\partial q_2}{\partial x_1} - q_2\frac{\partial q_2}{\partial x_1} = 0. \quad (2.16)$$

Observe that

$$p_4 = x_2Lu + Lw + \sigma;$$
$$q_3 = x_2(2u + \frac{\partial q_2}{\partial x_1}) + 2w + \tau;$$
$$q_4 = cu^2 + x_2w' + \gamma',$$

where $u = cq_2 - \frac{\partial q_2}{\partial x_1}L$, $q_2 = x_3^2 - x_1x_4$, $L = a_3x_3$ (or $a_4x_4$), $x_1w$ and $\sigma, \tau \in \mathbb{C}[x_3, x_4], \gamma' \in \mathbb{C}[x_1, x_3, x_4]$. By (2.16), $\frac{\partial (q_2^2 - q_2\alpha)}{\partial x_1} = 0$. Since $\text{deg}_{x_1} q_2 = 1$, and $\text{deg}_{x_1} q_3 \leq 3$, we see that $\text{deg}_{x_1} q_3 \leq 2$ and $\text{deg}_{x_1} q_4 \leq 3$. Let

$$w = b_0x_3^2x_3 + c_0x_1^2x_4 + d_0x_1x_3^2 + e_0x_1x_3x_4 + f_0x_1x_4^2;$$
$$\gamma' = b_0'x_3^2x_3 + c_0'x_1^2x_4 + d_0'x_1x_3^2 + e_0'x_1x_3x_4 + f_0'x_1x_4^2 + \gamma'',$$

where $\text{deg}_{x_3} \gamma'' \leq 1$. Looking at the coefficients of the terms $x_3^2x_1^4$, $x_3^2x_1^4$, $x_3^2x_1^4$ and $x_3^4x_1^2$ of $\frac{1}{2}q_2^2 - q_2q_4$, we obtain that $b_0 = 0$, $c_0 + c_0' = 0$, $-c_0' + d_0 + 2c_0d_0 = 0$ and $-d_0' + d_0' = 0$, which implies that $c_0 + d_0 = 0$.

By (2.9) and (2.10), we have $p_4L\frac{\partial q_2}{\partial x_1} = q_2L\frac{\partial q_2}{\partial x_1} - q_3Lu$ and thus $p_4\frac{\partial q_2}{\partial x_1} = q_3u - q_2\frac{\partial q_2}{\partial x_1} = 0$. Looking at the part containing only $x_3, x_4$ in this equation, we see that

$$\sigma \cdot (-x_4) + \tau \cdot (cx_3^2 + x_4L) - x_3^2\alpha = 0, \quad (2.17)$$

for some polynomial $\alpha$.

We are done when $L|\sigma$, since if $L|\sigma$ then $L|p$, contradicting that $p$ is irreducible.

If $L = a_3x_3$, then it follows by (2.17) that $L|\sigma$.

If $L = a_4x_4$, then $u \equiv \frac{\partial q_2}{\partial x_1}L = a_4x_4^2 \pmod{q_2}$. Noticing that $b_0 = 0, c_0 + d_0 = 0$ and $x_3^2 \equiv x_1x_4 \pmod{q_2}$, we have $w \equiv c_0x_3^2x_4 + d_0x_1x_3^2 + e_0x_1x_3x_4 + f_0x_1x_4^2 = (e_0x_1x_3 + f_0x_1x_4)x_4$. Write $\sigma = k_0x_3^4 + l_0x_3^2x_4 + x_4^2\beta$, where $\beta \in \mathbb{C}[x_3, x_4]$. Then

$$\sigma \equiv k_0x_3^2x_4^2 + l_0x_1x_3x_4^2 + x_4^2\beta = (k_0x_1^2 + l_0x_1x_3 + \beta)x_4^2 \pmod{q_2}.$$
Therefore,
\[ p_4 = x_2 Lu + Lw + \sigma \]
\[ \equiv (a_4^2 x_2 x_4 + a_4(e_0 x_1 x_3 + f_0 x_1 x_4) + k_0 x_1^2 + l_0 x_1 x_3 + \beta) x_4^2 \]
\[ = (k_0 x_4^2 + \delta) x_4^2 \pmod{q_2}, \]
for some \( \delta \) with \( \deg x_1 \delta \leq 1 \). Since \( p_4(H) = 0 \), we have \( H_4 = 0 \) or \( k_0 H_4^2 + \delta(H) = 0 \). It suffices to consider the case \( k_0 H_4^2 + \delta(H) = 0 \). Noticing that \( p, q \) are algebraically independent and \( H_1 \equiv p^s + pq^{s-1}, H_2 \equiv q^s + c_2 pq^{s-1}, H_i \equiv 0, i = 3, 4 \pmod{p^2 q} \), we have \( k_0 = 0 \). It follows that \( L|\sigma \).

**Corollary 2.6** Let \( F = x + H \) be a homogeneous quasi-translation of degree \( d \) in dimension 5. If \( d < 15 \), then the components of \( H \) are linearly dependent.

**Proof.** We only need to consider irreducible homogeneous quasi-translation of the form (2.1) with the condition (*), and we may assume that \( s, t \geq 3 \) due to Theorem 2.1. Since \( d = st < 15 \), we have \( t \leq 4 \), and thus the conclusion follows by Theorem 2.3 and Theorem 2.5.

The following corollary follows by Proposition 1.1 and Corollary 2.6.

**Corollary 2.7** Let \( D \) be a homogeneous nice derivation of degree \( d \) in dimension 5. If \( d < 15 \), then \( \ker D \) contains a linear coordination.

### 3 Some remarks

The original motive for our research of the cases \( t = 3 \) and \( t = 4 \) was to find a counterexample to Problem 1.2, but the research led to a positive solution in these cases. We hope the investigations can help to understand the structure of quasi-translations in dimension 5.

Finally we make some observations in the general case. By Theorem 2.1 and Proposition 2.2, we may assume that \( 0 < \deg x_5 p < \deg x_5 q < t - 1 \), say
\[ p = p_t + p_{t-1} x_5 + p_{t-2} x_5^2 + \cdots + p_3 x_5^{t-3}, \]
\[ q = q_t + q_{t-1} x_5 + q_{t-2} x_5^2 + \cdots + q_2 x_5^{t-2}. \]

Let \( I \) be the ideal of \( \mathbb{C}[x_1, x_2, x_3, x_4] \) of relations between \( H_i, 1 \leq i \leq 4 \). Then \( p_i, q_i \in I \) for all \( i, j \).

Let \( G_i = h_i(p_t, q_t) \) for \( 1 \leq i \leq 4 \) and \( D = \sum_{i=1}^4 G_i \partial_i \). Then \( D \) is a derivation on \( \mathbb{C}[x_1, x_2, x_3, x_4] \). Observing the parts that have degree 0 with respect to \( x_5 \) in (2.2) and (2.3), we see that \( D(p_t) + r_d p_{t-1} = 0 \) and \( D(q_t) + r_d q_{t-1} = 0 \), where \( r_d \) is the part of \( r \) that has degree 0 with respect to \( x_5 \). Let \( I' \) be the ideal of \( \mathbb{C}[x_1, x_2, x_3, x_4] \) generated by \( D^i(p_{t-1}) \) and \( D^i(q_{t-1}) \) for \( 0 \leq i < t - 1 \). Then
$I' \subset I$ and $D$ induces naturally a derivation $\tilde{D}$ on $\mathbb{C}[x_1, x_2, x_3, x_4]/I'$. We see that $\tilde{D}(p_i) = 0$ and $\tilde{D}(q_i) = 0$, and thus $\tilde{D}^2(x_i) = \tilde{D}(G_i) = 0$ for $1 \leq i \leq 4$.

In particular, if $p_{i-1} = q_{i-1} = 0$, then $D^2(x_i) = D(G_i) = 0$, i.e., $D$ is a nice derivation on $\mathbb{C}[x_1, x_2, x_3, x_4]$, which implies that $G_i = h_i(p_i, q_i), 1 \leq i \leq 4$, are linearly dependent. But $h_i, 1 \leq i \leq 4$, are linearly independent, we have $p_i$ and $q_i$ are algebraic dependent and thus linearly dependent. Then we may assume that $q_i = 0$, whence $x_5|q$. Since $q \in \text{ker} \, D_H$ and $\text{ker} \, D_H$ is factorial closed, we have $x_5 \in \text{ker} \, D_H$, and thus $r = D_H(x_5) = 0$.

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References


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