Some Properties of Hyperstructure and Union Normal Fuzzy Subgroups

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Abstract

In this paper, we present a hyperstructure with fuzzy sets endowed with a membership functions on \( F(H) \) of all fuzzy subsets of \( H \) and analyze the properties of this new hyperstructure with union fuzzy subgroups. Generally the new hyperstructure is not commutative, we give some conditions such that the new hyperstructure is commutative. Several characterization theorems are obtained, particularly about the relation between hyperstructure and union normal fuzzy subgroups.

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1 Introduction

Zadeh introduced the notion of a fuzzy subset of a non-empty set \( X \), as a function from \( X \) to \([0, 1]\) in [13].
The theory of algebraic hyperstructures which is a generalization of ordinary algebraic structures was first introduced by Marty [9]. Since then, many researchers have studied the theory of hyperstructures and developed it. Moreover, the applications of this theory in other fields such as geometry, graphs and hypergraphs, lattices, fuzzy and rough sets, automata, cryptography, codes, etc
has been extensively studied, see [1, 2, 3, 4, 6].
There is a considerable amount of works initiated by Corsini in [5] on the relationship between algebraic hyperstructures and fuzzy sets. The study of the fuzzy algebraic structures has started in the pioneering paper of Rosenfeld [12] in 1971. Rosenfeld introduced the notion of fuzzy groups and showed that many results in groups can be extended in an elementary manner to develop the theory of fuzzy group. Since then the literature of various fuzzy algebraic concepts has been growing very rapidly. For example, the concept of a fuzzy ideal of a semigroup was introduced by Kuroki [7]. Liu in [8], and Mukherjee and Sen in [11], introduced and examined the notion of a fuzzy ideal of a ring.

This paper is structured as follows. After the introduction, in section 2, we recall some basic notions and results on hypergroups and fuzzy sets. In section 3, we introduce a hyperoperation commutative of fuzzy sets, we study some properties of this hyperoperation with union fuzzy subgroups, give several examples and we establish some characterization theorems. Finally, in Section 4, several characterization theorems are obtained, particularly about the relation between hyperstructure and union normal fuzzy subgroups.

2 Preliminaries

Definition 2.1. ([4],[6]) Let $H$ be a non-empty set and $P^*(H)$ be the family of all non-empty subsets of $H$. A hyperoperation or join operation is a map $\odot : H \times H \longrightarrow P^*(H)$. If $(a, b) \in H \times H$, then its image under “$\odot$” is denoted by $a \odot b$.

The join operation is extended to subsets of $H$ in a natural way, so that $A \odot B$ is given by

$$A \odot B = \bigcup \{a \odot b| a \in A, b \in B\}.$$ 

The notations $a \odot A$ and $A \odot a$ are used for $\{a\} \odot A$ and $A \odot \{a\}$ respectively. Generally, the singleton $\{a\}$ is identified by its element $a$.

A non-empty set $H$, endowed with a hyperoperation $\odot$ is called a hypergroupoid and it is denoted by $(H, \odot)$. If $x \odot (y \odot z) = (x \odot y) \odot z, \forall x, y, z \in H$,

then $(H, \odot)$ is called a semihypergroup.

A hypergroupoid $(H, \odot)$ is called a quasihypergroup, if $x \odot H = H = H \odot x$, for all $x \in H$.

Definition 2.2. A hypergroup is a semihypergroup and a quasihypergroup.

Example 2.3. Let $H = \{1, 2, 3\}$, the hypergroupoid $(H, \odot)$ defined by the following table:
A fuzzy set on a non-empty set $H$ is a function $\mu : H \rightarrow [0; 1]$. We denote by $F(H)$ the set all fuzzy subsets of $H$. Let $\mu, \lambda \in F(H)$, if $\mu(x) \leq \lambda(x)$ (resp. $\mu(x) < \lambda(x)$) for all $x \in H$, then we say that $\mu$ is contained in $\lambda$ and we write $\mu \subseteq \lambda$ (resp. $\mu \subset \lambda$). Clearly, the inclusion relation $\subseteq$ is a partial ordering on $F(H)$. Also, the mapping $1_F : H \rightarrow [0, 1]|x \rightarrow 1$ is the greatest element of $F(H)$, and the mapping $0_F : H \rightarrow [0, 1]|x \rightarrow 0$ is the zero element of $F(H)$ for all $x \in H$, i.e., $0_F \subseteq \mu \subseteq 1_F$.

Define, $\mu \cup \lambda$ and $\mu \cap \lambda$ by $\mu \cup \lambda = \mu \cup \lambda$ and $\mu \cap \lambda = \mu \cap \lambda$ as follows:

$$(\mu \cup \lambda)(x) = \max\{\mu(x), \lambda(x)\},$$

$$(\mu \cap \lambda)(x) = \min\{\mu(x), \lambda(x)\},$$

for all $x \in H$, then $\mu \cup \lambda$ and $\mu \cap \lambda$ are called the union and intersection of $\mu$ and $\lambda$, respectively.

In this paper, $H$ always denotes an arbitrary group with a multiplicative binary operation and identity $e$.

### 3 Relation between hyperstructure and union fuzzy subgroups

For any collection, $\{\mu_i : i \in I\}$, of fuzzy subsets of $H$, where $I$ is a non-empty index set, the least upper $\bigcup_{i \in I} \mu_i$ and the greatest lower bound $\bigcap_{i \in I} \mu_i$ of the $\mu_i$ are given by $(\bigcup_{i \in I} \mu_i)(x) = \inf_{i \in I} \mu_i(x)$ and $(\bigcap_{i \in I} \mu_i)(x) = \sup_{i \in I} \mu_i(x)$ for all $x \in H$, respectively. We write $\bigcup_{i \in I} \mu_i = \bigcup_{i \in I} \mu_i = \bigcup_{i \in I} \mu_i = \bigcap_{i \in I} \mu_i = \bigcap_{i \in I} \mu_i = \bigcap_{i \in I} \mu_i$ if $I = \{1, 2, \ldots, n\}$.

**Definition 3.1.** ([1]) Suppose $\mu, \lambda : H \rightarrow [0, 1]$, be two fuzzy sets defined on a nonempty set $H$. Let $S_x := \{(y, z) \in H^2| x = yz\}$ for all $x \in H$, we can define hyperstructure “$\odot$” on $F$ as follows:

$$\odot : F \times F \rightarrow P^*(F)$$

$$\odot : (\mu, \lambda) = \mu \odot \lambda,$$

where $\mu \odot \lambda$ is the mapping of $H$ into $P^*([0; 1])$ defined by:

$$(\mu \odot \lambda)(x) := \bigcup_{(y, z) \in S_x} \min\{\mu(y), \lambda(z)\}.$$
Clearly, \( \mu \odot 0_F = 0_F \odot \mu = 0_F \).

**Remark 3.2.** We have \((\mu \odot \lambda)(x) \neq \emptyset\) for all \(x \in H\), since \((x,e), (e,x) \in S_x\).

**Remark 3.3.** Generally the hyperstructure \(\mu \odot \lambda\) is not commutative, as we notice in the following example.

**Example 3.4.** Let \(H = \{1, 2, 3, 4, 5, 6\}\), the groupoid \((H,)\) defined by the following table:

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then \((H,)\) is a group, but it is not a commutative group. Let \(\mu(n) = \frac{1}{n}\), \(\lambda(n) = \frac{2}{n}\) for \(n \in \{2, 3, 4, 5, 6\}\) and \(\mu(1) = \lambda(1) = 1\). Then, we have \(S_1 = \{(4,1), (1,4), (2,5), (3,6), (5,3), (6,2)\}\), so \((\mu \odot \lambda)(4) = \{\min\{\mu(4), \lambda(1)\}, \min\{\mu(1), \lambda(4)\}, \min\{\mu(2), \lambda(5)\}, \min\{\mu(3), \lambda(6)\}\}

, \(\min\{\mu(5), \lambda(3)\}, \min\{\mu(6), \lambda(2)\}\} = \{\frac{1}{4}, \frac{1}{2}, \frac{2}{5}, \frac{1}{3}, \frac{1}{6}\}\). Similarly, we have \((\lambda \odot \mu)(4) = \{\frac{1}{2}, \frac{1}{4}, \frac{1}{5}, \frac{1}{3}, \frac{1}{6}\}\). Which results that \((\mu \odot \lambda)(4) \neq (\lambda \odot \mu)(4)\).

**Theorem 3.5.** ([10]) Let \(\mu_i \in F(H), i \in I\). Then the following statements hold:

(i) \((\bigcup_{i \in I} \mu_i)^{-1} = \bigcup_{i \in I} \mu_i^{-1}\),

(ii) \((\bigcap_{i \in I} \mu_i)^{-1} = \bigcap_{i \in I} \mu_i^{-1}\).

Where \(\mu^{-1}(x) = \mu(x^{-1})\), for all \(x, y \in H\).

**Definition 3.6.** ([10]) A fuzzy subset \(\mu\) of \(H\) is called a fuzzy subgroup of \(H\), if

(i) \(\mu(xy) \geq \min\{\mu(x), \mu(y)\}\),

(ii) \(\mu(x^{-1}) \geq \mu(x)\),

for all \(x, y \in H\).

By Definition 3.6, if \(\mu\) be a fuzzy subgroup of \(H\), we have \(\mu(x^{-1}) = \mu(x)\).

**Definition 3.7.** ([1]) Let \(\mu, \lambda, \nu \in F(H)\), we consider a relation "\(\sqsubseteq\)" as follows:

\(\mu \odot \lambda \sqsubseteq \nu\) if \(\min\{\mu(y), \lambda(z)\} \leq \nu(x)\) \(\forall (y,z) \in S_x\).

For all \(x \in H\).
Theorem 3.8. ([1]) Let \( \mu \in F(H) \). Then \( \mu \) is a fuzzy subgroup of \( H \) if and only if \( \mu \) satisfies the following conditions:

(i) \( \mu \odot \mu \subseteq \mu \).

(ii) \( \mu^{-1} \subseteq \mu \).

Example 3.9. Let \( \mu, \lambda \in F(\mathbb{Z}) \), and

\[
\mu(n) = \begin{cases} 
\frac{1}{2} & \text{if } n \leq 2 \\
0 & \text{otherwise,}
\end{cases} \quad \text{and} \quad \lambda(n) = \begin{cases} 
\frac{1}{3} & \text{if } n \leq 3 \\
0 & \text{otherwise.}
\end{cases}
\]

Where \( < 2 > = \{2z : z \in \mathbb{Z}\} \), \( < 3 > = \{3z : z \in \mathbb{Z}\} \).

It is clear that \( \mu, \lambda \) are fuzzy subgroups of \( \mathbb{Z} \) and \( \mu \nsubseteq \lambda, \lambda \nsubseteq \mu \).

Since, \( S_5 = \{(y, z) \in \mathbb{Z}^2 : y + z = 5\} \), \( (2, 3) \in S_5 \) and \( \min\{\max\{\mu(2), \lambda(2)\}, \max\{\mu(3), \lambda(3)\}\} = \frac{2}{5} \), then \( \max\{\mu(5), \lambda(5)\} = 0 \). So \( \mu \cup \lambda \) is not a fuzzy subgroup of \( \mathbb{Z} \).

Corollary 3.10. In general, union of two fuzzy subgroup of a group is not a fuzzy subgroup.

Theorem 3.11. Let \( \mu, \lambda \in F(H) \). If \( \mu \subseteq \lambda \) or \( \lambda \subseteq \mu \). Then the following statements hold:

(i) \( (\mu \cup \lambda) \odot (\mu \cup \lambda) \subseteq (\mu \cup \lambda) \).

(ii) \( (\mu \cup \lambda)^{-1} \subseteq (\mu \cup \lambda) \).

Proof (i). Suppose \( \mu \subseteq \lambda \) (similar argument is true for \( \lambda \subseteq \mu \)). If \( x \in H \), then

\[
((\mu \cup \lambda) \odot (\mu \cup \lambda))(x) = \bigcup_{(y,z) \in S_x} \min\{(\mu \cup \lambda)(y), (\mu \cup \lambda)(z)\} \\
= \bigcup_{(y,z) \in S_x} \min\{\max\{\mu(y), \lambda(y)\}, \max\{\mu(z), \lambda(z)\}\} \\
= \bigcup_{(y,z) \in S_x} \min\{\lambda(y), \lambda(z)\} \\
\leq \bigcup_{(y,z) \in S_x} \lambda(x) = \lambda(x) \\
= \max\{\mu(x), \lambda(x)\} = (\mu \cup \lambda)(x).
\]

Thus \( (\mu \cup \lambda) \odot (\mu \cup \lambda) \subseteq (\mu \cup \lambda) \).

(ii). By Theorem 3.5 we have \( (\mu \cup \lambda)^{-1} = \mu^{-1} \cup \lambda^{-1} \subseteq \lambda^{-1} \subseteq \lambda = (\mu \cup \lambda) \).

By the above theorem and the theorem 3.8, we can easily obtain the following result.

Corollary 3.12. Let \( \mu, \lambda \in F(H) \). If \( \mu \subseteq \lambda \) or \( \lambda \subseteq \mu \). Then \( \mu \cup \lambda \) is a fuzzy subgroup of \( H \).

Theorem 3.13. Let \( \mu_i \in F(H), i = 1, 2, ..., n \), be a family finite of fuzzy subgroups of \( H \). If \( \mu_1 \subseteq \mu_2 \subseteq \ldots \subseteq \mu_n \). Then the following statements hold:

(i) \( (\bigcup_{i=1}^n \mu_i) \odot (\bigcup_{i=1}^n \mu_i) \subseteq (\bigcup_{i=1}^n \mu_i) \).

(ii) \( (\bigcup_{i=1}^n \mu_i)^{-1} \subseteq (\bigcup_{i=1}^n \mu_i) \).
Then the following statements hold:

\[
\min\{\left(\bigcup_{i=1}^{n} \mu_i\right)(y), \left(\bigcup_{i=1}^{n} \mu_i\right)(z)\} = \min\{\max\{\mu_1(y), \ldots, \mu_n(y)\}, \max\{\mu_1(z), \ldots, \mu_n(z)\}\}
\]

\[
= \min\{\mu_n(y), \mu_n(z)\} 
\leq \mu_n(x) = \left(\bigcup_{i=1}^{n} \mu_i\right)(x).
\]

Thus \(\left(\bigcup_{i=1}^{n} \mu_i\right) \circ \left(\bigcup_{i=1}^{n} \mu_i\right) \subseteq \left(\bigcup_{i=1}^{n} \mu_i\right)\).

(ii). By Theorem 3.5 we have \(\left(\bigcup_{i=1}^{n} \mu_i\right)^{-1} = \bigcup_{i=1}^{n} \mu_i^{-1} \subseteq \mu_n = \bigcup_{i=1}^{n} \mu_i\).

**Corollary 3.14.** Let \(\mu_i \in F(H), \ i = 1, 2, \ldots, n\), be a family finite of fuzzy subgroups of \(H\). If \(\mu_1 \subseteq \mu_2 \subseteq \ldots \subseteq \mu_n\). Then \(\bigcup_{i=1}^{n} \mu_i\) is a fuzzy subgroup of \(H\).

In a similar way we prove the following:

**Theorem 3.15.** Let \(\mu_i \in F(H), \ i \in I\), be a family of fuzzy subgroups of \(H\). If \(\mu_1 \subseteq \mu_2 \subseteq \ldots \subseteq \mu_n\), where for all \(i \geq 1\), \(\mu_{n+i} = \mu_n\). Then the following statements hold:

(i) \(\left(\bigcup_{i \in I} \mu_i\right) \circ \left(\bigcup_{i \in I} \mu_i\right) \subseteq \left(\bigcup_{i \in I} \mu_i\right)\).

(ii) \(\left(\bigcup_{i \in I} \mu_i\right)^{-1} \subseteq \left(\bigcup_{i \in I} \mu_i\right)\).

**Corollary 3.16.** Let \(\mu_i \in F(H), \ i \in I\), be a family of fuzzy subgroups of \(H\). If \(\mu_1 \subseteq \mu_2 \subseteq \ldots \subseteq \mu_n\), where for all \(i \geq 1\), \(\mu_{n+i} = \mu_n\). Then \(\bigcup_{i \in I} \mu_i\) is a fuzzy subgroup of \(H\).

**Theorem 3.17.** Let \(\mu_i \in F(H), \ i \in I\), be a family of fuzzy subgroups of \(H\). If \(\mu_1 \supseteq \mu_2 \supseteq \ldots \). Then the following statements hold:

(i) \(\left(\bigcup_{i \in I} \mu_i\right) \circ \left(\bigcup_{i \in I} \mu_i\right) \subseteq \left(\bigcup_{i \in I} \mu_i\right)\).

(ii) \(\left(\bigcup_{i \in I} \mu_i\right)^{-1} \subseteq \left(\bigcup_{i \in I} \mu_i\right)\).

**Corollary 3.18.** Let \(\mu_i \in F(H), \ i \in I\), be a family of fuzzy subgroups of \(H\). If \(\mu_1 \supseteq \mu_2 \supseteq \ldots \). Then \(\bigcup_{i \in I} \mu_i\) is a fuzzy subgroup of \(H\).

### 4 Relation between hyperstructure and union normal fuzzy subgroups

This is obvious from Example 3.4 that generally hyperstructures \((H, \circ)\) is not commutative, but in this section, we give some conditions such that the hyperstructure have also this property.

**Definition 4.1.** ([10]) Let \(\mu \in F(H)\). Then we say that \(\mu\) is an abelian fuzzy subset of \(H\) if \(\mu(xy) = \mu(yx)\) for all \(x, y \in H\).
Theorem 4.2. ([1]) Let \( \mu \in F(H) \), \( \mu \neq \emptyset \). Then \( \mu \) is an abelian fuzzy subset of \( H \) if and only if \( \mu \odot \lambda = \lambda \odot \mu \) for all \( \lambda \in F(H) \).

Definition 4.3. Let \( \mu \in F(H) \) is a fuzzy subgroup of \( H \). Then \( \mu \) is called a normal fuzzy subgroup of \( H \) if it is an abelian fuzzy of \( H \). We write \( NF(H) \) the set of all normal fuzzy subgroup of \( H \).

Corollary 4.4. Let \( \mu \in F(H) \) is a fuzzy subgroup of \( H \). Then \( \mu \in NF(H) \) if and only if \( \mu \odot \lambda = \lambda \odot \mu \) for all \( \lambda \in F(H) \).

Theorem 4.5. Let \( \mu_1, \mu_2 \in NF(H) \). If \( \mu_1 \subseteq \mu_2 \) (or \( \mu_2 \subseteq \mu_1 \)). Then \((\mu_1 \cup \mu_2) \odot \lambda = \lambda \odot (\mu_1 \cup \mu_2) \) for all \( \lambda \in F(H) \).

Proof. Suppose \( \mu_1 \subseteq \mu_2 \) (similar argument is true for \( \mu_2 \subseteq \mu_1 \)) and \( \lambda \in F(H) \). Then for all \( x \in H \) we have

\[
[(\mu_1 \cup \mu_2) \odot \lambda](x) = \bigcup_{(y,z) \in S_x} \min\{(\mu_1 \cup \mu_2)(y), \lambda(z)\} \\
= \bigcup_{(y,z) \in S_x} \min\{\max\{\mu_1(y), \mu_2(y)\}, \lambda(z)\} \\
= \bigcup_{(y,z) \in S_x} \min\{\mu_2(y), \lambda(z)\} \\
= \bigcup_{(y,z) \in S_x} \min\{\lambda(y), \mu_2(z)\} \quad \text{(since } \mu_2 \odot \lambda = \lambda \odot \mu_2) \\
= \bigcup_{(y,z) \in S_x} \min\{\lambda(y), \max\{\mu_1(z), \mu_2(z)\}\} \\
= \bigcup_{(y,z) \in S_x} \min\{\lambda(y), (\mu_1 \cup \mu_2)(z)\} \\
= [\lambda \odot (\mu_1 \cup \mu_2)](x).
\]

By the above theorem and Theorem 3.8, we have the following result.

Corollary 4.6. Let \( \mu_1, \mu_2 \in NF(H) \). If \( \mu_1 \subseteq \mu_2 \) (or \( \mu_2 \subseteq \mu_1 \)). Then \( \mu_1 \cup \mu_2 \in NF(H) \).

Theorem 4.7. Let \( \mu_i \in NF(H), \ i = 1,2, ..., n \). If \( \mu_1 \subseteq \mu_2 \subseteq ... \subseteq \mu_n \). Then \((\bigcup_{i=1}^n \mu_i) \odot \lambda = \lambda \odot (\bigcup_{i=1}^n \mu_i)\).

Proof. Suppose \( x \in H \) and \( (y,z) \in S_x \). Then

\[
\min\{(\mu_1 \cup \mu_2)(y), \lambda(z)\} = \min\{\max\{\mu_1(y), \mu_2(y)\}, \lambda(z)\} \\
= \min\{\mu_2(y), \lambda(z)\} \\
= \min\{\lambda(y), \mu_2(z)\} \\
= \min\{\lambda(y), \max\{\mu_1(z), \mu_2(z)\}\} \\
= \min\{\lambda(y), (\mu_1 \cup \mu_2)(z)\}\}
\]

Corollary 4.8. Let \( \mu_i \in NF(H), \ i = 1,2, ..., n \). If \( \mu_1 \subseteq \mu_2 \subseteq ... \subseteq \mu_n \). Then \( \bigcup_{i=1}^n \mu_i \in NF(H) \).

In a similar way we prove the following:

Theorem 4.9. Let \( \mu_i \in NF(H), \ i \in I \). If \( \mu_1 \subseteq \mu_2 \subseteq ... \) and there exist \( n \in \mathbb{N} \), where for all \( i \geq 1 \), \( \mu_{n+i} = \mu_n \). Then \((\bigcup_{i \in I} \mu_i) \odot \lambda = \lambda \odot (\bigcup_{i \in I} \mu_i)\).
Corollary 4.10. Let $\mu_i \in NF(H)$, $i \in I$. If $\mu_1 \subseteq \mu_2 \subseteq ...$ and there exist $n \in \mathbb{N}$, where for all $i \geq 1$, $\mu_{n+i} = \mu_n$. Then $\bigcup_{i \in I} \mu_i \in NF(H)$.

Theorem 4.11. Let $\mu_i \in F(H)$, $i \in I$, be a family of normal fuzzy subgroups of $H$. If $\mu_1 \supseteq \mu_2 \supseteq ...$. Then $(\bigcup_{i \in I} \mu_i) \odot \lambda = \lambda \odot (\bigcup_{i \in I} \mu_i)$.

Corollary 4.12. Let $\mu_i \in F(H)$, $i \in I$, be a family of normal fuzzy subgroups of $H$. If $\mu_1 \supseteq \mu_2 \supseteq ...$. Then $\bigcup_{i \in I} \mu_i \in NF(H)$.

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