Option on a CPPI

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Abstract

In this paper we obtain closed-form expressions for the price of an European Call option on constant-proportion portfolio insurance strategies (CPPI). CPPIs are path-dependent derivatives themselves where the underlying typically is a market index or a fund portfolio. We describe and explain the functionality of CPPIs, showing closed-form expression for the price of a CPPI assuming a Geometric Brownian Motion and continuous as well as discrete rebalancing for the fund investment. The sensitivities of the option to the various parameters of the model are also derived.

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1 Introduction

The CPPI (Constant Proportion Portfolio Insurance) is an important representative of the so-called portfolio insurance strategies, which fits within the
framework of dynamic asset allocation. It was first proposed in 1986 and 1988 respectively by Andre Perold for bonds (See [14]) and in 1987 by Black and Jones for equity-investments (See [6]). Portfolio insurance first appeared in the US at the beginning of the eighties and has gained popularity over the last years due to various downturns of the stock markets and a stronger focus on risk management in the asset management industry. The distinct bear market from 2001 to 2003 strongly contributed to the rise of portfolio insurance strategies as both institutional and private investors have become more risk averse in their investment policy. The aim of a portfolio insurance strategy is to protect the portfolio against bigger losses in downside markets by guaranteeing a certain percentage of the initial capital while maintaining the chance for an upside participation if the assets of the portfolio do well (See, e.g., [3] or [9]).

The properties of continuous–time CPPI strategies have been studied in the literature. Some clarifying papers on the topic of continuous CPPIs are [4] and [8]. The literature on CPPI also deals with the effects of jump processes, stochastic volatility models and extreme value approaches. Nevertheless the issue of discrete-time CPPI has been barely covered; in a working paper [2] analyze a discrete–time version of a general CPPI strategy which is used for risk management purposes, therefore risk measures statistics like shortfall and expected shortfall given default are all computed under the real-world measure. In [8] a particular case of CPPI, called a Constant Leverage Strategy (CLS) in discrete time, was analyzed in the context of a Hedge Fund application.

In this paper we study the pricing of call options on a CPPI strategy. We price the Option on a CPPI and examine how its price depends on various market and CPPI-parameters for continuous and discrete rebalancing.

The paper proceeds as follows. In Section 2 we summarize the main results on CPPI. Section 3 focuses on pricing an Option on a CPPI where the CPPI is rebalanced continuously. To get a better understanding of the sensitivity of this derivative to market parameters, we study its Greeks in Section 4. Section 5 deals with an Option on a discrete CPPI. Finally, Section 6 concludes.

2 CPPI in Continuous Time

The CPPI is a dynamic asset allocation strategy. A CPPI portfolio consists of two assets, the risky asset $S_t$ (e.g. a stock index) and the riskless asset $B_t$, typically a bank account. The CPPI works as follows. At first, two parameters have to be specified: the constant multiplier and the floor (the insurance level at maturity). The amount which is invested in a risky asset is determined by the product of the multiplier and the excess of the portfolio value over the floor. This product is called exposure. The remaining part, i.e. the difference of the portfolio value and the asset exposure, is invested in the riskless asset.
This implies that the strategy is self-financing. In theory the portfolio allocation is adjusted continuously. The assumption of continuous rebalancing of the CPPI ensures that the portfolio value does not fall below the floor and thus the initially specified insurance level is guaranteed.

In the following sections we first describe the functionality of the CPPI, present a mathematical formulation and derive a formula for the portfolio value of the CPPI. Finally we give an expression for the leverage of the CPPI.

Let us now describe the functionality of the CPPI in five steps. To do this, we assume a constant riskfree rate $r$.

1. Parameter Specification:
   In $t = 0$ the investor has to specify the multiplicator $m \geq 0$ and the floor or insurance level $F = F_T$ which represents the minimum portfolio value at maturity $T$. With $V_0$ denoting the portfolio value in $t = 0$ it must hold that
   
   $$F_T \leq e^{rT}V_0,$$

   since the maximal riskfree return on the portfolio can be $r$. The current floor $F_t$ is obtained by discounting $F$ for the remaining time $T - t$

   $$F_t = e^{-r(T-t)}F_T.$$

2. The cushion $C_t$, the excess of the portfolio value $V_t$ over the floor $F_t$, at time $t$, is defined as follows:

   $$C_t = \begin{cases} V_t - F_t & \text{if } V_t \geq F_t \\ 0 & \text{if } V_t < F_t \end{cases}$$

   $$C_t = \max\{V_t - F_t, 0\}$$

3. The exposure $E_t$ is the product of the multiplicator $m$ and the cushion $C_t$, i.e.

   $$E_t = m \cdot C_t$$

4. Now the exposure $E_t$ is invested in (borrowed from) the risky asset $S_t$, the remaining part of the portfolio is invested in the riskless asset $B_t$.

5. The portfolio is rebalanced continuously which means that the exposure $E_t$ and the investment in the riskless asset $B_t$ are adjusted at continuous time.

Continuous rebalancing ensures that the portfolio value $V_t$ is always greater than the discounted floor $F_t$. This in particular guarantees that the minimum
portfolio value $V_T$ is greater than the floor $F_T$ at time $T$.
The above description demonstrates that the CPPI is a procyclical investment strategy. In the case the risky asset performs well the cushion and thus the exposure increase. This means that more money is invested in the risky asset. In the case of declining stock prices the cushion shrinks and money is shifted from the risky asset to the riskless asset. Furthermore, we can see that the CPPI is a fairly simple and flexible asset allocation strategy. It is individually adjustable to meet the investors’ needs. Both, the floor and the multiple reflect the investor’s risk tolerance and are exogenous to the model. The more risk-loving the investor is the higher the multiplicator and the lower the floor. This combination of the multiplicator and the floor leads to a greater investment in the risky asset. The higher the multiplicator, the more the investor will participate in an increase in stock prices, but on the other hand the faster the portfolio will approach the floor when there is a sustained decrease in stock prices. As the cushion approaches zero, the exposure approaches zero, too.

Note that for certain combinations of the multiplicator, the floor, and the performance of the risky asset, the exposure $E_t = mC_t = m(V_t - F_t)$ can exceed the portfolio value $V_t$. This means that more than the current portfolio value has to be invested in the risky asset. The additional financing comes from taking a loan from the bank account. In this case the CPPI exhibits leverage.

In general to avoid that the portfolio exhibits too great leverages the CPPI can be modified by introducing a leverage constraint.

Next we describe the CPPI mathematically. First we give a mathematical definition of the CPPI. Based on this definition we derive a closed-form expression for the portfolio value $V_t$ of the CPPI.

**Definition 2.1.** Given the investment period $[0, T]$, the floor $F_T$, the multiplicator $m$, and the riskless rate $r$, the portfolio value $V_t^{CPPI}$ of the CPPI in $t$ is given by

$$V_t^{CPPI} = \varphi_{CPPI,B}(t) \cdot B_t + \varphi_{CPPI,S}(t) \cdot S_t, \quad t \in [0, T]$$

(2)

where

$$\varphi_{CPPI,S}(t) = \frac{mC_t}{S_t}$$

$$\varphi_{CPPI,B}(t) = \frac{V_t - \varphi_{CPPI,S}(t)S_t}{B_t} = \frac{V_t - mC_t}{B_t}, \quad t \in [0, T]$$

To simplify the notation we will refer to $V_t$ as the value $V_t^{CPPI}$ of the CPPI in $t$. Furthermore we will write $\varphi_{LS}$ and $\varphi_{LB}$ instead of $\varphi_{CPPI,S}(t)$ and
\( \varphi_{CPPI,B}(t) \).
Let us now present two results for the CPPI with continuous rebalancing. First
we derive a formula for the process of the cushion \( C_t \) and, using this result, we
obtain a formula for the process of the portfolio value \( V_t \).

**Lemma 2.2.** In the risk-neutral world the process for the cushion \( C_t \) is
given by
\[
\frac{dC_t}{C_t} = r \, dt + m \sigma \, dW_t. \tag{3}
\]

**Proof.** The proof is carried out according to Bertrand [5]. Recall that \( V_t = C_t + F_t \), \( E_t = m C_t \) and \( dF_t = r F_t dt = F_t \frac{dB_t}{B_t} \). Thus the cushion \( C_t \) must satisfy:
\[
dC_t = d(V_t - F_t) = dV_t - dF_t = (V_t - E_t) \frac{dB_t}{B_t} + E_t \frac{dS_t}{S_t} - dF_t
\]
\[
= (C_t + F_t - m C_t) \frac{dB_t}{B_t} + m C_t \frac{dS_t}{S_t} - dF_t = (C_t - m C_t) \frac{dB_t}{B_t} + m C_t \frac{dS_t}{S_t}
\]
\[
= (C_t - m C_t) r \, dt + m C_t (r \, dt + \sigma \, dW_t) = C_t (r \, dt + m \sigma \, dW_t)
\]
The result is obtained by dividing by \( C_t \).
\[ \square \]

Using this Lemma we can derive a closed-form expression for the portfolio
value \( V_t \) of a CPPI with continuous rebalancing.

**Proposition 2.3.** Let \( m \geq 0 \) be the multiplicator, \( r \) the riskless rate, \( F_t = e^{-r(T-t)} F_T \) the floor, and \( C_0 \) the cushion in \( t = 0 \). Then the portfolio value \( V_t \)
of a CPPI with continuous rebalancing is given by
\[
V_t(m, S_t) = \alpha_t S_t^m + F_t, \tag{4}
\]
where \( \alpha_t = \frac{C_0}{S_0^m} \cdot e^{\beta t} \) and \( \beta = \left( r - m (r - \frac{\sigma^2}{2}) - m^2 \frac{\sigma^2}{2} \right) \).

**Proof.** The proof is carried out according to Bertrand [5]. Recall that the
solution of the stochastic differential Equation (3) for \( C_t \) is
\[
C_t = C_0 \, e^{(r - m \frac{\sigma^2}{2})t + m \sigma W_t}. \tag{5}
\]
We know that \( W_t = \frac{1}{\sigma} \left[ \ln \frac{S_t}{S_0} - \left( r - \frac{\sigma^2}{2} \right) t \right] \). By substituting this expression for \( W_t \) into (5) we get

\[
C_t = C_0 \cdot e^{(r - \frac{\sigma^2}{2})t + m\sigma^2 \left( \ln \frac{S_t}{S_0} - \left( r - \frac{\sigma^2}{2} \right) t \right)} = C_0 \cdot \left( \frac{S_t}{S_0} \right)^m \cdot e^{(r - m(r - \frac{\sigma^2}{2}) - m^2 \sigma^2) t} = \alpha_t S_t^m \tag{6}
\]

where \( \alpha_t = \frac{C_0}{S_0^m} \cdot e^{\beta t} \) and \( \beta = \left( r - m(r - \frac{\sigma^2}{2}) - m^2 \sigma^2 \right) \).

We know that \( V_t = C_t + F_t \) and thus with Equation (6) we get

\[
V_t = C_t + F_t = \alpha_t S_t^m + F_t.
\]

Having derived the value \( V_t \) of a CPPI in \( t \) we can now present the properties of the option on CPPI in the next section.

### 3 Option on a Continuous CPPI

In this section we develop a closed-form solution for the price of a Call Option on a CPPI in continuous time. Afterwards we conduct a sensitivity analysis to show how changes in the market parameters influence the price of the Option on a CPPI. Therefore, we calculate the Greeks of the price of the Option on a CPPI, i.e. the derivatives of the price with respect to these parameters. Furthermore, we show which impact the parameters that determine the CPPI strategy, i.e. the multiplicator and the floor, have on the Option on a CPPI.

Let us first calculate the price of the Option on a CPPI. A Call Option on a CPPI is an option on a portfolio that is managed according to the continuous CPPI strategy which we have presented in Section 2. The payoff at maturity \( T \) of this option is \((V_T - K)^+\), where \( V_T \) denotes the value of the CPPI portfolio in \( T \) and \( K \) is the strike price of the option. We assume that the CPPI and the Option on a CPPI have the same maturity \([0, T]\). Furthermore, we only allow for values of the strike price \( K \) which are greater than the floor \( F = F_T \) at maturity \( T \). This seems a reasonable assumption as the continuous CPPI by definition cannot fall below the floor, and in particular it holds \( V_T > F_T \). Thus a strike price which is smaller than the floor \((K < F_T)\) would result in a payoff for the option which is certainly greater than \( F_T - K \). This does not correspond to the typical payoff of the option. As we will show later this
payoff can be produced by the Option on a CPPI plus a zero coupon bond. In Section 2 we have shown that

$$V_t = \alpha_t S_t^m + F_t,$$  \hspace{1cm} (7)

where $\alpha_t = \frac{C_0}{S_0} \cdot e^{\beta t}$ and $\beta = \left(r - m(r - \frac{\sigma^2}{2}) - m^2 \frac{\sigma^2}{2}\right)$. Recall that $F_t$ denotes the floor of the CPPI and is given by $F_t = e^{-r(T-t)} F_T$. The cushion $C_t$ is defined as $C_t = V_t - F_t$.

In order to obtain the price of the Option on a CPPI in $t$ we have to calculate the discounted expectation of the payoff $V_T - K$. The following proposition shows the result.

**Proposition 3.1.** Let the dynamics of $S_t$ be a Geometric Brownian Motion in the risk-neutral world. Given a CPPI on $S_t$ specified by the multiplicator $m$, the floor $F = F_T$ and the investment period $[0,T]$, the price of the Option on this CPPI in $t$ with a strike price of $K$ and the maturity $T$ is

$$OoC(t) = \alpha_t S_t^m N(d_1) - e^{-r(T-t)} (K - F_T)N(d_2),$$  \hspace{1cm} (8)

where

$$d_1 = d_1(t) = \frac{\ln \left( \frac{\alpha_t S_t^m}{K - F_T} \right) + (r + m^2 \frac{\sigma^2}{2})(T-t)}{m \sigma \sqrt{T-t}},$$  \hspace{1cm} (9)

$$d_2 = d_2(t) = d_1(t) - m \sigma \sqrt{T-t},$$  \hspace{1cm} (10)

$$\alpha_t = \frac{C_0}{S_0} \cdot e^{\beta t}, \quad \beta = \left(r - m(r - \frac{\sigma^2}{2}) - m^2 \frac{\sigma^2}{2}\right), \quad C_t = V_t - F_t, \text{ and } F_t = e^{-r(T-t)} F.$$

The proof is shown in the Appendix.

If we have a closer look at the price of the Option on a CPPI in Equation (8), we can see that this formula resembles the price of a standard call option in the Black-Scholes Model.\(^1\) Recall that $\alpha_t S_t^m = C_t$ and $C_t = V_t - F_t$. Thus, Equation (8) becomes

$$OoC(t) = (V_t - F_t) N(d_1) - e^{-r(T-t)} (K - F_T)N(d_2),$$  \hspace{1cm} (11)

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\(^1\)See, e.g., [10] p. 361
and the Option on a CPPI can be interpreted as an option on $V_t$ which is shifted by $F_t$. The volatility of this option is $m\sigma$ compared to $\sigma$ in the Black-Scholes Model and the strike price is $K - F_T$ instead of $K$. With this, $d_1$ and $d_2$ of the Option on a CPPI equal $d_1$ and $d_2$ from the Black-Scholes Model.

To get a better understanding of the Option on a CPPI we now give an example on how its price depends on the specification of its underlying, the CPPI. Figure 1 shows the price of the Option on a CPPI in $t = 0$ with the maturity $T = 3$ years and the volatility $\sigma = 20\%$ for the underlying $S_t$ of the CPPI. The interest rate $r$ is 5\%, the value of the CPPI in $t = 0$ is $V_0 = 100$ and the strike price of the Option on a CPPI is $K = 100$. In the graph the multiplicator and the floor of the CPPI range from 1 to 10 and from 0 to 100 respectively. Note that this option quotes at the money, as $V_t = K = 100$.

![Price Option on CPPI](image)

**Figure 1**: Price Option on CPPI in $t = 0$, $T = 3$ years, $\sigma = 0.2$, $r = 0.05$, $V_0 = 100$ and $K = 100$

As we can see both, the multiplicator and the floor, have a strong impact on the price of the Option on a CPPI. The price for this particular option ranges from about 13 for $m = 1$ and $F = 100$ to more than 80 for $m = 10$ and $F = 0$. Recall that the price of the CPPI is 100 which means that in the latter
case the option is almost as expensive as its underlying (the CPPI). This fact clarifies the enormous upside potential of a speculative CPPI and an option on this CPPI. The plot shows that the option price increases with an increasing multiplicator and a decreasing floor. This can be attributed to the fact that an increase in the multiplicator and a decrease in the floor basically have the same effect which is that the exposure and thus the investment in the risky asset $S$ increases. Consequently, in this case the CPPI becomes more risky and strongly participates in scenarios where the underlying $S$ rises. Here we can clearly see the option character of the Option on a CPPI. In bad scenarios for the underlying $S$ the CPPI will approach the floor so that the option is virtually worthless. However, in good scenarios the payoff of the option will be the greater the more risky the CPPI is. Especially for extreme specifications of the CPPI, like $m = 10$ and $F = 0$, the CPPI is highly levered and its returns amount to a multiple of the returns of $S$. Thus, the distribution of $V_T$ is more skewed to the right the more risky the CPPI is. This explains the high prices for the Option on a CPPI for risky CPPIs. Furthermore, we can see from Equation (8) that the price of the Option on a CPPI depends on the constant interest rate $r$.

Figure 2: Price Option on CPPI in $t = 0$, $T = 3$ years, $r = 0.05$, $V_0 = 100$ and $K = 120$

Figure 2 shows the price of the Option on a CPPI with a strike price of $K = 120$ for varying $\sigma$ and floor. Not surprisingly, a higher volatility of the underlying leads to a greater option price. Again we can see that the higher the floor the lower the option price. Thus Figure 2 confirms that the option
price rises as the CPPI becomes more speculative.

Let us make an interesting remark here. For specific Options on a CPPI where the floor of the CPPI equals the strike price of the Option on a CPPI (i.e. \( K = F_T \)) the price of the option can be derived on an easier way than by using Formula (8), which is by setting up an arbitrage portfolio. Imagine we have two portfolios, Portfolio 1 consisting of a CPPI with multiplicator \( m \) and floor \( F \), and Portfolio 2 which contains a Option on a CPPI on this specific CPPI from Portfolio 1 with strike price \( K = F \) plus a zero-coupon bond \( ZB(t) \) with the nominal value of \( F \). The maturity of the CPPI, the Option on the CPPI and the zero-coupon bond is \( T \).

Comparing the payoffs of these two portfolios at maturity \( T \) we can see that they are equal. The terminal value of Portfolio 1 is \( V_T \) where \( V_T > F = K \), the terminal value of Portfolio 2 is the payoff \( \max\{V_T - K, 0\} = V_T - K \) of the Option on the CPPI plus the terminal-value \( K \) of the zero-coupon bond. Note that with continuous rebalancing the CPPI can approach the floor \( F \) but \( V_t \) will never actually equal \( F \) as the cushion is always positive. In other words, the payoff of the Option on a CPPI is always positive. Table 1 shows the values of the two portfolios in \( t \) and \( T \).

<table>
<thead>
<tr>
<th></th>
<th>Portfolio 1</th>
<th>Portfolio 2</th>
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</thead>
<tbody>
<tr>
<td>( t )</td>
<td>( V_t )</td>
<td>( OoC(t) + ZB(t) )</td>
</tr>
<tr>
<td>( T )</td>
<td>( V_T )</td>
<td>( \max{V_T - K, 0} + ZB(T) = V_T - K + K = V_T )</td>
</tr>
</tbody>
</table>

Table 1: Derivation of price of the Option on a CPPI using an arbitrage portfolio

If the value of Portfolio 1 equals the value of Portfolio 2 in \( T \) their prices have to be the same for all \( t < T \). Otherwise arbitrage opportunities would exist. Thus as we know the price \( V_t \) of the CPPI from Formula (4) in Section 2 and the price of the zero-coupon bond in \( t \) which is \( ZB(t) = e^{-r(T-t)} ZB(T) = e^{-r(T-t)} K \), the price of the Option on a CPPI in \( t \) must be \( OoC(t) = V_t - ZB(t) = V_t - e^{-r(T-t)} K = V_t - F_t \).

Yet, we have to keep in mind that this method to calculate the price of the Option on a CPPI only works in the case where the strike price \( K \) of the Option on a CPPI is equal to the floor \( F \) of the CPPI that represents the underlying of the Option on a CPPI. As it must hold for the floor that \( F \leq e^{rT} V_0 \) the use of an arbitrage portfolio to price the Option on a CPPI is restricted to the cases where \( K \leq e^{rT} V_0 \).
4 The Greeks of the Option on a CPPI

For the Greeks we have calculated the price of the Option on a CPPI and shown how it is influenced by the parameters of the CPPI. In this next section we will study its exposure to movements in the market parameters.

These sensitivities are known as the Greeks and are crucial for the issuer (seller) of the option to control his risks resulting from his short position in the option. Each of the Greeks measures a different dimension of the risk of the short position. These sensitivities indicate how much the price of the Option on a CPPI changes for a marginal change in the respective parameter. It is important to note that the sensitivities with respect to one parameter are calculated under the assumption that all other parameters remain constant. Furthermore, the sensitivities are not constant but change over time. The aim of the issuer is to hedge his position by keeping his exposure to these risks in an acceptable range (See, e.g., [10] p. 421 f).

We have to keep in mind that we are dealing with an option on the CPPI strategy which itself is based on an underlying $S_t$. The parameters which determine the CPPI, i.e. the multiplicator $m$ and the floor $F$, are given and constant as the Option on a CPPI is an option on one specific CPPI strategy. Thus, the risk factors of the Option on a CPPI are the underlying $S_t$ which is traded in the market, the volatility of $S_t$, and the interest rate $r$. The last factor that influences the price of the option is the maturity $T - t$. Later we will see that the maturity is not a risk factor like the ones mentioned above. As the CPPI is not an asset that is traded in the market but on its own has to be replicated using the underlying $S_t$ and the riskless asset we calculate the sensitivity of the Option on a CPPI with respect to $S_t$ instead of $V_t$, the value of the CPPI. A trader who wants to hedge the Option on a CPPI would not make the long way round to replicate the CPPI and then hedge the Option on a CPPI using the replicated CPPI. Furthermore, we have to be aware that CPPIs with different specifications for the multiplicator and the floor respond differently to changes in the risk factors. Accordingly, we will show how the Greeks depend on varying characteristics for the CPPI.

In the following $N'(x)$ shall denote the derivative of the cumulative standard normal distribution with respect to $x$, i.e. the density function of the standard normal distribution at $x$, where $x \sim N(0, 1)$. $N'(x)$ is given by

$$N'(x) = \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{1}{2} x^2}.$$

Furthermore, we will also need $N''(x)$ which is given by

$$N''(x) = -\frac{x}{\sqrt{2\pi}} \cdot e^{-\frac{1}{2} x^2}.$$
To simplify our notation we write $d_1$ and $d_2$ which we have derived in Equations (9) and (10) instead of $d_1(t)$ and $d_2(t)$.

4.1 The Delta

As we have mentioned above we calculate the delta of the Option on a CPPI with respect to $S_t$ and not, as it is usual for standard options, to the direct underlying $V_t$. $\Delta_S^{OoC}$ is the most important sensitivity for the issuer of the option who wants to hedge his exposure towards movements of $S_t$. It shows how much the price of the Option on a CPPI changes with a marginal change of the underlying $S_t$. Therefore, the delta is the slope of the curve which describes the relationship between the option price and the price of the underlying $S_t$. Let us assume for example that the delta of the option is 0.4. This means that for a marginal change in the underlying $S_t$ the change of the option amounts to 40% of this change. In the following proposition we calculate the delta of the Option on a CPPI.

**Proposition 4.1.** The $\Delta_S^{OoC}$ of the Option on a CPPI, i.e. the sensitivity of the Option on a CPPI with respect to changes of the value $S_t$ of the CPPI, is given by:

$$\Delta_S^{OoC} = \frac{\partial O_{OoC}}{\partial S_t} = m \alpha_t S_t^{m-1} \cdot N(d_1).$$

(12)

*Proof.* The proof follows from calculating the delta of the CPPI with respect to $S_t$ (denoted by $\Delta_S$) and then deriving the delta of the Option on a CPPI with respect to $V_t$ (denoted by $\Delta_V^{OoC}$). 

Let us now give an example for the delta of the Option on a CPPI. In the following we will study an Option on a CPPI with maturity $T = 3$ years and a strike price $K = 100$. The underlying of the Option on a CPPI will be a CPPI with maturity $T = 3$ years, $V_0 = 100$ and $F = 80$. Furthermore, we will consider three different multiplicators $m = 2, 5$ and $8$ for the CPPI. The risky asset $S_t$ has a volatility of $\sigma = 0.2$ where $S_0 = 100$ and the interest rate on the riskless asset is $r = 0.05$. As we want to study how the Option on a CPPI depends on the evolution of $S_t$, we choose $t = 1$ for the point of our examination (any other value could have been chosen).

Figure 3 shows the value $V_t$ of the CPPI described above after one year with multiplicators of $m = 2, 5$ and $8$. We can see that the payoff of the CPPI becomes more convex with a higher multiplicator. In the case of a downturn
of $S_t$ the CPPI with multiplicator $m = 8$ has considerably approached the discounted floor $F_t = e^{-r(T-t)} F = 76.1$ which means that the exposure is almost zero and the CPPI is for the most part invested in the riskless asset. On the other hand, this CPPI generates remarkable returns for high stock prices. In the scenario where the risky asset climbed from 100 to 120 in $t = 1$ the value $V_1$ of the CPPI with multiplicator 8 went up to 180. Here the investment in the risky asset, i.e. the exposure, amounts to $E_1 = m\cdot C_1 = 8(180 - 76.1) = 831$.

Let us now demonstrate how the $\Delta^{O_oC}_S$ of the Option on a CPPI depends on the price of the risky asset $S_t$ and the multiplicator $m$. Figure 4 shows the $\Delta^{O_oC}_S$ of the Option on a CPPI in $t = 1$ on the same CPPIs as before. The strike price of the option is $K = 100$ again.

We can see that the $\Delta^{O_oC}_S$ of the Option on a CPPI resembles the $\Delta_S$ and becomes more convex with increasing values for the multiplicator. Note that for low values of $S_t$ the $\Delta^{O_oC}_S$ is higher for low multiplicators whereas this relation reverses for high values of $S_t$. It is evident that the $\Delta^{O_oC}_S$ approaches zero for declining prices for $S_t$. This effect is stronger for higher multiplicators because here the cushion is shrinking faster and thus the respective CPPI is invested in the riskless asset to a greater extent than a CPPI with lower multiplicator. In contrast to standard call options, the $\Delta^{O_oC}_S$ can adopt values greater than 1. Whereas a standard option can never move by more than the change of the underlying, this is very well possible for the Option on a CPPI with re-
spect to $S_t$. The reason for this was pointed out in Figure 3 in the case of a high multiplicator combined with high prices for $S_t$. Here the CPPI is highly levered due to its great exposure. Hence, merely minor changes of $S_t$ cause greater movements of $V_t$ and thus the Option on a CPPI, which is where the high values for the $\Delta_{S}^{OoC}$ stem from.

To understand the meaning of these values for the $\Delta_{S}^{OoC}$ let us give some numbers here. For $S_t = 120$ and $m = 8$ the $\Delta_{S}^{OoC}$ amounts to approximately 2.5 in $t = 1$. In this case the price of the Option on a CPPI is 25. A $\Delta_{S}^{OoC}$ of 2.5 means that the option price changes by 2.5 for a movement of $S_t$ by 1. Thus, a return of 0.8% ($= (121 - 120)/120$) is accompanied by a return of 10% ($= (27.5 - 25)/25$) for the Option on a CPPI which is about twelve times as much.

This immense power of the Option on a CPPI for high multiplicators and rising prices for the underlying is pointed out further by the $\Gamma$ of the Option on a CPPI which we will derive in the next section.

4.2 The Gamma

In the next proposition we give the gamma ($\Gamma_{S}^{OoC}$) of the Option on a CPPI. The $\Gamma$ of an option is the sensitivity of its delta with respect to the price of the underlying $S_t$. It measures the curvature of the relationship between the
option price and $S_t$. With a gamma-neutral position the influence of this curvature on the performance of the delta-hedging can be reduced. A high $\Gamma$ of a delta-neutral portfolio indicates that the hedging has to occur more frequently to keep the portfolio delta-neutral compared to a portfolio with a low $\Gamma$. The reason for this is that in the case of a high $\Gamma$ changes of the underlying $S_t$ lead to higher changes of the delta.

**Proposition 4.2.** The $\Gamma^{OoC}_S$ of the Option on a CPPI, i.e. the sensitivity of the $\Delta^{OoC}_S$ with respect to $S_t$, is given by

$$
\Gamma^{OoC}_S = \frac{N'(d_1) (m - 1)}{\sigma \sqrt{T - t} \cdot S_t^2} + \frac{m (m - 1) \alpha_t}{m^2 - 2} \cdot N(d_1).
$$

(13)

**Proof.** The proof follows from standard calculus derivations. \qed

![Figure 5: $\Gamma$ of the Option on a CPPI in $t = 1$, $T = 3$ years, $\sigma = 0.2$, $r = 0.05$, $V_0 = 100$ and $K = 100$](image)

Figure 5 shows the $\Gamma^{OoC}_S$ for the example we have studied above. We can see that the $\Gamma^{OoC}_S$ is always positive which means that the $\Delta^{OoC}_S$ is monotonically increasing. Furthermore, it turns out that the $\Gamma^{OoC}_S$ is increasing for $m = 5$ and $m = 8$ and decreasing for $m = 2$. Hence the $\Delta^{OoC}_S$ is convex for higher and concave for lower multiplicators. The convexity of the first derivative of the Option on a CPPI for high multiplicators again demonstrates the power of the Option on a CPPI. A $\Delta^{OoC}_S$ that is growing even more the higher $S_t$ gets shows
the enormous upside potential of the Option on a CPPI. This behaviour is a result of the combination of an option on the CPPI which itself can generate large returns due to its possibility to build up a high leverage. Note that in contrast to the Option on a CPPI the \( \Gamma_{OoC}^S \) of a standard call option approaches zero for options which are far in the money. Here the option approximately behaves like the underlying whereas the Option on a CPPI moves exponentially with the underlying \( S_t \).

We will now show the impact of the volatility of \( S_t \) on the price of the Option on a CPPI.

### 4.3 The Vega

The vega of an option is the sensitivity of the option price with respect to the volatility of the underlying.

**Proposition 4.3.** The vega \( \frac{\partial O_oC_t}{\partial \sigma} \) of the Option on a CPPI is given by

\[
vega = \frac{\partial O_oC_t}{\partial \sigma} = \alpha_t S^m_S (tm\sigma(1 - m) N(d_1) + N'(d_1) \cdot \frac{\partial d_1}{\partial \sigma})
\]

\[ -e^{-r(T-t)}(K - F_T) N'(d_2) \cdot \frac{\partial d_2}{\partial \sigma}, \tag{14} \]

where

\[
\frac{\partial d_1}{\partial \sigma} = \frac{1}{2} m T + t(1 - \frac{3}{2}m) \frac{\sqrt{T-t}}{\sigma} - \frac{\ln \left( \frac{\alpha_t S^m_S}{K - F_T} \right)}{m \sigma^2 \sqrt{T-t}} \frac{\sqrt{T-t}}{m \sigma^2}
\]

\[
\frac{\partial d_2}{\partial \sigma} = \frac{\partial d_1}{\partial \sigma} - m \sqrt{T-t}.
\]

**Proof.** The proof follows from standard calculus derivations. \( \square \)

The vega of the Option on a CPPI in \( t = 1 \) is presented in Figure 6. The vega is decreasing with increasing prices for \( S_t \). Furthermore, higher multipliers lead to a greater influence of \( S_t \) (from Equation 4, \( S^m_S, S_t > 1 \)), implying lower vegas, with higher absolute values, which means that these options are more sensitive to changes in the volatility. For \( m = 2 \) and \( m = 5 \) the vega is positive for low values of \( S_t \) which can be explained as follows. In these cases the cushion of the CPPI has diminished as \( V_t \) has approached the floor. Thus, a strong performance of \( S_t \) is required for the option to get into the money again. This performance becomes more likely if the volatility of \( S_t \) increases. On the other hand of course, an increasing volatility also increases the probability of lower prices for \( S_t \). But this risk doesn’t affect the option price negatively as
the downside potential of the CPPI is protected by the floor and the Option on a CPPI quotes out of the money in this case. Accordingly, it will expire worthless unless $S_t$ rises significantly. Hence, a higher volatility enhances the chances of the option ending in the money at maturity while the risks that come along with a higher volatility do not hurt the option price much in this particular case. To sum it up, the Option on a CPPI profits from a rising volatility if the option quotes out of the money, and thus the vega is positive for this case.

To give a complete picture of the influence of $\sigma$ on the price of the Option on a CPPI let us have a look at Figure 7. It shows how the price of the Option on a CPPI depends on $\sigma$ and $t$. Here we examine an at the money call for all $t$ as $V_t = K = 100$. Recall that in Figure 6 we presented the vega for this example at one specific point in time, which was $t = 1$, for varying prices of $S_{t=1}$. Figure 7 demonstrates that the impact of $\sigma$ on the price of the option changes over time. Obviously the price converges to zero with decreasing maturity (as $V_t = K \forall t$). However, the interesting conclusion this plot presents is the hump in the curve for the price of the Option on a CPPI. For most values of $t$ an increase in $\sigma$ leads to higher option prices (starting from low values for $\sigma$) while this relation is reversed for higher $\sigma$s, i.e. a further increase in the volatility results in a decrease of the option price. In the most cases an increase in the volatility leads to lower option prices. Yet, for $t = 0$ the price
of the Option on a CPPI increases with rising $\sigma$.
Furthermore, the hump of the curve is moving backwards in terms of increasing $\sigma$ and increasing maturity, i.e. for one fixed $t$ the peak of the price moves to higher $\sigma$s if we decrease $t$ or increase the maturity respectively. This tells us that on a certain level for $\sigma$ an increase in $\sigma$ may be favorable for the option price for longer maturities but not for shorter maturities.

Together these inspections of the relationship of $\sigma$ and the price of the Option on a CPPI clarify that the Option on a CPPI is a complex product which depends on many parameters. These parameters influence each other so that varying combinations can yield in different results. In the next section we will present the rho of the Option on a CPPI and derive how it responds to changes of the interest rate.

### 4.4 The Rho

The rho measures how changes in the interest rate $r$ influence the price of the Option on a CPPI.
Proposition 4.4. The rho $\frac{\partial O_o C_t}{\partial r}$ of the Option on a CPPI is given by

$$\rho = \frac{\partial O_o C_t}{\partial r} = \alpha_t S^m_t \left( (1 - m) t N(d_1) + N'(d_1) \cdot \frac{\partial d_1}{\partial r} \right) + e^{-r(T-t)} (K - F_T) \left( (T - t) N(d_2) - N'(d_2) \cdot \frac{\partial d_2}{\partial r} \right) \tag{15}$$

where $\frac{\partial d_1}{\partial r}$ and $\frac{\partial d_2}{\partial r}$ are given by

$$\frac{\partial d_1}{\partial r} = \frac{\partial d_2}{\partial r} = \frac{1}{\sigma \sqrt{T-t}} \left( \frac{T}{m} - t \right).$$

Proof. The proof follows from standard calculus derivations.

4.5 The Theta

The theta ($\Theta$) measures the options exposure to the passage of time. Specifically, it describes how the price of the Option on a CPPI changes with the time provided that all other parameters remain constant. The basic difference of the $\Theta$ compared to the other sensitivities we presented above is that the future prices of $S_t$, the future volatility and interest rate are uncertain whereas the passage of time is not. This implies that it makes no sense to hedge the option against a change of $T - t$.

Proposition 4.5. The $\Theta$, i.e. the sensitivity of the price of the Option on a CPPI with respect to the passage of time, is given by

$$\Theta = \frac{\partial O_o C_t}{\partial (T-t)} = \alpha_t S^m_t N'(d_1) \cdot \frac{\partial d_1}{\partial (T-t)} + e^{-r(T-t)} (K - F_T) \cdot \left( r N(d_2) - N'(d_2) \frac{\partial d_2}{\partial (T-t)} \right) \tag{16}$$

where $\frac{\partial d_1}{\partial (T-t)}$ and $\frac{\partial d_2}{\partial (T-t)}$ are given by

$$\frac{\partial d_1}{\partial (T-t)} = \frac{r + m^2 \sigma^2}{2m \sigma \sqrt{T-t}} - \frac{\ln \left( \frac{\alpha_t S^m_t}{K - F_T} \right)}{m^2 \sigma^2 (T-t)}$$

$$\frac{\partial d_2}{\partial (T-t)} = \frac{\partial d_1}{\partial (T-t)} - \frac{m \sigma}{2 \sqrt{T-t}}.$$

Proof. The proof follows from standard calculus derivations.
5 Option on a Discrete CPPI

In the previous section we conducted a detailed examination of the Option on a CPPI in continuous time. We found a closed-form solution for the option price and calculated the Greeks of the option. This section deals with an Option on a CPPI where the underlying CPPI portfolio is rebalanced in discrete time. As we know from the previous section the continuous-time application of the CPPI ensures that the portfolio value does not fall below the floor if the price process of the risky asset does not permit jumps. However, in practice continuous rebalancing is not feasible. In turbulent markets it can happen that the underlying asset falls steeply before the investor is able to rebalance his portfolio adequately. Hence it is no longer ensured that the strategy outperforms the prescribed floor if there is a sudden drop in market prices like for example in the 1987 crash. The risk of falling under the floor is called gap risk. It depends on the specification of the CPPI and usually is marginal. Nevertheless, imposing the gap risk on the investor is not a reasonable option because that would contradict the idea of the portfolio insurance. We can assume that the buyer of a CPPI is completely risk averse for the terminal value of the strategy to end up below the floor and thus would not invest in a CPPI where he bears the gap risk. A better way to deal with the gap risk is a hard guarantee which is given by the issuer of the CPPI or the institution that carries out the actual trading of the assets. The issuer takes the gap risk and considers this in the pricing of the CPPI. Accordingly a premium upon the value of the CPPI portfolio arises.

In this section we first calculate the value of a CPPI portfolio with discrete rebalancing including this premium. With this result we can give recommendations on how frequently the rebalancing should occur depending on the multiplier and the volatility of the underlying to minimize the probability of falling below the floor and thus the premium. To model the gap risk we introduce trading restrictions, i.e. we only allow for trading of the underlying portfolio at specific dates. Thus we keep our continuous stochastic process to model the underlying S but restrict trading to discrete time. This means that possibly the CPPI cannot be adjusted adequately.

Let us now define the discrete CPPI. As before we consider a maturity period of $[0,T]$ and $t_0 = 0$. Rebalancing occurs in $t_s$, $s = 1, ..., N - 1$ and $N = \frac{T}{\Delta t}$, where $\Delta t$ denotes the period between two rebalancing dates. Note that in this setup $\varphi^S$ and $\varphi^B$, i.e. the number of shares held in the risky and the riskless asset, are constant between two rebalancing dates whereas the value of the investments in those two assets changes with movements in the prices for $S$ and $B$.

Recall that the value of the CPPI in $t$ is given by $V_t = \varphi^S_t S_t + \varphi^B_t B_t$. As we

---

2See, e.g. [2]
do not want to allow for short positions in the risky asset, $\varphi^S_t$ for $t \in [t_s, t_{s+1})$, $s = 0, \ldots, N - 1$, is given by

$$
\varphi^S_t = \max \left\{ \frac{mC_{t_s}}{S_{t_s}}, 0 \right\}.
$$

In order for the CPPI to be self-financing it must hold

$$
\varphi^S_{t_s} S_{t_{s+1}} + \varphi^B_{t_s} B_{t_{s+1}} = \varphi^S_{t_{s+1}} S_{t_{s+1}} + \varphi^B_{t_{s+1}} B_{t_{s+1}}
$$

for all $s = 0, \ldots, N - 1$.

In the next proposition we derive the value $V_{t_s}$ of a CPPI portfolio in $t_s$. Here we have to consider that the CPPI might fall below the floor before the next rebalancing date.

**Proposition 5.1.** Let

$$
t_k := \min \{ t_s | s = 1, \ldots, N - 1, V_{t_s} - F_{t_s} \leq 0 \}
$$

and $t_k = \infty$ if the minimum is not attained. Then the value $V_{t_{s+1}}$ of a CPPI portfolio in $t_{s+1}$ in the case of discrete rebalancing is given by

$$
V_{t_{s+1}} = e^{r(t_{s+1} - \min\{t_k, t_{s+1}\})} \cdot \left( (V_0 - F_0) \prod_{i=1}^{\min\{k, s+1\}} \left( \frac{mS_{t_i}}{S_{t_{i-1}}} - (m - 1)e^{r\Delta t} \right) + F_{t_{\min\{k, s+1\}}} \right).
$$

(17)

**Proof.** If we define $r_s = \ln \frac{S_{t_s}}{S_{t_{s-1}}}$, we get for $V_{t_{s+1}}$

$$
V_{t_{s+1}} = \max \left\{ mc_{t_{s+1}} S_{t_{s+1}}, 0 \right\} + (V_{t_s} - \max \{ mc_{t_s}, 0 \}) e^{r\Delta t} = \begin{cases} 
mc_{t_s} S_{t_{s+1}} + (V_{t_s} - mc_{t_s} F_{t_s}) e^{r\Delta t} & \text{if } V_{t_s} - F_{t_s} > 0 \\
V_{t_s} e^{r\Delta t} & \text{if } V_{t_s} - F_{t_s} \leq 0 
\end{cases}
$$

$$
= \begin{cases} 
(V_{t_s} - F_{t_s}) \left( mc_{t_{s+1}} S_{t_{s+1}} - mc_{t_s} F_{t_s} \right) + V_{t_s} e^{r\Delta t} & \text{if } V_{t_s} - F_{t_s} > 0 \\
V_{t_s} e^{r\Delta t} & \text{if } V_{t_s} - F_{t_s} \leq 0 
\end{cases}
$$

$$
= \begin{cases} 
(V_{t_s} - F_{t_s}) \left( mc_{t_{s+1}} S_{t_{s+1}} - (m - 1)c_{t_s} F_{t_s} \right) + F_{t_s} e^{r\Delta t} & \text{if } V_{t_s} - F_{t_s} > 0 \\
V_{t_s} e^{r\Delta t} & \text{if } V_{t_s} - F_{t_s} \leq 0 
\end{cases}
$$

Iterative application of this step leads to the result.

\[\Box\]
Remark 5.2. If $t_k < t_N = T$, i.e. the portfolio falls below the floor in $t_k$, the process for $V_{ts}$ switches from

$$V_{ts} = (V_0 - F_0) \cdot \prod_{i=1}^{ts} \left( m \frac{S_{ti}}{S_{ti-1}} - (m-1)e^{r \Delta t} \right) + F_{ts}, t_s < t_k$$

to

$$V_{ts} = V_{tk} e^{r(t_s-t_k)}, t_s > t_k$$
in $t_k$. For $V_T$ we get

$$V_T = \begin{cases} (V_0 - F_0) \cdot \prod_{i=1}^{N} \left( m \frac{S_{ti}}{S_{ti-1}} - (m-1)e^{r \Delta t} \right) + F_T & \text{if } t_k \geq t_{N-1} \\ V_{tk} e^{r(T-t_k)} & \text{if } t_k \leq t_{N-1}. \end{cases}$$

As the next step, we want to calculate the value of the discrete CPPI portfolio in $0$. The difficulty here is that the portfolio can fall below the floor in each $t_i$, $i = 1, ..., N$, and that in this case the process for $V_{ts}$, $s = i+1, ..., N$ switches because the whole portfolio is invested in the riskless asset.

Recall that $\frac{S_{ti}}{S_{ti-1}} = e^{(r-\frac{\sigma^2}{2})\Delta t + \sigma W_{ti}}$, where $W_{ti} \sim N(0, \Delta t)$.

The process for the CPPI portfolio switches if the portfolio falls below the discounted floor $F_{ts}$ in $t_s$ for $s = 1, ..., N - 1$. This happens if $C_{ts+1} \leq 0$ under the assumption that $C_{ts} > 0$. For the cushion $C_{ti}$ to be positive in $t_i$, $i = 1, ..., N$ it must hold:

$$C_{ti+1} > 0 \iff mC_{ti} \frac{S_{ti+1}}{S_{ti}} + (V_{ti} - mC_{ti}) e^{r \Delta t} > F_{ti+1}$$

$$\iff (V_{ti} - F_{ti}) \left( m \frac{S_{ti+1}}{S_{ti}} - (m-1)e^{r \Delta t} \right) > 0 \quad (18)$$

$$C_{ti} > 0 \iff W_{ti} > \frac{\ln \frac{m-1}{m} + \frac{\sigma^2}{2} \Delta t}{\sigma} =: \hat{z} \quad (19)$$

If we want to calculate the price of a CPPI portfolio with discrete rebalancing, we have to distinguish between two cases, which is the case where the portfolio never falls below the floor during the investment period (Case 1) and the case where the portfolio does fall below the floor (Case 2). Example 5.3 illustrates the functionality of the discrete CPPI.

Example 5.3. Figure 8 gives an example of a discrete CPPI with two rebalancing dates. In $t = 1$ the cushion can be either positive or negative depending on the performance of the risky asset in $[0, 1]$. In the case the CPPI has fallen under the floor, the whole portfolio is invested in the riskless asset and the
Option on a CPPI

Figure 8: Three step example for the value process of a discrete CPPI

terminal value of the CPPI is $V_3 = V_1 e^{2r\Delta t}$. In the other case, the portfolio is rebalanced according to the rebalancing rules of the CPPI. Now the same procedure is repeated in $t = 2$.

Recall that, according to our definition, the discrete CPPI is self-financing. Hence, under the risk-neutral measure the expected terminal value of the discrete CPPI is

$$E[V_T] = e^{rT}V_0.$$ (20)

In the following we will point out that the value of the CPPI in 0 for the investor is not $V_0$. This is due to the fact that a CPPI is sold as a product with a capital guarantee. Thus, we can imply that the buyer of the CPPI is completely risk-averse for terminal values below the floor when buying a portfolio insurance product. Hence, the issuer of the CPPI should guarantee a payoff equal to or greater than the floor and therefore has to carry the gap risk which arises in Case 2 (see below). Consequently, the payoff of the discrete CPPI for the investor is

$$CPPI_T = \max\{V_T, F_T\}.$$ (21)

Let us briefly clarify the notation we will use. In the following, $CPPI_t$ shall denote the value of the CPPI for the investor while $V_t$ denotes the portfolio value of the CPPI portfolio as we have shown in Example 5.3.

Equation (21) constitutes that the investor receives the terminal value $V_T$ of the CPPI if the floor has not been hit during the maturity. In the case the floor has been hit, the issuer of the CPPI pays the guaranteed amount $F_T$. Note that if the CPPI has fallen below the floor, it holds $V_T < F_T$, i.e. the issuer has to pay out more than the CPPI is actually worth. Thus $CPPI_T$ and $V_T$ differ in case the CPPI has fallen below the floor. Figure 9 illustrates the relation of $CPPI_T$ and $V_T$. 
Next, we want to calculate the price of the discrete CPPI. The issuer should sell the CPPI at a price which corresponds to the expected payoff for the investor. Hence, the price in 0 should be $e^{-rT} E[CPPI_T]$. As we can see from Figure 9 this price is composed of two parts. According to the calculation rules for the expected value, we get

$$E[CPPI_T] = E[CPPI_T|C1] + E[CPPI_T|C2].$$  \hspace{1cm} (22)

Let us first examine $E[CPPI_T|C1]$, where the portfolio does not fall below the floor during the maturity, or more precisely in $t_s$, $s = 1, ..., N$. In this case it holds that $CPPI_T = V_T$ and we know from Remark 5.2 that $V_T$ is given by

$$V_T = (V_0 - F_0) \cdot \prod_{i=1}^{N} \left( m \frac{S_{t_i}}{S_{t_{i-1}}} - (m - 1)e^{r\Delta t} \right) + F_T. \hspace{1cm} (23)$$

To obtain the final value $V_T$ in Case 1, $V_T|C1$, we have to multiply $V_T$ with the indicator of the event that the portfolio does not fall below the floor in $t_s$, $s = 1, ..., N$.

$$V_T|C1 = \left( (V_0 - F_0) \cdot \prod_{i=1}^{N} \left( m \frac{S_{t_i}}{S_{t_{i-1}}} - (m - 1)e^{r\Delta t} \right) + F_T \right) \cdot \prod_{l=1}^{N} 1_{\{V_{t_l} > F_{t_l}\}}. \hspace{1cm} (24)$$

Next, we examine $E[CPPI_T|C2]$. According to our definition of the discrete CPPI, the issuer guarantees a payoff of $F_T$ even if the CPPI has fallen below the floor during the maturity period. Hence, to obtain $CPPI_T|C2$ we have to multiply $F_T$ with the indicator of the event that the portfolio does fall below the floor for one $t_s$, $s = 1, ..., N$. This event is the complement of the event that the CPPI never falls below the floor (see Case 1). Thus we get

$$CPPI_T|C2 = F_T \cdot \left( 1 - \prod_{l=1}^{N} 1_{\{V_{t_l} > F_{t_l}\}} \right).$$

Figure 9: Relation of $CPPI_T$ and $V_T$
Let us now calculate $e^{-rT} E[CPPI_T|C1] = e^{-rT} E[V_T|C1]$.

**Proposition 5.4.** The expected value of $CPPI_T$ (and $V_T$) discounted to 0 for Case 1, where the portfolio does not fall below the floor during the investment period is:

\[ e^{-rT} E[CPPI_T|C1] = F_0 N(d_2)^N + (V_0 - F_0) [mN(d_1) - (m - 1)N(d_2)]^N, \]  

(25)

where

\[ d_1 = \frac{\ln \frac{m}{m-1} + \frac{\sigma^2}{2} \Delta t}{\sigma \sqrt{\Delta t}}, \quad d_2 = d_1 - \sigma \sqrt{\Delta t}. \]

The proof can be found in the Appendix.

Having derived $e^{-rT} E[CPPI_T|C1]$, we now want to calculate $e^{-rT} E[CPPI_T|C2]$ according to Equation (25).

**Proposition 5.5.** The expected value for $CPPI_T$ discounted to 0 in Case 2, i.e. in the case where the CPPI has fallen under the floor $F_{t_i}$ in $t_i$, $i = 1, ..., N$, is

\[ e^{-rT} E[CPPI_T|C2] = F_0 \cdot (1 - N(d_2)^N), \]

(26)

where $d_2$ is again given by $d_2 = \frac{\ln \frac{m}{m-1} - \frac{\sigma^2}{2} \Delta t}{\sigma \sqrt{\Delta t}}$.

**Proof.** In Equation (25) we derived

\[ E[CPPI_T|C2] = F_T \cdot \left(1 - \prod_{l=1}^{N} 1_{\{V_{t_l} > F_{t_l}\}}\right). \]

(27)

Now recall Equation (19) where we showed that the CPPI falls below the floor in $t_i$, $i = 1, ..., N$ if

\[ W_{t_i} < \frac{\ln \frac{m}{m-1} + \frac{\sigma^2}{2} \Delta t}{\sigma} =: z \]

where $W_{t_i} \sim N(0, \Delta t)$. Thus $P(W_{t_i} < z)$ is given by

\[ P(W_{t_i} < z) = N\left(\frac{z}{\sqrt{\Delta t}}\right) \]
and \( E[CPPI_T|C2] \) is given by

\[
E[CPPI_T|C2] = F_T \cdot \left( 1 - \left( 1 - N \left( \frac{z}{\sqrt{\Delta t}} \right) \right)^N \right)
\]

\[
= F_T \cdot \left( 1 - N \left( \frac{-z}{\sqrt{\Delta t}} \right)^N \right).
\]

With \( F_0 = e^{-rT} F_T \), inserting \( z \) leads to the result.

Having done these calculations we can now give the price of a discrete CPPI.

**Proposition 5.6.** The price of a discrete CPPI in 0 is

\[
e^{-rT} E[CPPI_T] = F_0 + (V_0 - F_0) \left[ mN(d_1) - (m - 1)N(d_2) \right]^N
\]

where

\[
d_1 = \frac{\ln \frac{m}{m-1} + \frac{\sigma^2 \Delta t}{2}}{\sigma \sqrt{\Delta t}}
\]

\[
d_2 = \frac{\ln \frac{m}{m-1} - \frac{\sigma^2 \Delta t}{2}}{\sigma \sqrt{\Delta t}} = d_1 - \sigma \sqrt{\Delta t}.
\]

**Proof.** The result follows from Equation (22), using Propositions 5.4 and 5.5. □

### 5.1 Pricing an Option on a Discrete CPPI

The transition from continuous to discrete rebalancing involves the arising of a gap risk, i.e. the problem that the discrete CPPI can fall below the floor if the risky asset exhibits large sudden losses. Therefore the discrete CPPI is no longer path-independent but its value \( V_t \) depends on the course of \( S_t \) in \([0, t]\). Hence we have to check at every rebalancing date if \( V_t \) is still greater than the discounted floor \( F_t \) or not. Accordingly, the process for \( V_t \) can switch if the CPPI falls below the floor because in this case the whole portfolio is invested in the riskless asset. Furthermore, as in the continuous case, we assume that the strike price \( K \) of the Option on a CPPI is greater than the floor \( F \). This means that the Option on a CPPI expires worthless once the CPPI has fallen below the discounted floor \( F_t \) in \( t \) because then the portfolio is fully invested in the riskless asset and can never become greater than \( F \) again.
For the following, let the setting be the same as in previous section. Denote with $V_{ts}$ the value of a CPPI portfolio in $t_s$, where the rebalancing occurs in $t_s$, $s = 1, ..., N - 1$ and $N = \frac{T}{\Delta t}$. Furthermore, let $r$ be deterministic again.

Recall that we derived the value $V_{ts}$ of a CPPI with discrete rebalancing as

$$V_{ts} = e^{r(t_s - \min\{t_k, t_s\})} \cdot \left( (V_0 - F_0) \prod_{i=1}^{\min\{k, s\}} \left( m \frac{S_{ti}}{S_{ti-1}} - (m - 1)e^{r\Delta t} \right) + F_{t_{\min\{k, s\}}} \right),$$

where $t_k$ was defined as the point of time when the CPPI falls below the floor, $t_k := \min\{t_s, s = 1, ..., N - 1\} | V_{ts} - F_{ts} \leq 0 \}$ and $t_k = \infty$ if the minimum is not attained. Thus, for the value of the CPPI at maturity $T$ we get Remark 5.2. Furthermore, recall that $S_{ti}S_{ti-1} - 1 = e^{-(r - \frac{\sigma^2}{2})\Delta t - \sigma W_{ti}}$, where $W_{ti} \sim N(0, \Delta t)$.

Under the assumption that $K > F_T$ the option on the CPPI portfolio expires worthless either if the CPPI falls below the discounted floor $F_{ti}$ in $t_i$ for $i = 1, ..., N - 1$ during the investment period (Case 1) or if $V_T < K$ at maturity $T$ (Case 2).

Case 1 occurs if $C_{t_{i+1}} \leq 0$ under the condition that $C_{t_i} > 0$. For the cushion $C_{t_{i+1}}$ to be positive in $t_{i+1}$, $i = 1, ..., N$, Equation 18 must hold. Note that $z$ is constant and independent of $t$.

Now, if we consider the special case where $K = F_T$, the Option on a discrete CPPI ends up in the money at maturity $T$ if the CPPI has not defaulted until maturity $T$, i.e. $C_{t_i} \geq 0 \forall t_i, i = 1, ..., N$. Then, a portfolio which is composed of the Option on a CPPI with $K = F_T$ and a zero-coupon bond with nominal value $K$, is equal to a CPPI with floor $F$ as we have shown for the continuous case in Section 2. Hence, the price of the Option on a CPPI can be calculated as the difference of a discrete CPPI and $F_0$, where the price of the discrete CPPI is known, Equation (28). Thus, for $K = F_T$, the price of the Option on a discrete CPPI in $0$, denoted by $OoC^d(0)$, is given as

$$OoC^d(0) = (V_0 - F_0) [mN(d_1) - (m - 1)N(d_2)]^N. \quad (29)$$

Let us now focus on the general case where $K > F_T$. In this case the Option on a discrete CPPI ends up in the money at maturity $T$ if the CPPI has not fallen below the floor in $t_i, i = 1, ..., N - 1$ and if the value $V_T$ is greater than the strike price $K$. In order to calculate the price of the Option on a CPPI in $0$ we have to discount the expected payoff of the option to $0$. We have to check at each rebalancing date whether $V_t$ is above the floor or not. Furthermore, it
must hold $V_T > K$. With $\hat{z}$ we get

$$OoC^d(0) = e^{-rT}E \left[ (V_T - K)^+ | W_i > \hat{z}, i = 1, ..., N - 1 \right]$$

$$= E \left[ \left( F_T - K + (V_0 - F_0) \prod_{i=1}^{N} \left( me^{(r - \frac{\sigma^2}{2}) \Delta t + \sigma W_i} - (m - 1)e^{r \Delta t} \right) \right)^+ \right]$$

$$= e^{-rT} \prod_{i=1}^{N-1} 1 \{ W_i > \hat{z} \}.$$  \hspace{1cm} (30)

The price of the Option on a discrete CPPI can be obtained by using Monte Carlo simulations in Equation 30. Greeks could also be computed by MonteCarlo simulations as well within this setting.

6 Conclusion

In this paper we examined a dynamic asset allocation strategy and European call options on this strategy. We first dealt with the CPPI strategy with continuous rebalancing. We elaborated on Options on a CPPI finding a closed-form solution for the option price as well as closed-form expressions for several useful sensitivities like Delta, Gamma, Vega, Rho and Theta. In a second step, discrete rebalancing was considered. In the discrete case an expectation expression was found for the price of the derivative depending on the path of the underlying stock price. Therefore Monte Carlo simulations could be used to calculate the price as well as the sensitivities. This work could be generalized by studying other common derivatives on a CPPI, like Asian and American options or more hand-made products targeting the two dimensional joint behaviour at maturity of the CPPI and the underlying stock price.

A Appendix

Proof. Proposition 3.1

Recall that if $S$ follows a Geometric Brownian Motion under the risk-neutral measure, $S_t$ is given by

$$S_t = S_0 e^{(r - \frac{\sigma^2}{2})t + \sigma W_t}.$$  \hspace{1cm} (31)

Thus, in $t$ we get for $S_T$

$$S_T = S_t e^{(r - \frac{\sigma^2}{2})(T-t) + \sigma W_{T-t}}.$$
The payoff of the Option on a CPPI is positive if \( V_T > K \). With Equation (7) this becomes to

\[
V_T > K \iff \alpha T S^m_T + F_T > K \iff S^m_T > \frac{K - F_T}{\alpha T} \iff S_T > m \sqrt{\frac{K - F_T}{\alpha T}}.
\]

Substituting \( S_T \) from Equation (31) leads to

\[
S_T > m \sqrt{\frac{K - F_T}{\alpha T}} \iff S_t e^{(r - \frac{\sigma^2}{2})(T-t) + \sigma W_{T-t}} > m \sqrt{\frac{K - F_T}{\alpha T}}
\]

\[
\ln \left( \frac{m \sqrt{\frac{K - F_T}{\alpha T}}}{S_t} \right) - (r - \frac{\sigma^2}{2})(T-t)
\]

\[
\iff W_{T-t} > \frac{\ln \left( \frac{m \sqrt{\frac{K - F_T}{\alpha T}}}{S_t} \right) - (r - \frac{\sigma^2}{2})(T-t)}{\sigma}.
\]

Thus, we get a positive payoff for the Option on a CPPI at maturity \( T \) if

\[
W_{T-t} > \frac{1}{m} \ln \left( \frac{K - F_T}{\alpha T S^m_T} \right) - (r - \frac{\sigma^2}{2})(T-t) =: z.
\]

(32)

Now we can calculate the price of the Option on a CPPI in \( t \). Therefore, we have to discount the expected payoff in \( T \) to \( t \).

\[
OoC(t) = E \left[ e^{-r(T-t)} (V_T - K)^+ | \mathcal{F}_t \right] = e^{-r(T-t)} E \left[ (V_T - K)^+ | \mathcal{F}_t \right]
\]

\[
= e^{-r(T-t)} \int_0^\infty (\alpha T S^m_T + F_T - K)^+ f(s_T)ds_T
\]

\[
= e^{-r(T-t)} \int_0^\infty \left( \frac{K - F_T}{\alpha T} \right) f(s_T)ds_T
\]

\[
= e^{-r(T-t)} \int_{z}^\infty \left( \alpha T S^m_t e^{m((r - \frac{\sigma^2}{2})(T-t) + \sigma W_{T-t})} + F_T - K \right) f(w_{T-t})dw_{T-t}
\]
\[ 
= e^{-r(T-t)} \alpha T S^m_t \int_{\mathbb{F}} e^{m(r-\frac{\sigma^2}{2})(T-t)+\sigma w_{T-t}} f(w_{T-t}) dw_{T-t} \\
+ e^{-r(T-t)} (F_T - K) \int_{\mathbb{F}} f(w_{T-t}) dw_{T-t} \\
= e^{-r(T-t)} \alpha T S^m_t \int_{\mathbb{F}} e^{m(r-\frac{\sigma^2}{2})(T-t)+\sigma w_{T-t}} \frac{1}{\sqrt{2\pi(T-t)}} e^{-\frac{w^2_{T-t}}{2(T-t)}} dw_{T-t} \\
+ e^{-r(T-t)} (F_T - K) \left[ 1 - N \left( \frac{z}{\sqrt{T-t}} \right) \right] \\
= e^{-r(T-t)} \alpha T S^m_t e^{m(r-\frac{\sigma^2}{2})(T-t)+\sigma^2(T-t)} \\
\cdot \int_{\mathbb{F}} \frac{1}{\sqrt{2\pi(T-t)}} e^{-\frac{1}{2\sigma^2(T-t)}(w_{T-t}^2-2m\sigma(T-t)w_{T-t}+m^2\sigma^2(T-t)^2)+\frac{m^2\sigma^2(T-t)}{2}} dw_{T-t} \\
+ e^{-r(T-t)} (F_T - K) \left[ 1 - N \left( \frac{z}{\sqrt{T-t}} \right) \right] \\
= e^{-r(T-t)} \alpha T S^m_t e^{m(r-\frac{\sigma^2}{2})(T-t)+\frac{m^2\sigma^2(T-t)}{2}} \\
\cdot \int_{\mathbb{F}} \frac{1}{\sqrt{2\pi(T-t)}} e^{-\frac{1}{2\sigma^2(T-t)}(w_{T-t}^2-m\sigma(T-t))^2} dw_{T-t} \\
+ e^{-r(T-t)} (F_T - K) \left[ 1 - N \left( \frac{z}{\sqrt{T-t}} \right) \right] \\
= e^{-r(T-t)} \alpha T S^m_t e^{m(r-\frac{\sigma^2}{2})(T-t)+\frac{m^2\sigma^2(T-t)}{2}} \left[ 1 - N \left( \frac{z - m\sigma(T-t)}{\sqrt{T-t}} \right) \right] \\
+ e^{-r(T-t)} (F_T - K) \left[ 1 - N \left( \frac{z}{\sqrt{T-t}} \right) \right] \\
= e^{-r(T-t)} \frac{C_t}{S^m_t} \left( e^{(T-t)(r-m(r-\frac{\sigma^2}{2})-\frac{m^2\sigma^2}{2})} S^m_t e^{m(r-\frac{\sigma^2}{2})(T-t)+\frac{m^2\sigma^2(T-t)}{2}} \right) \\
\left[ 1 - N \left( \frac{z - m\sigma(T-t)}{\sqrt{T-t}} \right) \right] + e^{-r(T-t)} (F_T - K) \left[ 1 - N \left( \frac{z}{\sqrt{T-t}} \right) \right] \\
= C_t N \left( \frac{-z + m\sigma(T-t)}{\sqrt{T-t}} \right) + e^{-r(T-t)} (F_T - K) N \left( \frac{-z}{\sqrt{T-t}} \right) \quad (33) 
\]
By substituting $C_t$ by $C_t = V_t - F_t = \alpha_t S_t^m$ and $z$ from Equation (32) we get

$$\text{OoC}(t) = \alpha_t S_t^m \left( \frac{\frac{1}{m} \ln \left( \frac{\alpha_t S_t^m}{K-F_t} \right) + (r - \frac{\sigma^2}{2})(T-t) + m\sigma^2(T-t)}{\sigma \sqrt{T-t}} \right)$$

$$+ e^{-r(T-t)} (F_T - K) \left( \frac{\frac{1}{m} \ln \left( \frac{\alpha_T S_t^m}{K-F_t} \right) + (r - \frac{\sigma^2}{2})(T-t)}{\sigma \sqrt{T-t}} \right)$$

With $\alpha_t = \frac{C_0}{S_0} \cdot e^{\beta_t}$ and $\beta = \left( r - m(r - \frac{\sigma^2}{2}) - m^2 \sigma^2 \right)$, we get for $\alpha_T$:

$$\alpha_T = \frac{C_t}{S_t^m} \cdot e^{\beta(T-t)}.$$ 

Now, we can make the following transformation:

$$\ln (\alpha_T S_t^m) = \ln \left( \frac{C_t}{S_t^m} e^{\beta(T-t)} S_t^m \right) = \ln \left( C_t e^{\beta(T-t)} \right) = \ln C_t + \beta(T-t).$$

With this we define $d_1(t)$ and $d_2(t)$.

$$d_1(t) = \frac{-\ln(K - F_T) + \ln C_t + \beta(T-t) + m(r - \frac{\sigma^2}{2})(T-t) + m^2 \sigma^2(T-t)}{m\sigma \sqrt{T-t}}$$

$$= \frac{\ln \left( \frac{\alpha_t S_t^m}{K-F_t} \right) + (r + m^2 \sigma^2)(T-t)}{m\sigma \sqrt{T-t}}$$

and

$$d_2(t) = d_1(t) - m\sigma \sqrt{T-t}.$$
Proof. Proposition 5.4 Let \( \hat{z} := \frac{\ln m - 1}{\sigma} \). Then,

\[
E[V_T|C1] = E[V_T|W_i > z, i = 1, \ldots, N]
\]

\[
= E\left[ F_T + (V_0 - F_0) \prod_{i=1}^{N} \left( me^{(r - \frac{\sigma^2}{2}) \Delta t + \sigma W_i} - (m - 1)e^{r \Delta t} \right) \right]
\]

\[
\cdot \prod_{i=1}^{N} 1\{W_i > \hat{z}\}
\]

\[
= \int_{\hat{z}}^{\infty} \ldots \int_{\hat{z}}^{\infty} \left( F_T + (V_0 - F_0) \prod_{i=1}^{N} \left( me^{(r - \frac{\sigma^2}{2}) \Delta t + \sigma W_i} - (m - 1)e^{r \Delta t} \right) \right)
\]

\[
f(w_1) \ldots f(w_N) \, dw_1 \ldots dw_N
\]

\[
= F_T \prod_{i=1}^{N} \int_{\hat{z}}^{\infty} f(w_{t_i}) \, dw_{t_i} + (V_0 - F_0)
\]

\[
\cdot \prod_{i=1}^{N} \int_{\hat{z}}^{\infty} \left( me^{(r - \frac{\sigma^2}{2}) \Delta t + \sigma W_i} - (m - 1)e^{r \Delta t} \right) f(w_{t_i}) \, dw_{t_i}
\]

\[
= F_T \prod_{i=1}^{N} \int_{\hat{z}}^{\infty} \frac{1}{\sqrt{2\pi \Delta t}} e^{-\frac{w_{t_i}^2}{2\Delta t}} \, dw_{t_i} + (V_0 - F_0)
\]

\[
\cdot \prod_{i=1}^{N} \int_{\hat{z}}^{\infty} \left( me^{(r - \frac{\sigma^2}{2}) \Delta t + \sigma W_i} - (m - 1)e^{r \Delta t} \right) \frac{1}{\sqrt{2\pi \Delta t}} e^{-\frac{w_{t_i}^2}{2\Delta t}} \, dw_{t_i}
\]

\[
= F_T \left[ 1 - N\left( \frac{\hat{z}}{\sqrt{\Delta t}} \right) \right]^N + (V_0 - F_0)
\]

\[
\cdot \prod_{i=1}^{N} \left[ me^{r \Delta t} \left( 1 - N\left( \frac{\hat{z} - \sigma \Delta t}{\sqrt{\Delta t}} \right) \right) - (m - 1)e^{r \Delta t} \left( 1 - N\left( \frac{\hat{z}}{\sqrt{\Delta t}} \right) \right) \right]
\]

\[
= F_T \left[ 1 - N\left( \frac{\hat{z}}{\sqrt{\Delta t}} \right) \right]^N + (V_0 - F_0)
\]

\[
\cdot e^{r_T} \left[ m \left( 1 - N\left( \frac{\hat{z} - \sigma \Delta t}{\sqrt{\Delta t}} \right) \right) - (m - 1) \left( 1 - N\left( \frac{\hat{z}}{\sqrt{\Delta t}} \right) \right) \right]^N.
\]
Substituting $\hat{z} = \frac{\ln \frac{m-1}{m} + \sigma^2 \Delta t}{\sigma}$ leads to the result.

$$e^{-rT} E[V_T|C1] = F_0 N \left( \frac{\ln \frac{m-1}{m} - \frac{\sigma^2}{2} \Delta t}{\sigma \sqrt{\Delta t}} \right)^N + (V_0 - F_0) \cdot \left[ m N \left( \frac{\ln \frac{m-1}{m} + \sigma^2 \Delta t}{\sigma \sqrt{\Delta t}} \right) - (m - 1) N \left( \frac{\ln \frac{m-1}{m} - \frac{\sigma^2}{2} \Delta t}{\sigma \sqrt{\Delta t}} \right) \right]^N$$

References


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