A Note on the Symmetric Nonnegative Inverse Eigenvalue Problem

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Abstract

The symmetric nonnegative inverse eigenvalue problem is the problem of characterizing all possible spectra of $n \times n$ symmetric entrywise nonnegative matrices. The problem remains open for $n \geq 5$. A number of realizability criteria or sufficient conditions for the problem to have a solution are known. In this paper we show that most of these sufficient conditions can be obtained by the use of a result by Soto, Rojo, Moro, Borobia in [ELA 16 (2007) 1-18]. Moreover, by applying this result we may always compute a solution matrix.

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1 Introduction

The symmetric nonnegative inverse eigenvalue problem (hereafter SNIEP) is the problem of finding necessary and sufficient conditions for a list $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ of real numbers to be the spectrum of an $n \times n$ symmetric nonnegative matrix. If there exists a symmetric nonnegative matrix $A$ with spectrum $\Lambda$, we say that $\Lambda$ is symmetrically realizable and that $A$ is the realizing matrix. When $\Lambda$ is a set of real numbers and the realizing matrix $A$ is required to be nonnegative, not necessarily symmetric, the problem is called the real nonnegative inverse eigenvalue problem (RNIEP). This problem remains unsolved. A complete solution is only known for $n \leq 4$ [9, 11, 25]. Necessary conditions have been found in [9, 7, 1]. However, they are very far from the sufficient conditions, which are known in the literature about the problem. A

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number of sufficient conditions for the existence of a solution to the \textit{RNIEP} are known [24, 12, 13, 15, 6, 2, 17, 16, 19]. In [10] the authors construct a map of the sufficient conditions for the \textit{RNIEP} and establish inclusion relations or independency relations between them. Sufficient conditions for the symmetric case (\textit{SNIEP}) have been obtained in [3, 23, 14, 20, 8, 21]. The \textit{RNIEP} and the \textit{SNIEP} are equivalent for $n \leq 4$ (see [4]), otherwise they are different (see [5]).

Our aim in this paper is to show the relevance of the following symmetric realizability criterion given in [21, Theorem 3.1], which we shall call the \textit{SRMB} criterion. In particular we will show that most of known results, which give realizability criteria for the problem to have a solution, may be obtained by applying the result in [21]. Moreover, the \textit{SRMB} criterion has a constructive proof, in the sense that it generates an algorithmic procedure to compute a realizing matrix.

**Theorem 1.1 SRMB criterion** [21]Let $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$ be a set of real numbers with $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ and, for some $t \leq n$, let $\omega_1, \ldots, \omega_t$ be real numbers satisfying $0 \leq \omega_k \leq \lambda_1$, $k = 1, \ldots, t$. Suppose there exist

i) a partition $\Lambda = \Lambda_1 \cup \ldots \cup \Lambda_t$, with $\Lambda_k = \{\lambda_{k1}, \lambda_{k2}, \ldots, \lambda_{kp_k}\}$, $\lambda_{11} = \lambda_1$, $\lambda_{k1} \geq 0$; $\lambda_{k1} \geq \ldots \geq \lambda_{kp_k}$, such that for each $k = 1, \ldots, t$ the set $\Gamma_k = \{\omega_k, \lambda_{k2}, \ldots, \lambda_{kp_k}\}$ is realizable by a symmetric nonnegative matrix of order $p_k$, and

ii) a symmetric nonnegative $t \times t$ matrix with eigenvalues $\lambda_{11}, \lambda_{21}, \ldots, \lambda_{t1}$ and diagonal entries $\omega_1, \omega_2, \ldots, \omega_t$.

Then $\Lambda$ is realizable by a symmetric nonnegative matrix of order $n$.

The \textit{SRMB} criterion was obtained by applying the following result by Soto, Rojo, Moro, Borobia [21, Theorem 2.6]

**Theorem 1.2** [21] Let $A$ be an $n \times n$ symmetric matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Let $\{x_1, \ldots, x_r\}$ an orthonormal set of eigenvectors of $A$ such that $AX = X\Omega$, where $X = [x_1 \mid x_2 \mid \cdots \mid x_r]$ and $\Omega = \text{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$. Let $C$ be any $r \times r$ symmetric matrix. Then the symmetric matrix $A + X C X^T$ has eigenvalues $\mu_1, \ldots, \mu_r, \lambda_{r+1}, \ldots, \lambda_n$, where $\mu_1, \ldots, \mu_r$ are eigenvalues of the matrix $\Omega + C$.

Theorem 1.2 is a symmetric version of a result of Rado, introduced by Perfect in [13], which shows how to modify $r$ eigenvalues of a matrix of order $n$, via a rank-$r$ perturbation, without changing any of the remaining $(n-r)$ eigenvalues.
Rado Theorem was applied by Perfect in [13], to derive an efficient realizability criterion for the RNIEP. Surprisingly, this result was somehow ignored in the literature about the problem, until in [19], the authors rescue it and extend it to a new realizability criterion. In [22], by using the SRMB criterion, with an special partition of the list \( \Lambda \), the authors obtain efficient sufficient conditions.

2 Symmetric realizability criteria

It is well known that the RNIEP and the SNIEP are equivalent for \( n \leq 4 \) [4]), otherwise they are different [5]. In this section, our aim is to show that most of known symmetric realizability criteria can be obtained by applying Theorem 1.1. It was proved in [21] that SRMB criterion contains, strictly, all previous symmetric realizability criteria. The first results about symmetric nonnegative realization are due to Fiedler [3]. Several realizability criteria obtained for the RNIEP have later been shown to be realizability criteria for the SNIEP as well. Fiedler [3] and Radwan [14] proved, respectively, that Kellogg [6] and Borobia [2] realizability criteria are also symmetric realizability criteria. In [20] it was shown that Soto criterion in [17] is also a symmetric realizability criterion. Fiedler first showed that Suleimanova criterion for the RNIEP, is also a symmetric realizability criterion. Next, by using Theorem 1.1, we give an alternative proof of this fact:

**Theorem 2.1** (Symmetric Suleimanova): Let \( \Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\} \) be satisfying

\[
\lambda_1 + \lambda_2 + \cdots + \lambda_n \geq 0 \quad \text{and} \quad \lambda_k < 0, \quad k = 2, \ldots, n.
\]

Then \( \Lambda \) is symmetrically realizable.

**Proof.** (SRMB using proof): For \( n = 1 \), the result is clear. Suppose \( n = 2 \) and \( \Lambda = \{\lambda_1, \lambda_2\} \) with \( \lambda_2 < 0 \). Consider the partition (as in [22, Theorem 4.1])

\[
\Lambda_1 = \{\lambda_1\}, \quad \Lambda_2 = \{\lambda_2\} \quad \text{with} \quad \Gamma_2 = \{\lambda_1, \lambda_2\}.
\]

Then, \( \Gamma_2 \) is realizable by the symmetric nonnegative matrix

\[
A_2 = \begin{bmatrix}
\frac{\lambda_1 + \lambda_2}{2} & \frac{\lambda_1 - \lambda_2}{2} \\
\frac{\lambda_1 - \lambda_2}{2} & \frac{\lambda_1 + \lambda_2}{2}
\end{bmatrix}.
\]

Now we look for a symmetric nonnegative matrix \( B \) with eigenvalue \( \lambda_1 \) and diagonal entry \( \lambda_1 \), that is \( B = [\lambda_1] \). Then

\[
A = A_2 + \begin{bmatrix}
\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\
\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}
\end{bmatrix} \begin{bmatrix} 0 & \frac{\sqrt{2}}{2} \end{bmatrix} = A_2.
\]
Suppose the result holds for $2 \leq k < n$. Then there exists a symmetric nonnegative matrix $A_1$ with spectrum $\{\lambda_1, \lambda_2, \ldots, \lambda_k\}$. Let $\Lambda' = \{\lambda_1, \ldots, \lambda_k, \lambda_{k+1}\}$ and consider the partition

$$
\Lambda'_1 = \{\lambda_1, \ldots, \lambda_k\}, \quad \Lambda'_2 = \{\lambda_{k+1}\}, \quad \Lambda'_j = \emptyset
$$

with $\Gamma_2 = \{\lambda_1 + \cdots + \lambda_k, \lambda_{k+1}\}$, $\Gamma_j = \{0\}$, $j = 3, \ldots, k + 1$.

Note that $\{-\lambda_{k+1}, \lambda_{k+1}\}$ is symmetrically realizable and $\lambda_1 + \cdots + \lambda_k \geq -\lambda_{k+1}$. Let $A_j$, $j = 2, \ldots, k + 1$, a symmetric nonnegative matrix realizing $\Gamma_j$. Let

$$
A_j = A_2 \oplus \cdots \oplus A_{k+1}; \quad X = [x_1 | \cdots | x_k] \quad \text{with}
$$

$$
x_1 = \left( \begin{array}{cccc}
\sqrt{2} & \sqrt{2} & 0 & \cdots & 0 \\
\end{array} \right)^T, \quad x_j = e_{j+1}, \quad j = 2, \ldots, k.
$$

The following sufficient conditions, due to Fiedler [3, Theorem 4.4], guarantee the existence of a symmetric nonnegative matrix with eigenvalues $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_t$ and diagonal entries $\omega_1 \geq \omega_2 \geq \cdots \geq \omega_t$.

Let the vectors $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_t)$ and $\omega = (\omega_1, \omega_2, \ldots, \omega_t)$. If

$$
\begin{align}
&i) \quad \alpha \text{ majorizes } \omega \quad \text{and} \\
&ii) \quad \omega_{k-1} \geq \lambda_k, \quad k = 2, \ldots, t - 1,
\end{align}
$$

then there exists a symmetric nonnegative matrix $B$ with eigenvalues $\alpha_1, \ldots, \alpha_t$ and diagonal entries $\omega_1, \ldots, \omega_t$. Then it is clear from (1), that we may construct a symmetric nonnegative matrix $B$ (see [21, Remarks 3.5, 3.7]) with spectrum $\{\lambda_1, \ldots, \lambda_k\}$ and diagonal entries $\lambda_1 + \cdots + \lambda_k, 0, \ldots, 0$.

Let $C = B - \text{diag}\{\lambda_1 + \cdots + \lambda_k, 0, \ldots, 0\}$. Then $A = M + XCX^T$ is symmetric nonnegative with spectrum $\Lambda'$.

Next we show that Perfect criterion in [12] is also a symmetric realizability criterion:

**Theorem 2.2** [12] Let

$$
\Lambda = \{\lambda_0, \lambda_1, \lambda_{11}, \ldots, \lambda_{1p_1}, \ldots, \lambda_r, \lambda_{r1}, \ldots, \lambda_{rp_r}, \delta\},
$$

where $\lambda_0 \geq |\lambda|$, for $\lambda \in \Lambda$, $\sum_{\lambda \in \Lambda} \lambda \geq 0$, $\delta \leq 0$.

$\lambda_j \geq 0$ and $\lambda_{ji} \leq 0$ for $j = 1, \ldots, r$ and $i = 1, \ldots, p_j$. If

$$
\lambda_j + \delta \leq 0 \quad \text{and} \quad \lambda_j + \sum_{i=1}^{p_j} \lambda_{ji} \leq 0 \quad \text{for} \quad j = 1, \ldots, r,
$$

then $\Lambda$ is symmetrically realizable.
Proof. (SRMB using proof) Let us consider the partition \( \Lambda = \Lambda_0 \cup_{i=1}^{r} \Lambda_i \), where

\[
\begin{align*}
\Lambda_0 &= \{\lambda_0, \delta\}, \text{ with } \Gamma_0 = \{-\delta, \delta\} \text{ and } \\
\Lambda_i &= \{\lambda_i, \lambda_{i1}, \ldots, \lambda_{ip_i}\}, \ i = 1, \ldots, r, \text{ with } \\
\Gamma_i &= \{-\sum_{j=1}^{p_i} \lambda_{ij}, \lambda_{i1}, \ldots, \lambda_{ip_i}\}, \ i = 1, \ldots, r.
\end{align*}
\]

We may assume, without loss of generality, that \( \sum_{\lambda \in \Lambda} \lambda = 0 \). Clearly, the lists \( \Gamma_i \) are symmetrically realizable (they are Suleimanova lists), say by matrices \( A_i, i = 0, \ldots, r \). Then

\[
M = A_0 \oplus A_1 \oplus \cdots \oplus A_r
\]

is symmetric nonnegative. Now we need to show that there exists a symmetric nonnegative matrix \( B \) with eigenvalues and diagonal entries

\[
\lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_r
\]

\[
-\delta, -\sum_{j=1}^{p_i} \lambda_{ij}, \ldots, -\sum_{j=1}^{p_r} \lambda_{rj},
\]

respectively. We may re-order these eigenvalues and diagonal entries in such a way that

\[
\begin{align*}
\lambda_0 &\geq \lambda_1 \geq \cdots \geq \lambda_r \\
-\sum_{j=1}^{p_1} \lambda_{1j} &\geq \cdots \geq -\delta \geq \cdots \geq -\sum_{j=1}^{p_r} \lambda_{rj},
\end{align*}
\]

with \(-\delta\) in the corresponding place in between the sequence of \(-\sum_{j=1}^{p_i} \lambda_{ij}, i = 1, \ldots, r\). Since \( \sum_{\lambda \in \Lambda} \lambda = 0 \), the sufficient conditions in (1) are satisfied and there exists the required matrix \( B \). Then from Theorem 1.2, for appropriate matrices \( X \) and \( C \), \( A = M + XCX^T \) is symmetric nonnegative with spectrum \( \Lambda \).

Next, by using Theorem 1.2, we show that Kellogg criterion is also a symmetric realizability criterion. Besides, different from Kellogg and Fiedler, our proof generate an algorithmic procedure to compute the realizing matrix.

Theorem 2.3 [3] (Symmetric Kellogg) Let \( \Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\} \) be a list of real numbers with \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \) and let \( p \) be the greatest index \( j \) (1 \( \leq j \leq n \)) for which \( \lambda_j \geq 0 \). Let the set of indices

\[
K = \{i : \lambda_i \geq 0 \text{ and } \lambda_i + \lambda_{n-i+2} < 0, \ i \in \{2, 3, \ldots, \left\lfloor \frac{n+1}{2} \right\rfloor\}\}.
\]
If
\[ \lambda_1 + \sum_{i \in K, i < k} (\lambda_i + \lambda_{n-i+2}) + \lambda_{n-k+2} \geq 0 \quad \text{for all } k \in K, \]  
(2)
and
\[ \lambda_1 + \sum_{i \in K} (\lambda_i + \lambda_{n-i+2}) + \sum_{j=p+1}^{n-p+1} \lambda_j \geq 0, \quad \text{provided that } n \geq 2p, \]  
(3)
then \( \Lambda \) is the spectrum of an \( n \times n \) symmetric nonnegative matrix.

**Proof.** (SRMB using proof). Suppose conditions (2) and (3) are satisfied and let \( K = \{k_1, k_2, \ldots, kt\} \) be the Kellogg set of indices. Consider the partition (SRMB partition in [21, Theorem 3.1]) \( \Lambda = \Lambda_1 \cup \bigcup_{i=1}^{t} \Lambda_{ki} \cup \Lambda_R \), where
\[
\Lambda_1 = \{\lambda_1, \lambda_{p+1}, \ldots, \lambda_{n-p+1}\} \\
\Lambda_{ki} = \{\lambda_{ki}, \lambda_{n-ki+2}\}, \quad k_i \in K, \quad i = 1, \ldots, t \\
\Lambda_R = \Lambda - \Lambda_1 - \bigcup_{i=1}^{t} \Lambda_{ki},
\]
with \( \lambda_{k_1} \geq \lambda_{k_2} \geq \cdots \geq \lambda_{kt} \). It is clear from (3) that
\[ \Gamma_1 = \{\lambda_1 + \sum_{i=1}^{t} (\lambda_{ki} + \lambda_{n-ki+2}), \lambda_{p+1}, \ldots, \lambda_{n-p+1}\} \]
is symmetrically realizable (Observe that \( \Gamma_1 \) is a Suleimanova list). Let \( A_1 = A_1^T \geq 0 \) realizing \( \Gamma_1 \).

We associate, to each list \( \Lambda_{ki} \), a symmetric realizable list
\[ \Gamma_{ki} = \{-\lambda_{n-ki+2}, \lambda_{n-ki+2}\} \]
with realizing \( 2 \times 2 \) symmetric nonnegative matrix \( A_{ki}, \ i = 1, 2, \ldots, t \). Then \( M = A_1 \oplus A_{k_1} \oplus \cdots \oplus A_{kt} \) is symmetric nonnegative with spectrum \( \Gamma_1 \cup \bigcup_{i=1}^{t} \Gamma_{ki} \).

Observe that elements of the list \( \Lambda_R \), if there exist some-ones, are of the form \( \{\lambda_{ki}, \lambda_{n-ki+2}\} \) with \( \lambda_{ki} + \lambda_{n-ki+2} \geq 0 \). Then \( \Lambda_R \) is symmetrically realizable, say by \( A_R \).

Now we show that there exists a symmetric nonnegative matrix \( B \), of order \((t + 1)\), with eigenvalues and diagonal entries
\[ \left\{ \begin{array}{l} \lambda_1, \lambda_{k_1}, \lambda_{k_2}, \ldots, \lambda_{kt} \quad \text{and} \\ \lambda_1 + \sum_{i=1}^{t} (\lambda_{ki} + \lambda_{n-ki+2}), -\lambda_{n-k_1+2}, \ldots, -\lambda_{n-kt+2}, \end{array} \right. \]  
(4)
respectively. To do this, we show that sufficient conditions (1) are satisfied. We know that
\[ -\lambda_{n-k_1+2} \geq -\lambda_{n-k_2+2} \geq \cdots \geq -\lambda_{n-kt+2} \geq 0. \]
Then we re-order the decreasing sequence of possible diagonal entries by setting
\[ \omega = \lambda_1 + \sum_{i=1}^{t} (\lambda_{ki} + \lambda_{n-k_i+2}) \]
in the right place, say in position \( r + 1 \). Thus (4) becomes
\[
\left\{ \begin{array}{l}
\lambda_1, \lambda_{k_1}, \lambda_{k_2}, \ldots, \lambda_{kt} \\
-\lambda_{n-k_1+2}, \ldots, \omega, \ldots, -\lambda_{n-kt+2}.
\end{array} \right. 
\tag{5}
\]
From (3) we have, respectively for \( k_1, k_2, \ldots, k_t \) in \( K \), that
\[
\begin{align*}
\lambda_1 & \geq -\lambda_{n-k_1+2} \\
\lambda_1 + \lambda_{k_1} & \geq -\lambda_{n-k_1+2} - \lambda_{n-k_2+2} \\
\lambda_1 + \lambda_{k_1} + \lambda_{k_2} & \geq -\lambda_{n-k_1+2} - \lambda_{n-k_2+2} - \lambda_{n-k_3+2} \\
& \vdots \\
\lambda_1 + \sum_{i=1}^{s-1} \lambda_{k_i} & \geq -\sum_{i=1}^{s} \lambda_{n-k_i+2}, \ s = 1, \ldots, t - 1
\end{align*}
\]
and
\[
\lambda_1 + \sum_{i=1}^{t} (\lambda_{k_i} + \lambda_{n-k_i+2}) - \sum_{i=1}^{r} \lambda_{n-k_i+2} = \lambda_1 + \lambda_{k_1} + \cdots + \lambda_{kt}.
\]
Observe that for \( \omega = \lambda_1 + \sum_{i=1}^{t} (\lambda_{k_i} + \lambda_{n-k_i+2}) \) in position \( r + 1 \) in the decreasing sequence of possible diagonal entries, we have
\[
\begin{align*}
\lambda_1 + \sum_{i=1}^{r} \lambda_{k_i} & \geq \lambda_1 + \sum_{i=1}^{r} \lambda_{k_i} + \sum_{i=r+1}^{t} (\lambda_{k_i} + \lambda_{n-k_i+2}) \\
& = -\sum_{i=1}^{r} \lambda_{n-k_i+2} + \omega,
\end{align*}
\]
and
\[
\lambda_1 + \sum_{i=1}^{r+1} \lambda_{k_i} \geq -\sum_{i=1}^{r+1} \lambda_{n-k_i+2} + \omega.
\]
Thus, \( i \) in (1) holds. Finally, since \( -\lambda_{n-k_i+2} \geq \lambda_{ki} \) for all \( ki \in K \), and for \( \omega \) in position \( r + 1 \) we have \( \omega \geq -\lambda_{n-k(r+1)+2} \geq \lambda_{k(r+1)} \), then \( ii \) in (1) holds. Hence \( B \) there exists and it can be constructed (see [21, Remarks 3.5, 3.7]). Let
\[
M' = A_{k_1} \oplus \cdots \oplus A_1 \oplus \cdots \oplus A_{kt} \text{ with } A_1 \text{ in position } r + 1.
\]
Therefore, from Theorem 1.2, for \( C = B - \text{diag}B \), the matrix \( M' + XCX^T \) is symmetric nonnegative with spectrum \( \Lambda_1 \cup \cup_{i=1}^{t} \Lambda_{ki}. \) Hence \( A = (M' + XCX^T) \oplus A_R \) is the desired matrix realizing \( \Lambda. \) ☑
Now we show that Borobia realizability criterion \cite{2} is also a symmetric realizability criterion. Different from Borobia and Radwan, our proof generate an algorithmic procedure to construct the realizing matrix.

**Theorem 2.4** \cite{14} (Symmetric Borobia): Let \( \Lambda = \{ \lambda_1, \lambda_2, \ldots, \lambda_n \} \) be a list of real numbers with \( \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \) and let \( p \) be the greatest index \( j \) (\( 1 \leq j \leq n \)) for which \( \lambda_j \geq 0 \). If there exists a partition \( J_1 \cup J_2 \cup \ldots \cup J_t \) of \( J = \{ \lambda_{p+1}, \lambda_{p+2}, \ldots, \lambda_n \} \), for some \( 1 \leq t \leq n - p + 1 \), such that

\[
\begin{align*}
\lambda_1 & \geq \lambda_2 \geq \ldots \geq \lambda_p \geq \sum_{\lambda \in J_1} \lambda \\
& \geq \sum_{\lambda \in J_2} \lambda \geq \ldots \geq \sum_{\lambda \in J_t} \lambda
\end{align*}
\]  

(6)

satisfies the Kellogg conditions (2) and (3), then \( \Lambda \) is the spectrum of a symmetric nonnegative matrix of order.

First we need the following immediate lemma:

**Lemma 2.1** \cite{18} Let \( B_k \) a \( 2 \times 2 \) symmetric nonnegative matrix realizing \( \{ -\lambda_{n-k+2}, \lambda_{n-k+2} \} \) (Suleimanova list), where

\[
\lambda_{n-k+2} = \mu_1 + \ldots + \mu_{s-1}, \quad \mu_i < 0, \quad i = 1, \ldots, s-1.
\]

Then there exists an \( s \times s \) symmetric nonnegative matrix \( C_k \) realizing \( \{ -\lambda_{n-k+2}, \mu_1, \ldots, \mu_{s-1} \} \) (Suleimanova list).

**Proof.** Theorem 2.4 (SRMB using proof). The same proof of Theorem 2.3 holds here, except for one detail: the list in (6) has less elements than the original list \( \Lambda \) and we need to obtain an \( n \times n \) symmetric nonnegative matrix realizing \( \Lambda \). Then, before to apply Theorem 1.2, we apply, if it is necessary, to each \( 2 \times 2 \) matrix \( A_{ki} \) realizing \( \Gamma_{ki} = \{ -\lambda_{n-k+2}, \lambda_{n-k+2} \} \), \( ki \in K \) (and possibly to the matrix \( A_1 \) realizing \( \Gamma_1 \)), the Lemma 2.1 to obtain a symmetric nonnegative matrix \( \tilde{A}_{ki} \) with spectrum \( \{ -\lambda_{n-k+2}, \mu_1, \ldots, \mu_{s-1} \} \), where

\[
\lambda_{n-k+2} = \mu_1 + \ldots + \mu_{s-1}, \quad \mu_i < 0, \quad i = 1, \ldots, s-1.
\]

Next, we consider the following result of Laffey-Šmigoc \cite{8}, and give an alternative proof of it.

**Theorem 2.5** \cite{8} Let \( A \) be an \( n \times n \) irreducible symmetric nonnegative matrix with spectrum \( \{ \lambda_1, \ldots, \lambda_n \} \) and a diagonal \( c \). Let \( B \) be an \( m \times m \) symmetric nonnegative matrix with spectrum \( \{ \mu_1, \ldots, \mu_m \} \).

i) If \( \mu_1 \leq c \), then there exists a symmetric nonnegative matrix \( C \), of order \( n + m - 1 \), with spectrum

\[
\{ \lambda_1, \ldots, \lambda_n, \mu_2, \ldots, \mu_m \}.
\]

ii) If \( c \leq \mu_1 \), then there exists a symmetric nonnegative matrix \( C \), of order \( n + m - 1 \), with spectrum

\[
\{ \lambda_1 + \mu_1 - c, \lambda_2, \ldots, \lambda_n, \mu_2, \ldots, \mu_m \}.
\]
Proof. (SRMB using proof):
i) Let \( \Lambda = \{ \lambda_1, \ldots, \lambda_n, \mu_2, \ldots, \mu_m \} \) and consider the partition (as in [22, Theorem 4.1])

\[
\Lambda = \Lambda_1 \cup \Lambda_2 \cup \cdots \cup \Lambda_{n+1}, \quad \text{where} \quad \\
\Lambda_1 = \{ \lambda_1, \ldots, \lambda_n \} \\
\Lambda_2 = \{ \mu_2, \ldots, \mu_m \} \\
\Lambda_k = \emptyset, \quad k = 3, \ldots, n+1.
\]

We know from the hypothesis that \( \Lambda_1 \) is symmetrically realizable by \( A \). Let \( c, c_2, \ldots, c_n \) be the diagonal entries of \( A \) and consider the associated symmetrically realizable lists

\[
\Gamma_2 = \{ c, \mu_2, \ldots, \mu_m \}; \quad \Gamma_k = \{ c_{k-1} \}, \quad k = 3, \ldots, n+1.
\]

Since \( c \geq \mu_1 \) and \( \{ \mu_1, \ldots, \mu_m \} \) is the spectrum of \( B \) (symmetric nonnegative), and \( c_{k-1} \geq 0 \), then all list \( \Gamma_k \) are symmetrically realizable. Let \( A_2 \) be symmetric nonnegative realizing \( \Gamma_2 \) and let \( [c_{k-1}] \) be the \( 1 \times 1 \) matrix realizing \( \Gamma_k, \ k = 3, \ldots, n+1 \). Then

\[
M = A_2 \oplus [c_2] \oplus \cdots \oplus [c_n]
\]

is symmetric nonnegative of order \( n + m - 1 \) with spectrum

\[
\{ c, \mu_2, \ldots, \mu_m, c_2, \ldots, c_n \}.
\]

Now, since \( A \) is symmetric nonnegative with spectrum \( \Lambda_1 \) and diagonal entries \( c, c_2, \ldots, c_n \), then from Theorem 1.2 with \( C = A - \text{diag}\{ c, c_2, \ldots, c_n \} \) and \( X \) as in Theorem 1.2, the matrix \( M + XCX^T \) is symmetric nonnegative with spectrum \( \Lambda \).

ii) Now, let \( c \leq \mu_1 \) and let \( \Lambda = \{ \lambda_1 + \mu_1 - c, \lambda_2, \ldots, \lambda_n, \mu_2, \ldots, \mu_m \} \). Consider the partition (as in [22, Theorem 4.1])

\[
\Lambda = \Lambda_1 \cup \Lambda_2 \cup \cdots \cup \Lambda_{n+1}, \quad \text{where} \quad \\
\Lambda_1 = \{ \lambda_1 + \mu_1 - c, \lambda_2, \ldots, \lambda_n \} \\
\Lambda_2 = \{ \mu_2, \ldots, \mu_m \} \\
\Lambda_k = \emptyset, \quad k = 3, \ldots, n+1.
\]

It is clear that \( \Lambda_1 \) is symmetrically realizable since \( \mu_1 - c \geq 0 \). Consider the associated symmetrically realizable lists

\[
\Gamma_2 = \{ \mu_1, \mu_2, \ldots, \mu_m \}; \quad \Gamma_k = \{ c_{k-1} \}, \quad k = 3, \ldots, n+1.
\]
We know that the symmetric nonnegative matrix $B$ realizes $\Gamma_2$ and $[c_{k-1}]$ realizes $\Gamma_k, k = 3, \ldots, n + 1$. Let

$$M = B \oplus [c_2] \oplus \cdots \oplus [c_n].$$

Now we apply Lemma 6 in [8] to the matrix $A$ (with $t = \mu_1 - c$) to obtain a symmetric nonnegative matrix $A_0$ with spectrum $\Lambda_1$ and diagonal entries $\mu_1, c_2, \ldots, c_n$. Then from Theorem 1.2, with $C = A_0 - \text{diag}\{\mu_1, c_2, \ldots c_n\}$, the matrix $M + XCX^T$ is symmetric nonnegative with spectrum $\Lambda$. ■

In [14], the author presents the following unpublished result due to Loewy, which we prove here by using SRMB result:

**Theorem 2.6** Let $n \geq 4$, $\lambda_1 \geq \lambda_2 \geq 0 \geq \lambda_3 \geq \cdots \geq \lambda_n$, $\sum_{i=1}^{n} \lambda_i \geq 0$.

Suppose $K_1, K_2$ is a partition of $\{3, 4, \ldots, n\}$ such that $\lambda_1 \geq -\sum_{i \in K_1} \lambda_i \geq -\sum_{i \in K_2} \lambda_i$. Then there exists a symmetric nonnegative matrix with spectrum $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$.

**Proof.** Let $\sum_{i=1}^{n} \lambda_i = s \geq 0$. Consider the partition $\Lambda = \Lambda_0 \cup \Lambda_1 \cup \Lambda_2$, where

$$\Lambda_0 = \{\lambda_1, \lambda_2\}, \quad \Lambda_1 = \{\lambda_i : i \in K_1\} = \{\lambda_{11}, \lambda_{12}, \ldots, \lambda_{1r}\},$$
$$\Lambda_2 = \{\lambda_i : i \in K_2\} = \{\lambda_{21}, \lambda_{22}, \ldots, \lambda_{2t}\}, \quad r + t = n - 2.$$

with symmetrically realizable associated lists (Suleimanova lists)

$$\Gamma_1 = \{\omega_1, \lambda_{11}, \lambda_{12}, \ldots, \lambda_{1r}\}; \quad \Gamma_2 = \{\omega_2, \lambda_{21}, \lambda_{22}, \ldots, \lambda_{2t}\}$$

where $\omega_1 = -\sum_{i \in K_1} \lambda_i + \alpha, \omega_2 = -\sum_{i \in K_2} \lambda_i + \beta$ with $\alpha + \beta = s, 0 \leq \alpha, \beta \leq s$, in such a way that $0 \leq \omega_1, \omega_2 \leq \lambda_1$. Let $A_1$ and $A_2$ symmetric nonnegative matrices realizing $\Gamma_1$ and $\Gamma_2$, respectively. Since $\omega_1 + \omega_2 = \lambda_1 + \lambda_2$, then there exists a $2 \times 2$ symmetric nonnegative matrix $B$ with eigenvalues $\lambda_1, \lambda_2$ and diagonal entries $\omega_1, \omega_2$. Thus, from Theorem 1.2, with $C = B - \text{diag}B$,

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} + XCX^T$$

is symmetric nonnegative with spectrum $\Lambda$. ■

Next we give an alternative proof of the following result of Fiedler:
Lemma 2.2 (Fiedler, [3]). Let $A$ be a symmetric $m \times m$ matrix with spectrum $\Lambda_1 = \{\alpha_1, \ldots, \alpha_m\}$. Let $u$, $\|u\| = 1$, be a unit eigenvector of $A$ corresponding to $\alpha_1$. Let $B$ be a symmetric $n \times n$ matrix with spectrum $\Lambda_2 = \{\beta_1, \ldots, \beta_n\}$. Let $v$, $\|v\| = 1$, be a unit eigenvector of $A$ corresponding to $\beta_1$. Then for any $\rho$, the matrix

$$C = \begin{pmatrix} A & \rho uv^T \\ \rho vu^T & B \end{pmatrix}$$

has spectrum $\Lambda = \{\gamma_1, \gamma_2, \alpha_2, \ldots, \alpha_m, \beta_2, \ldots, \beta_n\}$, where $\gamma_1, \gamma_2$ are eigenvalues of the matrix

$$\hat{C} = \begin{pmatrix} \alpha_1 & \rho \\ \rho & \beta_1 \end{pmatrix}.$$

Proof. (SRMB using proof). The matrix $M = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ is symmetric of order $(m + n)$ with eigenvalues $\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n$. Let

$$x_1 = (u_1, \ldots, u_m, 0, \ldots, 0)^T,$$

$$x_2 = (0, \ldots, 0, v_1, \ldots, v_n)^T$$

$(m + n)$-dimensional vectors, where the $u'_i$'s and the $v'_i$'s are the entries of the vectors $u$ and $v$, respectively. Let $\Omega = \text{diag}\{\alpha_1, \beta_1\}$, and $X = [x_1 \mid x_2]$, where the columns $x_1, x_2$ form an orthonormal set. Then $AX = X\Omega$. Define the symmetric $2 \times 2$ matrix

$$C = \begin{pmatrix} 0 & \rho \\ \rho & 0 \end{pmatrix}.$$ 

Then

$$XCX^T = \begin{pmatrix} 0 & \rho uv^T \\ \rho vu^T & 0 \end{pmatrix}$$

and

$$M + XCX^T = \begin{pmatrix} A & \rho uv^T \\ \rho vu^T & B \end{pmatrix}.$$ 

From Theorem 1.2, $M + XCX^T$ is symmetric with spectrum

$$\Lambda = \{\gamma_1, \gamma_2, \alpha_2, \ldots, \alpha_m, \beta_2, \ldots, \beta_n\},$$

where $\gamma_1$ and $\gamma_2$ are eigenvalues of the matrix

$$\Omega + C = \begin{pmatrix} \alpha_1 & \rho \\ \rho & \beta_1 \end{pmatrix}.$$
for any $\rho$. ■

Observe that if the matrices $A$ and $B$ are symmetric nonnegative and $\rho > 0$, then $M + XCT$ is also symmetric nonnegative. Theorem 1.2 also allow us to generalize Lemma 2.2. In fact, if we have symmetric matrices $A_1, A_2, \ldots, A_p$, with corresponding spectra $\Lambda_i = \{\alpha_{i1}, \alpha_{i2}, \ldots, \alpha_{in}\}$, $i = 1, 2, \ldots, p$ and unitary eigenvectors $u^{(i)}$ associated, respectively, to the eigenvalues $\alpha_{i1}$, then we may obtain a symmetric $n \times n$ matrix

$$A = (A_1 \oplus A_2 \oplus \ldots \oplus A_p) + XCT,$$

with spectrum $\{\gamma_1, \ldots, \gamma_p, \alpha_{11}, \ldots, \alpha_{n1}, \alpha_{12}, \ldots, \alpha_{n2}, \ldots, \alpha_{np}, \ldots, \alpha_{np}\}$, where $\gamma_1, \ldots, \gamma_p$ are eigenvalues of the matrix $\Omega + C$, with $\Omega = \text{diag}\{\alpha_{11}, \alpha_{12}, \ldots, \alpha_{1p}\}$.

References


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