Limiting Distribution and Simulated Power
of Some Proposed Cramér-Von Mises Type Statistics

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Abstract

Parameterization is an useful approach to construct a goodness of fit test based on parametric approaches. We proposed some powerful tests to goodness of fit test problem. This idea will help to use the known likelihood ratio approach to goodness of fit test. Proof is given of the limiting null distribution of the proposed statistics. Based on simulation, we compare the results of the proposed tests with Anderson-Darling and $\chi^2$ tests.

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1 Introduction

As the purpose of a goodness of fit test these tests are intended as tests for distributional form, not as tests of parametric values. They have some strengths and weaknesses. In hypothesis testing the null hypothesis is that the sample
follows a known distribution function. Then goodness of fit tests are hypothesis testing problems. However there are some differences. Hypothesis testing is formulated in terms of null and alternative hypotheses, type one and type two errors and power of tests. Traditionally the goodness of fit tests formulated based on the cumulative distribution functions (cdf) of a random variable $Y$, denoted by $F(y)$, is defined by $F(y) = P(Y \leq y)$, for all $y$ belong to a specified set $\mathcal{A}$. When we are in simple goodness of fit test situation, we have a null hypothesis which is completely clear and an alternative which is completely vague as

$$H_0 : F(y) = F_0(y) \quad \forall y \in \mathcal{A} \quad (1)$$

against

$$H_1 : F(y) \neq F_0(y) \quad \text{for some} \quad y \in \mathcal{A}$$

where $F_0(.)$ is a known distribution function. Maybe we consider $F_0(.)$ as $F(., \theta_0)$ where $\theta_0 \in \Theta$ is a specified value of $\theta$. In the composite situation we wish to test

$$H_{0c} : F(y) = F(y; \theta) \quad \forall y \in \mathcal{A} \quad (2)$$

against

$$H_{1c} : F(y) \neq F(y; \theta) \quad \text{for some} \quad y \in \mathcal{A}$$

where $\theta \in \Theta$ is an unknown vector of parameters. To test this composite hypothesis we have to estimate $\theta$ by $\hat{\theta}_n$ which is a regular estimate of $\theta$. We denote the true value of $\theta$ by $\theta_0$. A natural approach to this testing problem is to use the empirical distribution function as an approximation to the true underlying distribution where the empirical distribution function $F_n(.)$ is defined by $F_n(y) = \frac{1}{n} \sum_{i=1}^{n} 1(Y_i \leq y)$. A test statistics can then be formed by looking at some measure of the deviation between $F_n(.)$ and hypothesized distribution function $F_0(.)$. Then $d(F_n(.), F_0(.))$, where $d$ is any metric or divergence criteria
could serve as a test statistic. As a decision, this discrepancy should be small if the data is in fact from the hypothesized distribution. Most notable goodness of fit tests based on empirical distribution function and \(d(F_n(.), F_0(.))\) are the Cramér (1928), Von-Mises (1931), Kolmogorov (1933), Smirnov (1939, 1941), Kuiper V test (1960), Anderson-Darling \(A_n^2\) test (1952) and Watson’s \(U^2\) test (1961). The first approach to the problem of testing fit to a fixed distribution is Pearson’s (1930) Chi-squared test. A way to improve Pearson’s statistics consists of employing a functional distance as \(d(F_n(.), F_0(.))\). Possibly the best known test statistics based on the empirical distribution function are the Cramér (1928), pages 145 – 147 and in a more general form Von-Mises (1931) statistics, pages 316 – 335. They proposed

\[
\omega_n^2 = n \int_{-\infty}^{\infty} (F_n(y) - F_0(y))^2 \zeta(y) \, dy
\]

for some weight function \(\zeta\) as an adequate measure of discrepancy. CsÖrgő and Faraway (1996) have studied the exact and asymptotic distribution of Cramér-von Mises statistics. The Kolmogorov test (1933) is the easiest and most natural non-parametric test. It is based on the \(L_\infty\) norm and computes the distance between an empirical and the hypothesized (theoretical) distribution function under the null hypothesis. Under \(\mathcal{H}_1\) the difference between the empirical and theoretical distribution functions will be noticeable. This statistic is given by

\[
D_n = \sqrt{n} \sup_{y \in \mathbb{R}} |F_n(y) - F_0(y)|.
\]

The p-values can be used to obtain the significance level when testing it to any continuous distribution. In search of this property for \(\omega_n^2\) has introduced a simple modification. A modification for Cramér-Von Mises distance is

\[
W_n^2(\psi) = n \int_{-\infty}^{\infty} \psi(F_0(y)) \{(F_n(y) - F_0(y))^2\} \, dF_0(y)
\]

which was proposed by Smirnov (1939) and Smirnov (1941). The parametric version of this statistics when related parameter is estimated by \(\hat{\theta}_n\) is given by

\[
\hat{W}_n^2(\psi) = n \int_{-\infty}^{\infty} \psi(F(y; \hat{\theta}_n)) \{(F_n(y) - F(y; \hat{\theta}_n))^2\} \, dF(y; \hat{\theta}_n).
\]
The property exhibited by $D_n$ and $W_n^2(\psi)$ of being distribution free does not carry over to them parametric cases. However, sometimes the distribution of $F(Y; \hat{\theta}_n)$ does not depend on $\theta$, but only on $\mathcal{F}$, the family of underline distributions. In those cases, the distributions of parametric goodness of fit tests is parameter free. This happens if $\mathcal{F}$ is a location scale family and $\hat{\theta}_n$ is an equivariant estimator; see, David and Johnson (1948). All the statistics which can be obtained by varying $\psi$ are usually refereed to as statistics of Cramér-Von Mises type, two of them are as follows:

The Cramér-Von Mises’s statistic obtained by $W_n^2$ for $\psi(.) = 1$,

$$W_n^2 = n \int_{-\infty}^{\infty} (F_n(y) - F_0(y))^2 \, dF_0(y)$$

and the Anderson-Darling’s statistic (1955) for $\psi(t) = (t(1 - t))^{-1}$,

$$A_n^2 = n \int_{-\infty}^{\infty} \frac{(F_n(y) - F_0(y))^2}{F_0(y)(1 - F_0(y))} \, dF_0(y)$$

with parametric version as

$$\hat{A}_n^2 = n \int_{-\infty}^{\infty} \frac{(F_n(y) - F(y; \hat{\theta}_n))^2}{F(y; \hat{\theta}_n)(1 - F(y; \hat{\theta}_n))} \, dF(y; \hat{\theta}_n).$$

Consideration of different weight functions $\psi$ allows the statistician to put special emphasis on the detection of particular sets of alternatives. Some people prefer employing Cramér-Von Mises statistics instead of Kolmogorov-Smirnov statistics. It is because Kolmogorov-Smirnov statistics accounts only for the largest deviation between $F_n(t)$ and $F(t)$, while the other one is a weighted average of all the deviations between $F_n(t)$ and $F(t)$. As a decision, we reject $\mathcal{H}_0$ if in each case the value of the statistic is large. The supremum version of the Anderson-Darling statistics given by

$$B_n^2 = \sup_{-\infty \leq y \leq \infty} \frac{|F_n(y) - F_0(y)|}{\sqrt{F_0(y)(1 - F_0(y))}}.$$

Eicker (1979) considered $\psi(t) = (t(1 - t))^{-1} = \{F_n(y)(1 - F_n(y))\}^{-1}$, rather than the hypothesized variance. In this work we will purpose some tests and we will pay attention to the parallel evolution of the theory of empirical processes.
and the asymptotic theory of the goodness of fit test. Section 2 contains a review on Kullback-Leibler risk. Section 3 presents our goal to using Berk-Jones idea and using this idea to construct a test. The main section, section 4, shows how we make our test. The asymptotic property of the proposed weight function is given in section 5. After construction union-intersection like the test, in section 6 we will search some good weight functions in means of a powerful test in two directions as simple and composite hypothesis. In fact we develop an approach for simple hypothesis and then we will use the results for simulation study in both situations. Also, the power comparisons are given in this section.

2 Kullback-Leibler Risk and Likelihood Function

In decision theory, estimators are chosen as minimizing some risk function. The most important risk functions are based on the Kullback-Leibler, (1951), KL, divergence. Let a probability $P'$ is absolutely continuous with respect to a probability $P$ and $\mathcal{F}_1$ a sub-$\sigma$-field of $\mathcal{F}$ the loss using $P'$ in place of $P$ is the $L_{\mathcal{F}}^{P/P'} = \log \frac{dP}{dP'}$. Its expectation is

$$E_P \{ L_{\mathcal{F}}^{P/P'} \} = KL(P, P'; \mathcal{F}).$$

This is the Kullback-Leibler risk. If $\mathcal{F}$ is the largest sigma-field on the space, then we omit it in the notation. If $Y$ is random variable with p.d.f. $f_Y$ and $g_Y$ under $P$ and $P'$ respectively we have $\frac{dP}{dP'|\mathcal{F}} = \frac{f_Y(y)}{g_Y(y)}$ and the divergence of the distribution $P'$ relative to $P$ can be written as

$$KL(P, P') = \int \log \frac{f_Y(y)}{g_Y(y)} f_Y(y) d(y).$$

We have that $KL(P, P'; \mathcal{F}) = KL(P, P')$ if $\mathcal{F}$ is the $\sigma$-field generated by $y$ on $(\Omega, \mathcal{F})$. Base on continuity arguments, we take $0 \log \frac{0}{r} = 0$ for all $r \in \mathbb{R}$ and $t \log \frac{t}{0} = \infty$ for all non-zero $t$. Hence $KL$ divergence takes its value in $[0, \infty]$
and $\mathcal{KL}(h, g^\beta) = 0$ implies that $h = g^\beta$. The $\mathcal{KL}$ divergence is not a metric, but it is additive over marginals of product measures, see, Cover and Thomas (1991).

Berk-Jones (1979) used the divergence function which prepare an approach which give us a test statistic using known likelihood ratio test. This idea is not in favor or against the known goodness of fit tests, but is an approach which help us to solve a problem with a different method which works for binned and unbind data. More precisely, the Berk-Jones statistics as the supremum of the Kullback-Leibler discrepancy between hypothesized and empirical distribution functions could be defined as a supreme of

$$K(F_n(y), F_0(y))) = \begin{cases} 
F_n(y) \left( \log \left( \frac{F_n(y)}{F_0(y)} \right) \right) + (1 - F_n(y)) \log \frac{1 - F_n(y)}{1 - F_0(y)} & \text{if A} \\
0 & \text{if B} \\
\infty & \text{otherwise,}
\end{cases}$$

where $A = \{ y; 0 \leq F_0(y) < F_n(y) \leq 1 \}$, $B = \{ y; 0 \leq F_n(y) \leq F_0(y) \leq 1 \}$ and $K(F_n(y), F_0(y))$ is the $\mathcal{KL}$ discrepancy between two distributions.

It is known that $K(F_n(y), F_0(y))$ behaves as $\frac{1}{2} \frac{(F_n(y) - F_0(y))^2}{F_0(y)(1 - F_0(y))}$, which is half of the Pearson statistics for $F_n(y)$. When we consider the goodness of fit test for multinomial distribution, the Pearson $\chi^2$ statistics is asymptotically equivalent to the likelihood ratio statistics, see, Theorem 9.1 Knight (1999). We may fix $y$ and construct a test statistic by likelihood ratio for goodness of fit problem. Consider a random sample $\underline{Y} = (Y_1, ..., Y_n)$, $F_n(y)$ is a distribution function as a function of $y \in \mathcal{A}$. On the other hand for each fixed value of $y$, it is known that $F_n(y)$ is an unbiased maximum likelihood estimator for $F(y)$. The variance of the empirical distribution, converges to zero as $n$ goes to infinity. Thus $F_n(.)$ is weakly and strongly consistent for estimation of $F(y)$. So $nF_n(y) \sim Bin(n, F(y))$ for a fixed $y$, so, under $\mathcal{H}_0$; $nF_n(y) \sim Bin(n, F_0(y))$.

By latter result we conclude that the likelihood ratio statistic for testing $\mathcal{H}_0$ against $\mathcal{H}_1$ in fixed $y \in \mathcal{A}$ is given by

$$\lambda_n(y) = \frac{\sup_{F(y)} \mathcal{L}_n(F(y))}{\mathcal{L}_n(F_0(y))},$$
where $\mathcal{L}_n(F(y))$ and $\mathcal{L}_n(F_0(y))$ are likelihood functions evaluated at $F(y)$ and $F_0(y)$ respectively. A suitable relation between Berk-Jones and likelihood ratio statistics is as follows

$$\sup_{y \in [0,1]} K(F_n(y), F_0(y)) = \sup_{y \in [0,1]} n^{-1} \log \lambda_n(y).$$

Einmahl and McKeague (2003) introduced an integral form of Berk-Jones statistic. They considered testing for symmetry, a change point, independence and for exponentiality. Wellner and Koltchinskii (2002) have given proofs of the limiting null distribution of the Berk-Jones (1973) statistic. The method which we want to discuss in this work maybe viewed as an application of the goodness of fit measure extended to the likelihood ratio test. We consider two directions as simple and composite goodness of fit tests.

3 Motivation

Generally we use two types of test statistics for testing $H_0$ against $H_1$ that can be defined by $T = \int_{-\infty}^{\infty} T_z \, dw(z)$ or $T_{\text{max}} = \sup_{z \in (-\infty, \infty)} \{T_z \, w(z)\}$. To construct the global test statistic $T$ (or $T_{\text{max}}$) we need to some local statistics as $T_z$. In this work, we focus on $T$, and we will illustrate our approach to construct the local test statistic. Consider a random sample as $\mathbf{Y} = (Y_1, Y_2, ..., Y_n)$ and a goodness of fit test procedure which introduces a likelihood ratio test for each fixed $z$ which could be between any of two $Y_i$’s. Here we must emphasis that $F(y)$ is an unknown distribution function, whereas $F(z)$ with fixed $z$ is an unknown parameter. The likelihood ratio statistics to testing $H_0$ against $H_1$ is given by

$$\lambda_n(z) = \frac{\sup_{F(z)} \mathcal{L}_n(F(z))}{\mathcal{L}_n(F_0(z))} = \frac{\mathcal{L}_n(F_n(z))}{\mathcal{L}_n(F_0(z))} = \left( \frac{F_n(z)}{F_0(z)} \right)^{nF_n(z)} \left( 1 - \frac{F_n(z)}{1 - F_0(z)} \right)^{n(1-F_n(z))}.$$

If we separate the null hypothesis $H_0 : F(y) = F_0(y) \ \forall y \in A$ (related to a local test) to several null hypotheses as $H_{0,z} : F(z) = F_0(z) \ \forall z \in Z$, we can
construct a likelihood ratio for each one of the $\mathcal{H}_{0z}$s for each fixed $z$, and then construct a test for essential hypothesis testing problem. Considering a weight function as $w(z)$. This weight function permit to construct different tests. As a choice, we consider $w(z) = \psi(F_n(z), F_0(z))$. We may write

$$H_0 : \theta \in \bigcap_{\gamma \in \Gamma} \Theta_\gamma$$

where $\Gamma$ is an arbitrary index set that maybe finite or infinite, depending on the problem. Then

$$H_1 : \theta \in \bigcup_{\gamma \in \Gamma} \Theta_\gamma^c.$$ 

Suppose that for each hypothesis $\mathcal{H}_{0\gamma} : \theta \in \Theta_\gamma$ against the alternative hypothesis $\mathcal{H}_{1\gamma} : \theta \in \Theta_\gamma^c$, we know that the rejection region for $\mathcal{H}_{0\gamma}$ is $\{y : T_\gamma(y) \in R_\gamma\}$, where $T_\gamma(.)$ is the test statistic. Thus, if any of the $\mathcal{H}_{0\gamma}$ is rejected, then $\mathcal{H}_0$ must also be rejected. It offers a rejection region for $\mathcal{H}_0$ as $\bigcup_{\gamma \in \Gamma} \{y : T_\gamma(y) \in R_\gamma\}$.

4 Proposed Approach to Construct New Tests

Consider $Y = (Y_1, Y_2, ..., Y_n)$ as an i.i.d. random sample with unknown distribution function $F(.)$. We set $F_0(.)$ as a known distribution function. The official goodness of fit test is containing testing $\mathcal{H}_0 : F(y) = F_0(y) \ \forall y \in A$, against $\mathcal{H}_1 : F(y) \neq F_0(y)$ for some $y \in A$. A key for proposing goodness of fit test is that the distribution function $F(z)$ for a fixed $z$ is an unknown parameter. It reduces the goodness of fit test to a likelihood ratio test, LRT, as $\mathcal{H}_{0z} : F(z) = F_0(z) \ \forall z \in Z$ against $\mathcal{H}_{1z} : F(z) \neq F_0(z)$ for some $z \in Z$. As it is a case with composite hypotheses testing problems, there may not be in general an optimal test for testing $\mathcal{H}_0$ against $\mathcal{H}_1$. However, for a general class of testing problems our idea is to rewrite this hypothesis testing as the union intersection test. Thus we consider $\mathcal{H}_0 : \bigcap_{z \in Z} \mathcal{H}_{0z}$ against $\mathcal{H}_1 : \bigcup_{z \in Z} \mathcal{H}_{1z}$. 
There is flexibility in the decomposition of the hypotheses and choice of appropriate test statistics. To do this, for each $z$ we can define a new random variable, see, Berk and Jones (1978), thus we have

$$Y_{iz} = 1\{Y_i \leq z\} = \begin{cases} 1 & \text{if } Y_i \leq z \\ 0 & \text{if } Y_i > z \end{cases}$$

for $i = 1, 2, ..., n$.

Now we have a parametric test with a binary variable with value in $\{0, 1\}^n$, i.e. $Y_{iz} \sim Bin(1, F(z))$ and $\sum_{i=1}^n Y_{iz} = nF_n(z) \sim Bin(n, F(z))$. The likelihood function is

$$L_n(F(z)) = L_n(F(z); Y_{iz}) = (F(z))^{nF_n(z)}(1 - F(z))^{n(1 - F_n(z))}$$

and the likelihood ratio test is given by

$$\lambda_n(z) \equiv \frac{\mathcal{L}_n(F_n(z)/F_0(z))}{\mathcal{L}_n(F_0(z))} = \frac{\mathcal{L}_n(F_n(z))}{\mathcal{L}_n(F_0(z))}$$

for the large value of $\lambda_n(z)$ we reject the null hypothesis. The log likelihood function is given by

$$\log \lambda_n(z) = \log \mathcal{L}_n(F_n(z)/F_0(z)) = nF_n(z) \log \left( \frac{F_n(z)}{F_0(z)} \right) + n(1 - F_n(z)) \log \left( \frac{1 - F_n(z)}{1 - F_0(z)} \right).$$

The propose test statistic for testing $\mathcal{H}_0$ against $\mathcal{H}_1$ is

$$U_n = \int_{\mathcal{R}} \log \mathcal{L}_n(F_n(z)/F_0(z)) d(w(z)) = \int_{\mathcal{R}} \log \mathcal{L}_n(F_n(z)/F_0(z)) d(\psi(F_n(z), F_0(z)))$$

Note that the decision rule is built from the logical equivalence that $\mathcal{H}_0$ is wrong if and only if any of its components $\mathcal{H}_{0z}$ is wrong or equivalently $\mathcal{H}_0$ is true if all the $\mathcal{H}_{0z}$'s are individually true. Also, Assume that we can test $\mathcal{H}_{0z}$ using a statistic $T_z(y)$ such that, for any hypothesis include in $\mathcal{H}_{0z}$, $p(\{y \in \mathcal{A}; T_z(y) \geq c\})$ is known, for all $c \in \mathcal{R}$ and $z$. Using this idea in the search of the more powerful test, we will consider the asymptotic behavior of the introduced test statistic and the different $\psi(F_n(z), F_0(z))'$ s for $U_n$ in the next section.
5 Asymptotic Property of the Proposed Tests

The simplest aspect of $F_n$ is that, for each fixed $t$, $F_n(t)$ serves as an estimator of $F(y)$. For example, $F_n(y)$ is unbiased: $\mathbb{E}F_n(t) = F(y)$. Moreover $F_n(y)$ consistent in mean square and weakly consistent for estimation of $F(y)$. Furthermore, by a direct application of the SLLN, $F_n(y)$ is strongly consistent.

Indeed, the latter convergence hold uniformly in $y$. It is known that for each fixed $y \in \mathcal{A}
$$
F_n(y) \overset{c}{\to} \mathcal{N}\left(F(y), \frac{F(y)[1 - F(y)]}{n}\right)
$$
By these statements
$$
\frac{F_n(y)[1 - F_n(y)]}{F(y)[1 - F(y)]} \overset{p}{\to} 1
$$
and if $F_n(.) \neq 0, 1$ we have
$$
P\left(\frac{F_n(y)[1 - F_n(y)]}{F(y)[1 - F(y)]} = 0\right) = 0
$$
For $F(.)[1 - F(.)] \neq 0$ and $\kappa > 0$ we may claim that
$$
(F_n(y)[1 - F_n(y)])^{-\kappa} \overset{p}{\to} (F(y)[1 - F(y)])^{-\kappa}
$$
and
$$
\left(\frac{F_n(y)[1 - F_n(y)]}{F(y)[1 - F(y)]}\right)^{-\kappa} \overset{p}{\to} 1.
$$
Following Wellner and Koltchinskii (2002), define
$$
K(F_n(y), F_0(y)) = F_n(y) \left(\log\left(\frac{F_n(y)}{F_0(y)}\right) + (1 - F_n(y)) \log \frac{1 - F_n(y)}{1 - F_0(y)}\right)
$$
and note that $\frac{\partial}{\partial z} K(t, y)|_{t=y} = \log\left(\frac{1}{y}\right) - \log \frac{1-t}{1-y}|_{t=y} = 0$ and $\frac{\partial^2}{\partial t^2} K(t, y) = \frac{1}{t} + \frac{1}{1-t}$, then
$$
K(t, y) = K(y, y) + (t - y) \frac{\partial}{\partial y} K(t, y) + \frac{1}{2} (t - y)^2 \frac{\partial^2}{\partial y^2} K(t, y)|_{t=y^*}
$$
$$
= \frac{1}{2} \frac{(t - y)^2}{y^*(1 - y^*)}
$$
for some \( y^* \) satisfying \( |y^* - t| \leq |y - t| \). This yields

\[
K(F_n(y), F_0(y)) \approx \frac{1}{2} \frac{(F_n(y) - F_0(y))^2}{(F^*_n(y))(1 - F^*_n(y))},
\]

for \( y \in [0, 1] \) where \( |F^*_n(y) - F(y)| \leq |F_n(y) - F(y)| \).

Let \( \beta \in (1/2, 1) \) and \( a_n = n^{-\beta} \), it is known that

\[
\int_{a_n}^{1-a_n} n \frac{(F_n(y) - F_0(y))^2}{(F^*_n(y))(1 - F^*_n(y))} dF(y) \xrightarrow{L} \int_{0}^{1} \frac{U^2}{(F_0(y))(1 - F_0(y))} dF(y),
\]

where \( U \) is a standard Brownian bridge. To show

\[
\int_{0}^{a_n} n \frac{(F_n(y) - F_0(y))^2}{(F^*_n(y))(1 - F^*_n(y))} dF(y) \xrightarrow{P} 0,
\]

fix \( \epsilon > 0 \) and choose \( \lambda = \lambda_\epsilon \) so large that

\[
p(\| \frac{F_n(y)}{F_0(y)} \|_0^1 > \lambda) = \lambda^{-1} < \epsilon.
\]

On the event \( \| \frac{F_n(y)}{F_0(y)} \|_0^1 \leq \lambda \) we have

\[
\int_{0}^{a_n} 2nK(F_n(y), F_0(y)) dF(y) = \int_{0}^{a_n} 2nF_n(y) \log \frac{F_n(y)}{F_0(y)} dF_0(y) + o(1)
\]

\[
\leq \int_{0}^{a_n} 2n\lambda x \log \lambda dF(y)
\]

\[
= \lambda \log \lambda n a_n^2 \longrightarrow 0.
\]

By symmetry

\[
\int_{1-a_n}^{1} 2nK(F_n(y), F_0(y)) dF(y) \xrightarrow{P} 0.
\]

Thus

\[
\int_{0}^{1} 2nK(F_n(y), F_0(y)) dF(y) \xrightarrow{L} \int_{0}^{1} \frac{U^2}{(F_0(y))(1 - F_0(y))} dF(y) \quad (3)
\]

In the next section we will use (3) to show that the new tests asymptotically is equivalent to the Anderson-Darling statistic.
6 New Tests and their Powers

6.1 Simulation study

To generate new tests, we have to choose appropriating weight function. Then we need to modify \( F_n(z) \) at its discontinuity points \( Y_{(i)} \) for \( i = 1, 2, ..., n \).

It is trivial that for a point \( Y_{(i)} = y \) there are \( \frac{i-0.5}{2} \) observations among \( Y_1, ..., Y_n \) which are less than \( y \). This leads us to consider \( F_n(Y_{(i)}) \) as \( \frac{i-0.5}{2} \).

This choice is in accord with the traditional test statistic. In this section we propose some weight function and simulate their power against several alternatives. For each alternative the power result was derived from \( 10^4 \) samples of size \( n = 50, 70, 100, 120, 150, 200, 250 \) depend on choosing \( \kappa \) (related to the weight function) and \( \psi(F_n(z), F_0(z)) \) for simple and composite situations.

6.1.1 Simple Hypothesis

We generate \( 10^4 \) observations from a Beta distribution, say, \( \beta(\eta, \theta) \). Consider the \( \beta(1.1) \) as the true (data generate) density. As \( F_0(.) \), we consider the Beta distributions with parameters as \( (\eta, \theta) = (1.5, 1.5), (0.8, 0.8), (0.6, 0.6), (1.1, 0.8) \) and \( \kappa = 0.1, 0.3, 0.5, 0.7, 1, 1.2, 1.4 \). For all tests we set \( \alpha = 0.05 \) as the level of test. At given level, the critical values of the tests are simulated independently.

The reasonable choose of the weight function will give us a reasonable test statistic. At the first we consider the weight function as

\[
d\psi(F_n(z), F_0(z)) = \frac{1}{2} \left( \frac{F_n(z)}{F_0(z)} \right)^{-\sqrt{\kappa}} dF_n(z),
\]

Which introduce a new test, say \( K_n \), where

\[
K_n = \int_{\mathcal{R}} \log \mathcal{L}^{F_n(z)/F_0(z)} d(w(z))
\]

\[
= \int_{\mathcal{R}} \log \mathcal{L}^{F_n(z)/F_0(z)} \left( \frac{F_n(z)}{F_0(z)} \right)^{-\sqrt{\kappa}} d(F_n(z)).
\]
After simplification, $K_n$ is given by

$$K_n = \frac{1}{2} \sum_{i=1}^{n} \log \frac{F_n(z) / F_0(z)}{1 - F_0(z)} \left\{ \frac{F_n(z) (1 - F_n(z))}{F_0(z) (1 - F_0(z))} \right\} \sqrt{n} =$$

$$\frac{1}{2} \sum_{i=1}^{n} (F_n(X_{(i)}) \log \frac{F_n(X_{(i)})}{F_0(X_{(i)})} + (1 - F_n(X_{(i)})) \log (1 - F_n(X_{(i)})) \left\{ \frac{F_n(X_{(i)}) (1 - F_n(X_{(i)}))}{F_0(X_{(i)}) (1 - F_0(X_{(i)}))} \right\} \sqrt{n}.$$ (4)

Based on (3)

$$K_n \xrightarrow{L} \frac{1}{4} \int_0^1 \frac{U^2}{(F_0(y))(1 - F_0(y))} dF(y)$$

which behave as Anderson-Darling statistic. Because of the asymptotic property of $\frac{F_n(x)(1 - F_n(x))}{F(x)(1 - F(x))}$ the different values of $\kappa$ does not change the convergence rate. Using this weight function, this test has a good power, more often better than the Anderson-Darling test, see Table 1 – 3 and table 9 which is contained the power of the Anderson-Darling statistic. On the other hand, always our test is more powerfull than $\chi^2$ test. The power of $\chi^2$ test for some value of $(\eta, \theta)$ is given in table 10.

### Table 1- Power computations of $\mathcal{H}_0 : F(\cdot) = \beta(1, 1)$ against $\mathcal{H}_1 : F(\cdot) = \beta(\eta, \theta)$ at level $\alpha = 0.05$ for $(\eta, \theta) = (1.5, 1.5)$ for $K_n$.

<table>
<thead>
<tr>
<th>$\kappa$</th>
<th>$n = 50$</th>
<th>$n = 70$</th>
<th>$n = 100$</th>
<th>$n = 120$</th>
<th>$n = 150$</th>
<th>$n = 200$</th>
<th>$n = 250$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.196</td>
<td>0.300</td>
<td>0.423</td>
<td>0.496</td>
<td>0.712</td>
<td>0.887</td>
<td>0.945</td>
</tr>
<tr>
<td>0.3</td>
<td>0.247</td>
<td>0.355</td>
<td>0.526</td>
<td>0.640</td>
<td>0.727</td>
<td>0.896</td>
<td>0.976</td>
</tr>
<tr>
<td>0.5</td>
<td>0.275</td>
<td>0.428</td>
<td>0.590</td>
<td>0.686</td>
<td>0.806</td>
<td>0.910</td>
<td>0.962</td>
</tr>
<tr>
<td>0.7</td>
<td>0.342</td>
<td>0.441</td>
<td>0.633</td>
<td>0.696</td>
<td>0.827</td>
<td>0.938</td>
<td>0.982</td>
</tr>
<tr>
<td>1.0</td>
<td>0.352</td>
<td>0.502</td>
<td>0.674</td>
<td>0.755</td>
<td>0.870</td>
<td>0.945</td>
<td>0.983</td>
</tr>
<tr>
<td>1.2</td>
<td>0.373</td>
<td>0.529</td>
<td>0.676</td>
<td>0.765</td>
<td>0.871</td>
<td>0.947</td>
<td>0.985</td>
</tr>
<tr>
<td>1.4</td>
<td>0.407</td>
<td>0.552</td>
<td>0.704</td>
<td>0.793</td>
<td>0.900</td>
<td>0.959</td>
<td>0.986</td>
</tr>
</tbody>
</table>

### Table 2- Power computations of $\mathcal{H}_0 : F(\cdot) = \beta(1, 1)$ against $\mathcal{H}_1 : F(\cdot) = \beta(\eta, \theta)$ at level $\alpha = 0.05$ for $(\eta, \theta) = (1.1, 0.8)$ for $K_n$.

<table>
<thead>
<tr>
<th>$\kappa$</th>
<th>$n = 50$</th>
<th>$n = 70$</th>
<th>$n = 100$</th>
<th>$n = 120$</th>
<th>$n = 150$</th>
<th>$n = 200$</th>
<th>$n = 250$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.504</td>
<td>0.645</td>
<td>0.776</td>
<td>0.854</td>
<td>0.914</td>
<td>0.974</td>
<td>0.989</td>
</tr>
<tr>
<td>0.3</td>
<td>0.437</td>
<td>0.567</td>
<td>0.755</td>
<td>0.840</td>
<td>0.911</td>
<td>0.971</td>
<td>0.990</td>
</tr>
<tr>
<td>0.5</td>
<td>0.432</td>
<td>0.597</td>
<td>0.755</td>
<td>0.834</td>
<td>0.913</td>
<td>0.969</td>
<td>0.989</td>
</tr>
<tr>
<td>0.7</td>
<td>0.404</td>
<td>0.585</td>
<td>0.728</td>
<td>0.814</td>
<td>0.910</td>
<td>0.964</td>
<td>0.989</td>
</tr>
<tr>
<td>1.0</td>
<td>0.418</td>
<td>0.560</td>
<td>0.708</td>
<td>0.819</td>
<td>0.893</td>
<td>0.964</td>
<td>0.987</td>
</tr>
<tr>
<td>1.2</td>
<td>0.369</td>
<td>0.531</td>
<td>0.729</td>
<td>0.801</td>
<td>0.882</td>
<td>0.962</td>
<td>0.983</td>
</tr>
<tr>
<td>1.4</td>
<td>0.357</td>
<td>0.490</td>
<td>0.703</td>
<td>0.804</td>
<td>0.890</td>
<td>0.956</td>
<td>0.986</td>
</tr>
</tbody>
</table>
Consider a weight function as
\[ d\psi(F_n(z), F_0(z)) = 2\{F_n(z)(1 - F_n(z))\}^{-\sqrt{\kappa}} dF_n(z) \quad \text{for} \quad \kappa \geq 0, \]
which is an empiric version of the \( \psi(F_n(z), F_0(z)) \).

By this choice we have
\[
T_n = \int_\mathbb{R} \log \frac{L^{F_n(z)/F_0(z)}}{d(w(z))} = \int_\mathbb{R} \log \frac{L^{F_n(z)/F_0(z)}}{2\{F_n(z)(1 - F_n(z))\}^{-\sqrt{\kappa}}} d(F_n(z)).
\]

Then \( T_n \) is given by
\[
T_n = 2\frac{1}{n} \sum_{i=1}^{n} \left\{ \log \frac{L^{F_n(X(i))/F_0(X(i))}}{F_n(X(i))}\{F_n(X(i))(1 - F_n(X(i)))\}^{-\sqrt{\kappa}} \right\}
\]
\[
= 2 \sum_{i=1}^{n} \{F_n(X(i)) \log \frac{F_n(X(i))}{F_0(X(i))}\} + (1 - F_n(X(i)) \log \frac{1 - F_n(X(i))}{1 - F_0(X(i))})\{F_n(X(i))(1 - F_n(X(i)))\}^{-\sqrt{\kappa}}.
\]

For \( \kappa = 0 \)
\[
T_n \xrightarrow{\mathcal{L}} \int_0^1 \frac{U^2}{(F_0(y))(1 - F_0(y))} dF(y).
\]

This test is more powerful than the known Anderson-Darling test except for \( \kappa = 0.1 \). Table 4 to table 8 show the result of simulations for \( T_n \). For any candidate \( \kappa \), the power of test, growth when the sample size increase and the powers are converged to 1 very fast. On the other hand the power of the new test is always greater than the power of Anderson-Darling test. When we set \( (\eta, \theta) = (1.5, 1.5) \) and \( \kappa \leq 0.3 \) the power of Anderson-Darling test is greater than \( T_n \).

<table>
<thead>
<tr>
<th>( \kappa )</th>
<th>( n = 50 )</th>
<th>( n = 70 )</th>
<th>( n = 100 )</th>
<th>( n = 120 )</th>
<th>( n = 150 )</th>
<th>( n = 200 )</th>
<th>( n = 250 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.368</td>
<td>0.545</td>
<td>0.757</td>
<td>0.862</td>
<td>0.919</td>
<td>0.965</td>
<td>0.999</td>
</tr>
<tr>
<td>0.3</td>
<td>0.238</td>
<td>0.414</td>
<td>0.623</td>
<td>0.753</td>
<td>0.885</td>
<td>0.972</td>
<td>0.996</td>
</tr>
<tr>
<td>0.5</td>
<td>0.144</td>
<td>0.257</td>
<td>0.529</td>
<td>0.668</td>
<td>0.827</td>
<td>0.956</td>
<td>0.988</td>
</tr>
</tbody>
</table>
Limiting distribution and simulated power

Table 4 - Power computations of $H_0 : F(.) = \beta(1,1)$ against $H_1 : F(.) = \beta(\eta, \theta)$ at level $\alpha = 0.05$ for $(\eta, \theta) = (1,1.5)$ for $T_n$.

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$n = 50$</th>
<th>$n = 70$</th>
<th>$n = 100$</th>
<th>$n = 120$</th>
<th>$n = 150$</th>
<th>$n = 200$</th>
<th>$n = 250$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.137</td>
<td>0.251</td>
<td>0.485</td>
<td>0.557</td>
<td>0.702</td>
<td>0.890</td>
<td>0.957</td>
</tr>
<tr>
<td>0.3</td>
<td>0.257</td>
<td>0.446</td>
<td>0.597</td>
<td>0.706</td>
<td>0.825</td>
<td>0.947</td>
<td>0.984</td>
</tr>
<tr>
<td>0.5</td>
<td>0.322</td>
<td>0.475</td>
<td>0.664</td>
<td>0.776</td>
<td>0.888</td>
<td>0.972</td>
<td>0.993</td>
</tr>
<tr>
<td>0.7</td>
<td>0.333</td>
<td>0.568</td>
<td>0.684</td>
<td>0.780</td>
<td>0.911</td>
<td>0.973</td>
<td>0.994</td>
</tr>
<tr>
<td>1.0</td>
<td>0.398</td>
<td>0.610</td>
<td>0.729</td>
<td>0.827</td>
<td>0.915</td>
<td>0.970</td>
<td>0.995</td>
</tr>
<tr>
<td>1.2</td>
<td>0.413</td>
<td>0.570</td>
<td>0.742</td>
<td>0.824</td>
<td>0.912</td>
<td>0.975</td>
<td>0.999</td>
</tr>
<tr>
<td>1.4</td>
<td>0.438</td>
<td>0.606</td>
<td>0.785</td>
<td>0.820</td>
<td>0.910</td>
<td>0.955</td>
<td>0.987</td>
</tr>
</tbody>
</table>

Table 5 - Power computations of $H_0 : F(.) = \beta(1,1)$ against $H_1 : F(.) = \beta(\eta, \theta)$ at level $\alpha = 0.05$ for $(\eta, \theta) = (0.8,0.8)$ for $T_n$.

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$n = 50$</th>
<th>$n = 70$</th>
<th>$n = 100$</th>
<th>$n = 120$</th>
<th>$n = 150$</th>
<th>$n = 200$</th>
<th>$n = 250$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.148</td>
<td>0.179</td>
<td>0.241</td>
<td>0.261</td>
<td>0.325</td>
<td>0.422</td>
<td>0.536</td>
</tr>
<tr>
<td>0.3</td>
<td>0.164</td>
<td>0.182</td>
<td>0.250</td>
<td>0.262</td>
<td>0.336</td>
<td>0.482</td>
<td>0.560</td>
</tr>
<tr>
<td>0.5</td>
<td>0.138</td>
<td>0.186</td>
<td>0.252</td>
<td>0.307</td>
<td>0.345</td>
<td>0.488</td>
<td>0.572</td>
</tr>
<tr>
<td>0.7</td>
<td>0.129</td>
<td>0.178</td>
<td>0.255</td>
<td>0.296</td>
<td>0.371</td>
<td>0.500</td>
<td>0.592</td>
</tr>
<tr>
<td>1.0</td>
<td>0.118</td>
<td>0.146</td>
<td>0.214</td>
<td>0.249</td>
<td>0.316</td>
<td>0.436</td>
<td>0.565</td>
</tr>
<tr>
<td>1.2</td>
<td>0.105</td>
<td>0.144</td>
<td>0.220</td>
<td>0.239</td>
<td>0.306</td>
<td>0.383</td>
<td>0.509</td>
</tr>
<tr>
<td>1.4</td>
<td>0.098</td>
<td>0.123</td>
<td>0.172</td>
<td>0.201</td>
<td>0.267</td>
<td>0.336</td>
<td>0.418</td>
</tr>
</tbody>
</table>

Table 6 - Power computations of $H_0 : F(.) = \beta(1,1)$ against $H_1 : F(.) = \beta(\eta, \theta)$ at level $\alpha = 0.05$ for $(\eta, \theta) = (0.6,0.6)$ for $T_n$.

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$n = 50$</th>
<th>$n = 70$</th>
<th>$n = 100$</th>
<th>$n = 120$</th>
<th>$n = 150$</th>
<th>$n = 200$</th>
<th>$n = 250$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.644</td>
<td>0.807</td>
<td>0.923</td>
<td>0.957</td>
<td>0.986</td>
<td>0.997</td>
<td>1.000</td>
</tr>
<tr>
<td>0.3</td>
<td>0.675</td>
<td>0.821</td>
<td>0.934</td>
<td>0.964</td>
<td>0.987</td>
<td>0.999</td>
<td>1.000</td>
</tr>
<tr>
<td>0.5</td>
<td>0.691</td>
<td>0.845</td>
<td>0.936</td>
<td>0.967</td>
<td>0.992</td>
<td>0.999</td>
<td>1.000</td>
</tr>
<tr>
<td>0.7</td>
<td>0.689</td>
<td>0.827</td>
<td>0.942</td>
<td>0.972</td>
<td>0.994</td>
<td>1.000</td>
<td>1.000</td>
</tr>
<tr>
<td>1.0</td>
<td>0.673</td>
<td>0.798</td>
<td>0.936</td>
<td>0.973</td>
<td>0.992</td>
<td>0.999</td>
<td>1.000</td>
</tr>
<tr>
<td>1.2</td>
<td>0.642</td>
<td>0.803</td>
<td>0.919</td>
<td>0.965</td>
<td>0.990</td>
<td>0.999</td>
<td>0.999</td>
</tr>
<tr>
<td>1.4</td>
<td>0.598</td>
<td>0.769</td>
<td>0.898</td>
<td>0.951</td>
<td>0.981</td>
<td>0.998</td>
<td>0.999</td>
</tr>
</tbody>
</table>

Table 7 - Power computations of $H_0 : F(.) = \beta(1,1)$ against $H_1 : F(.) = \beta(\eta, \theta)$ at level $\alpha = 0.05$ for $(\eta, \theta) = (1.1,0.8)$ for $T_n$.

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$n = 50$</th>
<th>$n = 70$</th>
<th>$n = 100$</th>
<th>$n = 120$</th>
<th>$n = 150$</th>
<th>$n = 200$</th>
<th>$n = 250$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.500</td>
<td>0.659</td>
<td>0.790</td>
<td>0.854</td>
<td>0.930</td>
<td>0.978</td>
<td>0.992</td>
</tr>
<tr>
<td>0.3</td>
<td>0.519</td>
<td>0.620</td>
<td>0.800</td>
<td>0.872</td>
<td>0.933</td>
<td>0.978</td>
<td>0.992</td>
</tr>
<tr>
<td>0.5</td>
<td>0.497</td>
<td>0.622</td>
<td>0.777</td>
<td>0.859</td>
<td>0.924</td>
<td>0.973</td>
<td>0.994</td>
</tr>
<tr>
<td>0.7</td>
<td>0.479</td>
<td>0.615</td>
<td>0.785</td>
<td>0.833</td>
<td>0.912</td>
<td>0.967</td>
<td>0.989</td>
</tr>
<tr>
<td>1.0</td>
<td>0.429</td>
<td>0.545</td>
<td>0.732</td>
<td>0.797</td>
<td>0.885</td>
<td>0.947</td>
<td>0.980</td>
</tr>
<tr>
<td>1.2</td>
<td>0.404</td>
<td>0.526</td>
<td>0.677</td>
<td>0.769</td>
<td>0.837</td>
<td>0.933</td>
<td>0.964</td>
</tr>
<tr>
<td>1.4</td>
<td>0.380</td>
<td>0.486</td>
<td>0.592</td>
<td>0.706</td>
<td>0.788</td>
<td>0.888</td>
<td>0.955</td>
</tr>
</tbody>
</table>

Now consider a simple weight as $\psi(F_n(z), F_0(z)) = 2dF_n$, which gives $T_n$ as empiric version of the likelihood ratio statistic, thus

$$E_n = \int_{\mathcal{R}} [nF_n(z) \log \left( \frac{F_n(z)}{F_0(z)} \right)] + n(1 - F_n(z)) \log(\frac{1 - F_n(z)}{1 - F_0(z)}) d(2F_n(z)) =$$

$$\sum_{i=1}^{n} 2\{F_n(X(i)) \log \left( \frac{F_n(X(i))}{F_0(X(i))} \right) + (1 - F_n(X(i))) \log(\frac{1 - F_n(X(i))}{1 - F_0(X(i))}) \} =$$
\[ 2 \sum_{i=1}^{n} \left\{ F_n(X(i)) \log \frac{F_n(X(i))}{F_0(X(i))} + \left(1 - F_n(X(i)) \right) \log \frac{1 - F_n(X(i))}{1 - F_0(X(i))} \right\}. \] (5)

This test is more powerful than the Anderson-Darling test when we set \((\eta, \theta) = (0.6, 0.6)\), or \((1.1, 0.8)\). When we set \((\eta, \theta) = (0.8, 0.8)\), our test is the same as the Anderson-Darling test. However for \((\eta, \theta) = (1.5, 1.5)\) the new test has a power which is a little (about 0.1) lower then the Anderson-Darling test, See Table 8.

<table>
<thead>
<tr>
<th>((\eta, \theta))</th>
<th>(n = 50)</th>
<th>(n = 70)</th>
<th>(n = 100)</th>
<th>(n = 120)</th>
<th>(n = 150)</th>
<th>(n = 200)</th>
<th>(n = 250)</th>
</tr>
</thead>
<tbody>
<tr>
<td>((1.5, 1.5))</td>
<td>0.083</td>
<td>0.166</td>
<td>0.254</td>
<td>0.420</td>
<td>0.566</td>
<td>0.747</td>
<td>0.894</td>
</tr>
<tr>
<td>((0.8, 0.8))</td>
<td>0.146</td>
<td>0.170</td>
<td>0.210</td>
<td>0.241</td>
<td>0.256</td>
<td>0.339</td>
<td>0.447</td>
</tr>
<tr>
<td>((0.6, 0.6))</td>
<td>0.580</td>
<td>0.725</td>
<td>0.876</td>
<td>0.922</td>
<td>0.973</td>
<td>0.995</td>
<td>0.999</td>
</tr>
<tr>
<td>((1.1, 0.8))</td>
<td>0.506</td>
<td>0.640</td>
<td>0.704</td>
<td>0.844</td>
<td>0.923</td>
<td>0.978</td>
<td>0.993</td>
</tr>
</tbody>
</table>

Table 9- Power computations of \(\mathcal{H}_0: F(\cdot) = \beta(1,1)\) against \(\mathcal{H}_1: F(\cdot) = \beta(\eta,\theta)\) at level \(\alpha = 0.05\) for \((\eta, \theta) = (1.5, 1.5), (0.8, 0.8), (0.6, 0.6), (1.1, 0.8)\) based on Anderson-Darling test.

<table>
<thead>
<tr>
<th>((\eta, \theta))</th>
<th>(n = 50)</th>
<th>(n = 70)</th>
<th>(n = 100)</th>
<th>(n = 120)</th>
<th>(n = 150)</th>
<th>(n = 200)</th>
<th>(n = 250)</th>
</tr>
</thead>
<tbody>
<tr>
<td>((1.5, 1.5))</td>
<td>0.092</td>
<td>0.395</td>
<td>0.622</td>
<td>0.640</td>
<td>0.829</td>
<td>0.933</td>
<td>0.975</td>
</tr>
<tr>
<td>((0.8, 0.8))</td>
<td>0.041</td>
<td>0.082</td>
<td>0.104</td>
<td>0.109</td>
<td>0.129</td>
<td>0.212</td>
<td>0.297</td>
</tr>
<tr>
<td>((0.6, 0.6))</td>
<td>0.261</td>
<td>0.393</td>
<td>0.570</td>
<td>0.690</td>
<td>0.889</td>
<td>0.955</td>
<td>0.992</td>
</tr>
<tr>
<td>((1.1, 0.8))</td>
<td>0.495</td>
<td>0.638</td>
<td>0.782</td>
<td>0.849</td>
<td>0.906</td>
<td>0.964</td>
<td>0.980</td>
</tr>
</tbody>
</table>

Table 10- Power computations of \(\mathcal{H}_0: F(\cdot) = \beta(1,1)\) against \(\mathcal{H}_1: F(\cdot) = \beta(\eta,\theta)\) at level \(\alpha = 0.05\) for \((\eta, \theta) = (1.5, 1.5), (0.8, 0.8), (0.6, 0.6), (1.1, 0.8)\) based on \(\chi^2\) test.

<table>
<thead>
<tr>
<th>((\eta, \theta))</th>
<th>(n = 50)</th>
<th>(n = 70)</th>
<th>(n = 100)</th>
<th>(n = 120)</th>
<th>(n = 150)</th>
<th>(n = 200)</th>
<th>(n = 250)</th>
</tr>
</thead>
<tbody>
<tr>
<td>((1.5, 1.5))</td>
<td>0.210</td>
<td>0.400</td>
<td>0.500</td>
<td>0.591</td>
<td>0.630</td>
<td>0.785</td>
<td>0.810</td>
</tr>
<tr>
<td>((0.8, 0.8))</td>
<td>0.102</td>
<td>0.170</td>
<td>0.175</td>
<td>0.185</td>
<td>0.200</td>
<td>0.280</td>
<td>0.320</td>
</tr>
<tr>
<td>((0.6, 0.6))</td>
<td>0.382</td>
<td>0.575</td>
<td>0.711</td>
<td>0.823</td>
<td>0.900</td>
<td>0.984</td>
<td>0.999</td>
</tr>
<tr>
<td>((1.1, 0.8))</td>
<td>0.085</td>
<td>0.145</td>
<td>0.155</td>
<td>0.165</td>
<td>0.183</td>
<td>0.212</td>
<td>0.264</td>
</tr>
</tbody>
</table>

### 6.1.2 Composite Hypothesis

In composite case we testing the goodness of fit for a family of distributions.

To goodness of test (2) our test function will be

\[ T_{nc} = 2 \frac{1}{n} \sum_{i=1}^{n} \log \mathcal{L}^{F_n(z)/F(z;\hat{\theta}_n)} \{ F_n(z)(1 - F_n(z)) \}^{-\sqrt{n}} \]

\[ 2 \sum_{i=1}^{n} \left\{ F_n(X(i)) \log \frac{F_n(X(i))}{F_0(X(i); \hat{\theta}_n)} + (1 - F_n(X(i)) \log \frac{1 - F_n(X(i))}{1 - F_0(X(i); \hat{\theta}_n)} \right\} \{ F_n(X(i))(1 - F_n(X(i))) \}^{-\sqrt{n}} \]

(6)
and

\[ K_{nc} = \frac{1}{2n} \sum_{i=1}^{n} \log \frac{L_{F_{n}(z)/F(z; \hat{\theta}_n)}}{\frac{F_n(z)}{F(z; \hat{\theta}_n)} \left( 1 - \frac{F_n(z)}{F(z; \hat{\theta}_n)} \right)^{-\sqrt{\kappa}}} \]

\[ = \frac{1}{2} \sum_{i=1}^{n} \left\{ F_n(X(i)) \log \frac{F_n(X(i))}{F_0(X(i); \hat{\theta}_n)} \right\} \]

\[ + \left( 1 - F_n(X(i)) \right) \log \left\{ \frac{1 - F_n(X(i))}{1 - F_0(X(i); \hat{\theta}_n)} \right\} \left\{ \frac{F_n(X(i))(1 - F_n(X(i)))}{F_0(X(i); \hat{\theta}_n)(1 - F_0(X(i); \hat{\theta}_n))} \right\}^{-\sqrt{\kappa}} \]

similar to (3) and (5). We assume that \( Y \) has a normal distribution, say \( N(\mu, \sigma^2) \) with \( \theta = (\mu, \sigma^2) \) unknown. We can estimate \( \theta \) by \( \hat{\theta}_n = (\bar{Y}_n, S_n^2) \), the mean and sample variance. Then \( T_{nc} \) and \( K_{nc} \) will apply to test the goodness of fit test for normality. To power study we verify the power for \( H_{0c} : Y \sim N(\mu, \sigma^2) \) against \( H_{1c} : Y \sim N(a + \hat{\mu}, b\hat{\sigma}^2) \) where \( a \in \mathcal{R} \) and \( b \in \mathcal{R}^+ \).

The power of this test for \( \kappa = .1, .5, 1.4, a = 0.3 \) and \( b = 1 \) using \( T_{nc} \) and \( K_{nc} \) are given in tables 11 and 12 respectively. As we see these tests have a good power to choose a model to describe the data at hand.

**Table 11:** Power computations of \( H_{0c} : Y \sim N(\mu, \sigma^2) \) against \( H_{1c} : Y \sim N(0.3 + \hat{\mu}, \hat{\sigma}^2) \) using test function \( T_{nc} \) at level \( \alpha = 0.05 \).

<table>
<thead>
<tr>
<th>( \kappa )</th>
<th>( n = 50 )</th>
<th>( n = 70 )</th>
<th>( n = 100 )</th>
<th>( n = 120 )</th>
<th>( n = 150 )</th>
<th>( n = 200 )</th>
<th>( n = 250 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.654</td>
<td>0.739</td>
<td>0.880</td>
<td>0.918</td>
<td>0.989</td>
<td>1.0</td>
<td>1.0</td>
</tr>
<tr>
<td>0.5</td>
<td>0.718</td>
<td>0.765</td>
<td>0.871</td>
<td>0.880</td>
<td>0.900</td>
<td>0.999</td>
<td>1.0</td>
</tr>
<tr>
<td>1.0</td>
<td>0.509</td>
<td>0.674</td>
<td>0.792</td>
<td>0.912</td>
<td>0.967</td>
<td>0.951</td>
<td>0.994</td>
</tr>
<tr>
<td>1.4</td>
<td>0.460</td>
<td>0.637</td>
<td>0.659</td>
<td>0.718</td>
<td>0.850</td>
<td>0.908</td>
<td>0.991</td>
</tr>
</tbody>
</table>

**Table 12:** Power computations of \( H_{0c} : Y \sim N(\mu, \sigma^2) \) against \( H_{1c} : Y \sim N(0.3 + \hat{\mu}, \hat{\sigma}^2) \) using test function \( K_{nc} \) at level \( \alpha = 0.05 \).

<table>
<thead>
<tr>
<th>( \kappa )</th>
<th>( n = 50 )</th>
<th>( n = 70 )</th>
<th>( n = 100 )</th>
<th>( n = 120 )</th>
<th>( n = 150 )</th>
<th>( n = 200 )</th>
<th>( n = 250 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.660</td>
<td>0.829</td>
<td>0.939</td>
<td>0.949</td>
<td>0.963</td>
<td>0.986</td>
<td>0.999</td>
</tr>
<tr>
<td>0.5</td>
<td>0.453</td>
<td>0.657</td>
<td>0.790</td>
<td>0.949</td>
<td>0.941</td>
<td>0.920</td>
<td>0.990</td>
</tr>
<tr>
<td>1.0</td>
<td>0.436</td>
<td>0.676</td>
<td>0.957</td>
<td>0.960</td>
<td>0.972</td>
<td>0.987</td>
<td>0.992</td>
</tr>
<tr>
<td>1.4</td>
<td>0.320</td>
<td>0.505</td>
<td>0.638</td>
<td>0.870</td>
<td>0.880</td>
<td>0.947</td>
<td>0.980</td>
</tr>
</tbody>
</table>


7 Conclusion

The goodness of fit tests are used for verifying whether the experimental data come from the postulated model. In this direction one must decide if theoretical and experimental distributions are the same. Then goodness of fit is a hypothesis testing problem and the problem is concerned with the choice of one of the alternative hypothesis. This problem could be containe the parameters or not. In this work we consider both simple situation where the distribution function is completely known, and the composite case. We have introduced an approach which is known to all statisticians, the likelihood ratio approach to hypothesis testing problem. For simple situation the family which we consider to simulation study is a simple family, but sensitive to choice of parameter. This family is U shaped if both of its parameters $(\eta, \theta)$ are less than one, is J shaped if $(\eta - 1)(\theta - 1) < 0$, and is otherwise unimodal. In the case $(\eta = 1, \theta = 1)$ this distribution is uniform distribution on $(0, 1)$. This sensibility to parameters let us verify our test to various situations. For composite situation we consider location- scale family as normal family. The limiting behavior of the proposed tests are studied. Develop of this approach to the other weight functions which could be morepowerful than Anderson-Darling test for any $\kappa$ is of interest.

References


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