Subclasses of Analytic Functions

Involving the Hurwitz-Lerch Zeta Function

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Abstract. In this paper, we introduce a generalized class of starlike functions and obtain the subordination results for various subclasses of starlike functions. Further, we obtain the integral means inequalities for various subclasses of starlike functions. Some interesting consequences of our results are also pointed out.

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1. Introduction and Preliminaries

Let \( \mathcal{A} \) denote the class of functions of the form

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n
\]

which are analytic in the open disc \( U = \{ z : |z| < 1 \} \). Also let \( \mathcal{T} \) be a subclass of \( \mathcal{A} \) consisting of functions of the form

\[
f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n, \quad z \in U,
\]

introduced and studied by Silverman [17].

For functions \( \phi \in \mathcal{A} \) given by \( \phi(z) = z + \sum_{n=2}^{\infty} \phi_n z^n \) and \( \psi \in \mathcal{A} \) given by \( \psi(z) = z + \sum_{n=2}^{\infty} \psi_n z^n \), we define the Hadamard product (or Convolution) of \( \phi \) and \( \psi \) by

\[
(\phi \ast \psi)(z) = z + \sum_{n=2}^{\infty} \phi_n \psi_n z^n, \quad z \in U.
\]

The following we recall a general Hurwitz-Lerch Zeta function \( \Phi(z, s, a) \) defined by (cf., e.g., [22], p. 121 et sep.)

\[
\Phi(z, s, a) := \sum_{n=0}^{\infty} \frac{z^n}{(n + a)^s}
\]

\((a \in \mathbb{C} \setminus \mathbb{Z}_0^-; s \in \mathbb{C}, \text{when} |z| < 1; \Re(s) > 1 \text{ when } |z| = 1)\) where, as usual, \( \mathbb{Z}_0^- := \mathbb{Z} \setminus \{\mathbb{N}\} \), \( (\mathbb{Z} := \{0, \pm1, \pm2, \pm3, \ldots\}); \mathbb{N} := \{1, 2, 3, \ldots\} \). Several interesting properties and characteristics of the Hurwitz-Lerch Zeta function \( \Phi(z, s, a) \) can be found in the recent investigations by Choi and Srivastava [4], Ferreira and Lopez [5], Garg et al. [8], Lin and Srivastava [13], Lin et al. [14], and others. In 2007, Srivastava and Attiya [21] (see also Raducanu and Srivastava [16], and Prajapat and Goyal [15]) introduced and investigated the linear operator \( \mathcal{J}_{\mu,b} : \mathcal{A} \rightarrow \mathcal{A} \) defined, in terms of the Hadamard product (or convolution), by

\[
\mathcal{J}_{\mu,b}f(z) = G_{b,\mu} \ast f(z)
\]

\((z \in U; b \in \mathbb{C} \setminus \{\mathbb{Z}_0^\pm\}; \mu \in \mathbb{C}; f \in \mathcal{A})\), where, for convenience,

\[
G_{\mu,b}(z) := (1 + b)^\mu [\Phi(z, \mu, b) - b^{-\mu}] \quad (z \in U).
\]
It is easy to observe from (1.5) and (1.6) that, for \( f(z) \) of the form (1.1), we have

\[
J_{\mu, b} f(z) = z + \sum_{n=2}^{\infty} C_n(b, \mu) a_n z^n \tag{1.7}
\]

where

\[
C_n(b, \mu) = |\left(\frac{1 + b}{n + b}\right)^\mu| \tag{1.8}
\]

and (throughout this paper unless otherwise mentioned) the parameters \( \mu, b \) are constrained as \( b \in \mathbb{C} \setminus \{0\}; \mu \in \mathbb{C} \). and various choices of \( \mu \) we get different operators as listed below.

\[
J_0^0 f(z) := f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.9}
\]

\[
J_0^1 f(z) := \int_0^z \frac{f(t)}{t} \, dt := \Lambda f(z) = z + \sum_{n=2}^{\infty} \left(\frac{1}{n}\right) a_n z^n, \tag{1.10}
\]

\[
J_\nu^1 f(z) := \frac{1 + \nu}{z^\nu} \int_0^z t^{1-\nu} f(t) \, dt := z + \sum_{n=2}^{\infty} \left(\frac{1 + \nu}{n + \nu}\right) a_n z^n, = F_\nu f(z), (\nu > -1), \tag{1.11}
\]

\[
J_1^\sigma f(z) := z + \sum_{n=2}^{\infty} \left(\frac{2}{n+1}\right)^\sigma a_n z^n = I^\sigma f(z) (\sigma > 0), \tag{1.12}
\]

where \( \Lambda f \) and \( F_\nu \) are the integral operators introduced by Alexander [1] and Bernardi [3], respectively, and \( I^\sigma(f) \) is the Jung-Kim-Srivastava integral operator [9] closely related to some multiplier transformation studied by Flett [6].

In this paper, by making use of the operator \( J_{\mu, b} \) we introduced a new subclass of analytic functions with negative coefficients and discuss some interesting properties of this generalized function class.
For $0 \leq \lambda < 1$, $0 \leq \gamma < 1$ and $k \geq 0$, we let $\mathcal{J}_b^\mu(\lambda, \gamma, k)$ be the subclass of $\mathcal{A}$ consisting of functions of the form (1.1) and satisfying the analytic criterion

$$\text{Re} \left\{ \frac{z(\mathcal{J}_b^\mu f(z))'}{(1-\lambda)\mathcal{J}_b^\mu f(z) + \lambda z(\mathcal{J}_b^\mu f(z))'} - \gamma \right\} > k \left| \frac{z(\mathcal{J}_b^\mu f(z))'}{(1-\lambda)\mathcal{J}_b^\mu f(z) + \lambda z(\mathcal{J}_b^\mu f(z))'} - 1 \right|, \quad z \in U,$$

(1.13)

where $\mathcal{J}_b^\mu f(z)$ is given by (1.5). We further let $T\mathcal{J}_b^\mu(\lambda, \gamma, k) = \mathcal{J}_b^\mu(\lambda, \gamma, k) \cap T$.

As illustrations, we present some examples.

**Example 1.** If $\mu = 0$ and $b = b$

$$\mathcal{J}_b^0(\lambda, \gamma, k) \equiv \mathcal{S}(\lambda, \gamma, k)$$

$$:= \left\{ f \in \mathcal{A} : \text{Re} \left\{ \frac{zf'(z)}{(1-\lambda)f(z) + \lambda zf'(z)} - \gamma \right\} > k \left| \frac{zf'(z)}{(1-\lambda)f(z) + \lambda zf'(z)} - 1 \right| , \quad z \in U \right\}.$$

Further $T\mathcal{S}(\lambda, \gamma, k) = \mathcal{S}(\lambda, \gamma, k) \cap T$, where $T$ is given by (1.2).

**Example 2.** If $\mu = 1$ and $b = \nu$ with $\nu > -1$, then

$$\mathcal{J}_b^1(\lambda, \gamma, k) \equiv B_\nu(\lambda, \gamma, k)$$

$$:= \left\{ f \in \mathcal{A} : \text{Re} \left( \frac{z(\mathcal{J}_\nu f(z))'}{(1-\lambda)\mathcal{J}_\nu f(z) + \lambda z(\mathcal{J}_\nu f(z))'} - \gamma \right) > k \left| \frac{z(\mathcal{J}_\nu f(z))'}{(1-\lambda)\mathcal{J}_\nu f(z) + \lambda z(\mathcal{J}_\nu f(z))'} - 1 \right| , \quad z \in U \right\},$$

where $\mathcal{J}_\nu$ is a Bernardi operator [3] given by (1.11).

Note that the operator $J_1$ was studied earlier by Libera [10] and Livingston [12]. Further, $TB_\nu(\lambda, \gamma, k) = B_\nu(\lambda, \gamma, k) \cap T$, where $T$ is given by (1.2).

**Example 3.** If $\mu = \sigma$ and $b = 1$ with $\sigma > 0$, then

$$\mathcal{J}_b^\sigma(\lambda, \gamma, k) \equiv \mathcal{I}_\sigma(\lambda, \gamma, k)$$

$$:= \left\{ f \in \mathcal{A} : \text{Re} \left( \frac{z(\mathcal{I}_\sigma f(z))'}{(1-\lambda)\mathcal{I}_\sigma f(z) + \lambda z(\mathcal{I}_\sigma f(z))'} - \gamma \right) > k \left| \frac{z(\mathcal{I}_\sigma f(z))'}{(1-\lambda)\mathcal{I}_\sigma f(z) + \lambda z(\mathcal{I}_\sigma f(z))'} - 1 \right| , \quad z \in U \right\},$$

where $\mathcal{I}_\sigma$ is the Jung-Kim-Srivastava integral operator [9] given by (1.12).

Motivated by earlier works of [2, 18],[23] in this paper, we investigate certain characteristic properties and obtain the subordination results for the class of functions $f \in \mathcal{J}_b^\mu(\lambda, \gamma, k)$ and integral means results for the class of functions $f \in T\mathcal{J}_b^\mu(\lambda, \gamma, k)$. We state some interesting results for functions in those classes defined in Examples 1 to 4.
In this section we obtain the characterization properties for the classes \( J_{b}^{\mu}(\lambda, \gamma, k) \) and \( T J_{b}^{\mu}(\lambda, \gamma, k) \).

**Theorem 2.1.** A function \( f(z) \) of the form \( (1.1) \) is in \( J_{b}^{\mu}(\lambda, \gamma, k) \) if

\[
\sum_{n=2}^{\infty} \left[ n(1 + k) - (\gamma + k)(1 + n\lambda - \lambda) \right] C_{n}(b, \mu) |a_{n}| \leq 1 - \gamma ,
\]

where \( 0 \leq \lambda < 1, \ 0 \leq \gamma < 1, \ k \geq 0, \) and \( C_{n}(b, \mu) \) is given by \( (1.8) \).

**Proof.** It suffices to show that

\[
k \left| \frac{z(J_{b}^{\mu} f(z))'}{(1 - \lambda)J_{b}^{\mu} f(z) + \lambda z(J_{b}^{\mu} f(z))'} - 1 \right| - \text{Re} \left\{ \frac{z(J_{b}^{\mu} f(z))'}{(1 - \lambda)J_{b}^{\mu} f(z) + \lambda z(J_{b}^{\mu} f(z))'} - 1 \right\}
\]

\[
\leq 1 - \gamma.
\]

We have

\[
k \left| \frac{z(J_{b}^{\mu} f(z))'}{(1 - \lambda)J_{b}^{\mu} f(z) + \lambda z(J_{b}^{\mu} f(z))'} - 1 \right| - \text{Re} \left\{ \frac{z(J_{b}^{\mu} f(z))'}{(1 - \lambda)J_{b}^{\mu} f(z) + \lambda z(J_{b}^{\mu} f(z))'} - 1 \right\}
\]

\[
\leq (1 + k) \left| \frac{z(J_{b}^{\mu} f(z))'}{(1 - \lambda)J_{b}^{\mu} f(z) + \lambda z(J_{b}^{\mu} f(z))'} - 1 \right| - \text{Re} \left\{ \frac{z(J_{b}^{\mu} f(z))'}{(1 - \lambda)J_{b}^{\mu} f(z) + \lambda z(J_{b}^{\mu} f(z))'} - 1 \right\}
\]

\[
\leq (1 + k) \sum_{n=2}^{\infty} (n - 1 - n\lambda + \lambda) C_{n}(b, \mu) |a_{n}| |z|^{n-1}
\]

\[
\leq \frac{(1 + k) \sum_{n=2}^{\infty} (n - 1 - n\lambda + \lambda) C_{n}(b, \mu) |a_{n}| |z|^{n-1}}{1 - \sum_{n=2}^{\infty} (1 + n\lambda - \lambda) C_{n}(b, \mu) |a_{n}| z^{n-1}}
\]

\[
< \frac{(1 + k) \sum_{n=2}^{\infty} (n - 1 - n\lambda + \lambda) C_{n}(b, \mu) |a_{n}| |z|^{n-1}}{1 - \sum_{n=2}^{\infty} (1 + n\lambda - \lambda) C_{n}(b, \mu) |a_{n}|}
\]

The last expression is bounded above by \((1 - \gamma)\) if

\[
\sum_{n=2}^{\infty} [n(1 + k) - (\gamma + k)(1 + n\lambda - \lambda)] C_{n}(b, \mu) |a_{n}| \leq 1 - \gamma
\]

and the proof is complete. \(\square\)
Various choices of $b$ and $\mu$ and in the view of Examples 1 and 2, we state the following corollaries.

**Corollary 2.1.** A function $f(z)$ of the form (1.1) is in $S(\lambda, \gamma, k)$ if

$$
\sum_{n=2}^{\infty} [n(1+k) - (\gamma + k)(1 + n\lambda - \lambda)] |a_n| \leq 1 - \gamma,
$$

where $0 \leq \lambda < 1$, $0 \leq \gamma < 1$ and $k \geq 0$.

**Corollary 2.2.** A function $f(z)$ of the form (1.1) is in $B_{\nu}(\lambda, \gamma, k)$ if

$$
\sum_{n=2}^{\infty} [n(1+k) - (\gamma + k)(1 + n\lambda - \lambda)] \left(\frac{\nu + 1}{\nu + n}\right) |a_n| \leq 1 - \gamma,
$$

where $0 \leq \lambda < 1$, $0 \leq \gamma < 1$, $k \geq 0$ and $\nu > -1$.

**Theorem 2.2.** Let $0 \leq \lambda < 1$, $0 \leq \gamma < 1$, $k \geq 0$, then a function $f$ of the form (1.2) to be in the class $T_{\mu}^{\lambda}(\lambda, \gamma, k)$ if and only if

$$
\sum_{n=2}^{\infty} [n(1+k) - (\gamma + k)(1 + n\lambda - \lambda)] C_n(b, \mu) |a_n| \leq 1 - \gamma,
$$

where $C_n(b, \mu)$ are given by (1.8).

**Proof.** In view of Theorem 2.1, we need only to prove the necessity. If $f \in T_{\mu}^{\lambda}(\lambda, \gamma, k)$ and $z$ is real then

$$
\text{Re} \left\{ \frac{1 - \sum_{n=2}^{\infty} nC_n(b, \mu) a_n z^{n-1}}{1 - \sum_{n=2}^{\infty} [1 + n\lambda - \lambda] C_n(b, \mu) a_n z^{n-1}} - \gamma \right\} > k \left| \frac{\sum_{n=2}^{\infty} (n - 1 - n\lambda + \lambda) C_n(b, \mu) a_n z^{n-1}}{1 - \sum_{n=2}^{\infty} [1 + n\lambda - \lambda] C_n(b, \mu) a_n z^{n-1}} \right|.
$$

Letting $z \rightarrow 1$ along the real axis, we obtain the desired inequality

$$
\sum_{n=2}^{\infty} [n(1+k) - (\gamma + k)(1 + n\lambda - \lambda)] C_n(b, \mu) |a_n| \leq 1 - \gamma,
$$

where $0 \leq \lambda < 1$, $0 \leq \gamma < 1$, $k \geq 0$, and $C_n(b, \mu)$ are given by (1.8).

**Corollary 2.3.** If $f \in T_{\mu}^{\lambda}(\lambda, \gamma, k)$, then

$$
|a_n| \leq \frac{1 - \gamma}{[n(1+k) - (\gamma + k)(1 + n\lambda - \lambda)] C_n(b, \mu)}, \quad 0 \leq \lambda < 1, \quad 0 \leq \gamma < 1, \quad k \geq 0,
$$

(2.5)
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where \( C_n(b, \mu) \) are given by (1.8). Equality holds for the function

\[
f(z) = z - \frac{1 - \gamma}{[n(1 + k) - (\gamma + k)(1 + n\lambda - \lambda)]C_n(b, \mu)} z^n.
\]

**Theorem 2.3. (Extreme Points)** Let

\[
f_1(z) = z \quad \text{and} \quad f_n(z) = z - \frac{1 - \gamma}{[n(1 + k) - (\gamma + k)(1 + n\lambda - \lambda)]C_n(b, \mu)} z^n, \quad n \geq 2,
\]

(2.6)

for \( 0 \leq \gamma < 1, \quad 0 \leq \lambda < 1, \quad k \geq 0, \quad C_n(b, \mu) \) are given by (1.8). Then \( f(z) \) is in the class \( T_{\mathcal{J}^\mu_b}(\lambda, \gamma, k) \) if and only if it can be expressed in the form

\[
f(z) = \sum_{n=1}^{\infty} \omega_n f_n(z),
\]

(2.7)

where \( \omega_n \geq 0 \) and \( \sum_{n=1}^{\infty} \omega_n = 1 \).

**Proof.** Suppose \( f(z) \) can be written as in (2.7). Then

\[
f(z) = z - \sum_{n=2}^{\infty} \omega_n \frac{1 - \gamma}{[n(1 + k) - (\gamma + k)(1 + n\lambda - \lambda)]C_n(b, \mu)} z^n.
\]

Now,

\[
\sum_{n=2}^{\infty} \omega_n \frac{1 - \gamma}{[n(1 + k) - (\gamma + k)(1 + n\lambda - \lambda)]C_n(b, \mu)} = \sum_{n=2}^{\infty} \omega_n = 1 - \omega_1 \leq 1.
\]

Thus \( f \in T_{\mathcal{J}^\mu_b}(\lambda, \gamma, k) \). Conversely, let us have \( f \in T_{\mathcal{J}^\mu_b}(\lambda, \gamma, k) \). Then by using (2.5), we set

\[
\omega_n = \frac{[n(1 + k) - (\gamma + k)(1 + n\lambda - \lambda)]C_n(b, \mu)}{1 - \gamma} a_n, \quad n \geq 2
\]

and \( \omega_1 = 1 - \sum_{n=2}^{\infty} \omega_n \). Then we have \( f(z) = \sum_{n=1}^{\infty} \omega_n f_n(z) \) and hence this completes the proof of Theorem 2.3. \( \Box \)

**Remark 2.1.** By suitably specializing the various parameters involved in Theorem 2.2 and Theorem 2.3, we can state the corresponding results for the new subclasses defined in Example 1 to 3 and also for many relatively more familiar function classes.
3. Subordination Results

In this section we obtain subordination results for the new class \( W^I_m(\lambda, \gamma, k) \)

**Definition 3.1.** (Subordination Principle) For analytic functions \( g \) and \( h \) with \( g(0) = h(0) \), \( g \) is said to be subordinate to \( h \), denoted by \( g \prec h \), if there exists an analytic function \( w \) such that \( w(0) = 0 \), \( |w(z)| < 1 \) and \( g(z) = h(w(z)) \), for all \( z \in U \).

**Definition 3.2.** (Subordinating Factor Sequence) A sequence \( \{b_n\}_{n=1}^{\infty} \) of complex numbers is said to be a subordinating sequence if, whenever \( f(z) = \sum_{n=1}^{\infty} a_n z^n \), \( a_1 = 1 \) is regular, univalent and convex in \( U \), we have

\[
\sum_{n=1}^{\infty} b_n a_n z^n \prec f(z), \quad z \in U.
\] (3.1)

**Lemma 3.1.** [23] The sequence \( \{b_n\}_{n=1}^{\infty} \) is a subordinating factor sequence if and only if

\[
\text{Re} \left\{ 1 + 2 \sum_{n=1}^{\infty} b_n z^n \right\} > 0, \quad z \in U.
\] (3.2)

**Theorem 3.1.** Let \( f \in W^I_m(\lambda, \gamma, k) \) and \( g(z) \) be any function in the usual class of convex functions \( C \), then

\[
\frac{(2 + k - \gamma - \lambda(k + \gamma))C_2(b, s)}{2[1 - \gamma + (2 + k - \gamma - \lambda(k + \gamma))C_2(b, \mu)]} (f \ast g)(z) \prec g(z)
\] (3.3)

where \( 0 \leq \gamma < 1 \); \( k \geq 0 \) and \( 0 \leq \lambda < 1 \), with

\[
C_2(b, \mu) = \left( \frac{1 + b}{2 + b} \right)^\mu
\] (3.4)

and

\[
\text{Re} \{ f(z) \} > -\frac{[1 - \gamma + (2 + k - \gamma - \lambda(k + \gamma))C_2(b, \mu)]}{(2 + k - \gamma - \lambda(k + \gamma))C_2(b, \mu)}, \quad z \in U.
\] (3.5)

The constant factor \( \frac{(2 + k - \gamma - \lambda(k + \gamma))C_2(b, \mu)}{2[1 - \gamma + (2 + k - \gamma - \lambda(k + \gamma))C_2(b, \mu)]} \) in (3.3) cannot be replaced by a larger number.

**Proof.** Let \( f \in W^I_m(\lambda, \gamma, k) \) and suppose that \( g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in C \). Then

\[
\frac{(2 + k - \gamma - \lambda(k + \gamma))C_2(b, \mu)}{2[1 - \gamma + (2 + k - \gamma - \lambda(k + \gamma))C_2(b, \mu)]} (f \ast g)(z)
\]

\[
= \frac{(2 + k - \gamma - \lambda(k + \gamma))C_2(b, \mu)}{2[1 - \gamma + (2 + k - \gamma - \lambda(k + \gamma))C_2(b, \mu)]} \left( z + \sum_{n=2}^{\infty} b_n a_n z^n \right).
\] (3.6)
Thus, by Definition 3.2, the subordination result holds true if

\[
\left\{ \frac{(2 + k - \gamma - \lambda(k + \gamma))C_2(b, \mu)}{2[1 - \gamma + (2 + k - \gamma - \lambda(k + \gamma))C_2(b, \mu)]} \right\}_{n=1}^{\infty}
\]

is a subordinating factor sequence, with \( a_1 = 1 \). In view of Lemma 3.1, this is equivalent to the following inequality

\[
\Re \left\{ 1 + \sum_{n=1}^{\infty} \frac{(2 + k - \gamma - \lambda(k + \gamma))C_2(b, \mu)}{1 - \gamma + (2 + k - \gamma - \lambda(k + \gamma))C_2(b, \mu)} a_n z^n \right\} > 0, \quad z \in U.
\]  

(3.7)

By noting the fact that \( \frac{\ln(1+k) - (\gamma+k)(1+n\lambda - \lambda)}{(1-\gamma)} \) is increasing function for \( n \geq 2 \) and in particular

\[
\frac{(2 + k - \gamma - \lambda(k + \gamma))C_2(b, \mu)}{(1 - \gamma)} \leq \frac{(n(1 + k) - (\gamma + k)(1 + n\lambda - \lambda))C_n(b, \mu)}{(1 - \gamma)}, \quad n \geq 2,
\]

therefore, for \( |z| = r < 1 \), we have

\[
\Re \left\{ 1 + \sum_{n=1}^{\infty} \frac{(2 + k - \gamma - \lambda(k + \gamma))C_2(b, \mu)}{1 - \gamma + (2 + k - \gamma - \lambda(k + \gamma))C_2(b, \mu)} a_n z^n \right\} 

= \Re \left\{ 1 + \frac{(2 + k - \gamma - \lambda(k + \gamma))C_2(b, \mu)}{1 - \gamma + (2 + k - \gamma - \lambda(k + \gamma))C_2(b, \mu)} z + \sum_{n=2}^{\infty} \frac{(2 + k - \gamma - \lambda(k + \gamma))C_2(b, \mu)a_n z^n}{1 - \gamma + (2 + k - \gamma - \lambda(k + \gamma))C_2(b, \mu)} \right\} 

\geq 1 - \frac{(2 + k - \gamma - \lambda(k + \gamma))C_2(b, \mu)}{1 - \gamma + (2 + k - \gamma - \lambda(k + \gamma))C_2(b, \mu)} r

- \frac{1}{1 - \gamma + (2 + k - \gamma - \lambda(k + \gamma))C_2(b, \mu)} \sum_{n=2}^{\infty} |[n(1 + k) - (\gamma + k)(1 + n\lambda - \lambda)]C_n(b, \mu)a_n| r^n

\geq 1 - \frac{(2 + k - \gamma - \lambda(k + \gamma))C_2(b, \mu)}{1 - \gamma + (2 + k - \gamma - \lambda(k + \gamma))C_2(b, \mu)} r

- \frac{1 - \gamma}{1 - \gamma + (2 + k - \gamma - \lambda(k + \gamma))C_2(b, \mu)} r

> 0, \quad |z| = r < 1,
\]

where we have also made use of the assertion (2.1) of Theorem 2.1. This evidently proves the inequality (3.7) and hence also the subordination result (3.3) asserted by Theorem 3.1. The inequality (3.5) follows from (3.3) by taking

\[
g(z) = \frac{z}{1 - z} = z + \sum_{n=2}^{\infty} z^n \in C.
\]

Next we consider the function

\[
F(z) := z - \frac{1 - \gamma}{(2 + k - \gamma - \lambda(k + \gamma))C_2(b, \mu)} z^2
\]
where \(0 \leq \gamma < 1\), \(k \geq 0\), \(0 \leq \lambda < 1\) and \(\sigma_2(\alpha_1)\) is given by (3.4). Clearly \(F \in W_{\alpha}^m(\lambda, \gamma, k)\). For this function (3.3) becomes

\[
\frac{(2 + k - \gamma - \lambda(k + \gamma))C_2(b, \mu)}{2[1 - \gamma + (2 + k - \gamma - \lambda(k + \gamma))C_2(b, \mu)]}F(z) \prec \frac{z}{1 - z}.
\]

It is easily verified that

\[
\min \left\{ \Re \left( \frac{(2 + k - \gamma - \lambda(k + \gamma))C_2(b, \mu)}{2[1 - \gamma + (2 + k - \gamma - \lambda(k + \gamma))C_2(b, \mu)]}F(z) \right) \right\} = -\frac{1}{2}, \quad z \in U.
\]

This shows that the constant \(\frac{(2 + k - \gamma - \lambda(k + \gamma))C_2(b, \mu)}{2[1 - \gamma + (2 + k - \gamma - \lambda(k + \gamma))C_2(b, \mu)]}\) cannot be replaced by any larger one.

By taking \(\mu\), and in view of the Examples 1 to 2 in Section 1, we state the following corollaries for the subclasses defined in those examples.

**Corollary 3.1.** If \(f \in S(\lambda, \gamma, k)\), then

\[
\frac{[2 + k - \gamma - \lambda(k + \gamma)]}{2[3 + k - 2\gamma - \lambda(k + \gamma)]}(f \ast g)(z) \prec g(z),
\]

where \(0 \leq \gamma < 1\), \(0 \leq \lambda < 1\), \(k \geq 0\), \(g \in C\) and

\[
\Re\{f(z)\} > -\frac{[3 + k - 2\gamma - \lambda(k + \gamma)]}{[2 + k - \gamma - \lambda(k + \gamma)]}, \quad z \in U.
\]

The constant factor

\[
\frac{[2 + k - \gamma - \lambda(k + \gamma)]}{2[3 + k - 2\gamma - \lambda(k + \gamma)]}
\]

in (3.8) cannot be replaced by a larger one.

**Corollary 3.2.** If \(f \in B_{\nu}(\lambda, \gamma, k)\), then

\[
\frac{(\nu + 1)[2 + k - \gamma - \lambda(k + \gamma)]}{2[(\nu + 2)(1 - \gamma) + (\nu + 1)[2 + k - \gamma - \lambda(k + \gamma)]]}(f \ast g)(z) \prec g(z),
\]

where \(0 \leq \gamma < 1\), \(0 \leq \lambda < 1\), \(k \geq 0\), \(\nu > -1\), \(g \in C\) and

\[
\Re\{f(z)\} > -\frac{[(\nu + 2)(1 - \gamma) + (\nu + 1)[2 + k - \gamma - \lambda(k + \gamma)]}{(\nu + 1)[2 + k - \gamma - \lambda(k + \gamma)]}, \quad z \in U.
\]

The constant factor

\[
\frac{(\nu + 1)[2 + k - \gamma - \lambda(k + \gamma)]}{2[(\nu + 2)(1 - \gamma) + (\nu + 1)[2 + k - \gamma - \lambda(k + \gamma)]]}
\]

in (3.9) cannot be replaced by a larger one.
Remark 3.1. We observe that Corollary 3.1, yields the results obtained by Frasin [7] and Singh [20] for the special values of $\lambda, \gamma$ and $k$.

4. Integral Means Inequalities

In this section, we obtain integral means inequalities for the functions in the family $T J b^\mu(\lambda, \gamma, k)$.

Lemma 4.1. [11] If the functions $f$ and $g$ are analytic in $U$ with $g \prec f$, then for $\eta > 0$, and $0 < r < 1$,

$$\int_0^{2\pi} |g(re^{i\theta})|^{\eta} d\theta \leq \int_0^{2\pi} |f(re^{i\theta})|^{\eta} d\theta. \quad (4.1)$$

In [17], Silverman found that the function $f_2(z) = z - \frac{z^2}{2}$ is often extremal over the family $T$ and applied this function to resolve his integral means inequality, conjectured in [18] and settled in [19], that

$$\int_0^{2\pi} |f(re^{i\theta})|^{\eta} d\theta \leq \int_0^{2\pi} |f_2(re^{i\theta})|^{\eta} d\theta,$$

for all $f \in T$, $\eta > 0$ and $0 < r < 1$. In [19], Silverman also proved his conjecture for the subclasses $T^s(\gamma)$ and $C(\gamma)$ of $T$.

Applying Lemma 4.1, Theorem 2.2 and Theorem 2.3, we prove the following result.

Theorem 4.1. Suppose $f \in T J b^\mu(\lambda, \gamma, k)$, $\eta > 0$, $0 \leq \lambda < 1$, $0 \leq \gamma < 1$, $k \geq 0$ and $f_2(z)$ is defined by

$$f_2(z) = z - \frac{1 - \gamma}{\Phi(\lambda, \gamma, k, 2)} z^2,$$

where

$$\Phi(\lambda, \gamma, k, 2) = [2 + k - \gamma - \lambda(k + \gamma)]C_2(b, \mu) \quad (4.2)$$

and $C_2(b, \mu)$ is given by (3.4). Then for $z = re^{i\theta}$, $0 < r < 1$, we have

$$\int_0^{2\pi} |f(z)|^{\eta} d\theta \leq \int_0^{2\pi} |f_2(z)|^{\eta} d\theta. \quad (4.3)$$
Proof. For \( f(z) = z - \sum_{n=2}^{\infty} |a_n|z^n \), (4.3) is equivalent to proving that
\[
\int_{0}^{2\pi} \left| 1 - \sum_{n=2}^{\infty} |a_n|z^{n-1} \right| d\theta \leq \int_{0}^{2\pi} \left| 1 - \frac{(1 - \gamma)}{\Phi(\lambda, \gamma, k, 2)} z \right| d\theta.
\]
By Lemma 4.1, it suffices to show that
\[
1 - \sum_{n=2}^{\infty} |a_n|z^{n-1} \prec 1 - \frac{1 - \gamma}{\Phi(\lambda, \gamma, k, 2)} z.
\]
Setting
\[
1 - \sum_{n=2}^{\infty} |a_n|z^{n-1} = 1 - \frac{1 - \gamma}{\Phi(\lambda, \gamma, k, 2)} w(z),
\]
and using (2.4), we obtain
\[
|w(z)| = \left| \sum_{n=2}^{\infty} \frac{\Phi(\lambda, \gamma, k, n)}{1 - \gamma} |a_n|z^{n-1} \right|
\leq |z| \sum_{n=2}^{\infty} \frac{\Phi(\lambda, \gamma, k, n)}{1 - \gamma} |a_n|
\leq |z|,
\]
where \( \Phi(\lambda, \gamma, k, n) = [n(1 + k) - (\gamma + k)(1 + n\lambda - \lambda)]C_n(b, \mu) \). This completes the proof by Theorem 4.1. \( \square \)

Remark 4.1. We observe that for \( \lambda = 0 \), if \( \mu = 0 \) the various results presented in this paper would provide interesting extensions and generalizations of those considered earlier for simpler and familiar function classes studied in the literature. The details involved in the derivations of such specializations of the results presented in this paper are fairly straightforward.

References

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