A Gap Function Approach for Concave Programming

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Abstract
We consider a convex maximization problem or equivalently, concave programming. We introduce a gap function for the problem and propose an algorithm for solving it based on the global optimality conditions by Strekalovsky [9]. A convergence of the algorithm has been shown.

1 Introduction

Concave programming or convex maximization problem plays an important role in theory of global optimization. Many engineering and economics problems can be formulated as concave programming [6]. The existing methods and algorithm for solving concave programming until 1987 were mainly cutting plane and branch and bound methods [8],[5] and [6]. Global optimality conditions for concave programming for the first time were obtained in [9] by Strekalovsky in 1987.

An algorithm for solving the problem based on the global optimality conditions using a gap function approach was proposed in [1].

A convergent algorithm for maximizing a strongly convex function has been studied in [1]. In this paper, we generalize the results of [1] for the convex maximization problem using the same gap function. The paper organized as
follows. In section 2, we consider the global optimality conditions for the convex maximization problem. Section 3 is devoted to construction of the algorithm for solving concave programming and its convergence.

2 Global Optimality Condition

Consider the convex maximization problem, or a concave programming

\[ f(x) \rightarrow \max, \; x \in D, \tag{1} \]

where \( f: \mathbb{R}^n \rightarrow \mathbb{R} \) is a convex and differentiable function and \( D \subset \mathbb{R}^n \) is a nonempty arbitrary subset of \( \mathbb{R}^n \). Then an optimality condition for this problem is given by the following theorem:

**Theorem 2.1** [2] Let \( z \) be a solution of problem (1), and let

\[ E_c(f) = \{ y \in \mathbb{R}^n \mid f(y) = c \} \]

Then

\[ \langle f'(y), x - y \rangle \leq 0 \text{ for all } y \in E_{f(z)}(f) \text{ and } x \in D. \tag{2} \]

If, in addition, \( f'(y) \neq 0 \) holds for all \( y \in E_{f(z)}(f) \), then condition (2) is sufficient for \( z \in D \) being a solution to problem (1).

**Remark 2.1** There is another related result obtained by Hiriart-Urruty and Ledyaev in 1996 [4].

**Theorem 2.2** [9] Suppose \( f \) is convex and \( D \) is closed convex in problem (1). Let a point \( z \in D \) satisfy \( -\infty \leq \inf_D f < f(z) \). Then \( z \) is a global maximizer of (1) if and only if

\[ \partial f(x) \subset N(x|D) \text{ for all } x \in D \text{ and } x \in E_{f(z)}(f), \]

where \( N(x|D) \) is the normal cone to \( D \) at \( x \) defined as:

\[ N(x|D) = \{ c \in \mathbb{R}^n \mid \langle c, y - x \rangle \leq 0, \; y \in D \}. \]

**Remark 2.2** If \( D \) is convex, we obtain from (2) with \( y = z \) the well-known local optimality condition [7]:

\[ \langle f'(z), x - z \rangle \leq 0, \; \forall x \in D. \]

Thus it is clear that the global optimality condition (2) is connected to the classical extremum theory. Note, however, that condition (2) does not require the convexity of \( D \) at all.

**Remark 2.3** In order to conclude that a point \( z' \) in \( D \) is not a solution to problem (1), Theorem 2.2 tells that we need to find a pair \( x, y \in \mathbb{R}^n \) such that

\[ \langle f'(y), x - y \rangle > 0, \; f(y) = f(z'), \; x \in D. \]
3 Algorithm and Its Convergence

Consider the convex maximization problem as a particular case of problem (1),

$$f(x) \rightarrow \max, \ x \in D, \tag{3}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is convex and continuously differentiable, $D \subset \mathbb{R}^n$ is a compact set.

Define the gap function $\pi(y)$ as follows:

$$\pi(y) = \max_{x \in D} \langle f'(y), x - y \rangle, \ y \in \mathbb{R}^n. \tag{4}$$

It has been shown in [1] that function $\pi(y)$ is continuous on $\mathbb{R}^n$.

Before presenting an algorithm for solving problem (1), it is useful to restate Theorem 2.1 in a convenient way via the following function $\theta(z)$ defined for $z \in D$.

$$\theta(z) = \sup_{y \in E_{f(z)}} \pi(y),$$

where $\pi(y) = \max_{x \in D} \langle f'(y), x - y \rangle$. We can note that

$$\pi(y) \leq \theta(z) \text{ for all } y \in E_{f(z)}.$$

**Lemma 3.1** If there is a point $y \in \mathbb{R}^n$ such that $\pi(y) > 0$ and $f(y) = f(z)$ for a feasible point $z \in D$, then

$$f(x(y)) > f(z)$$

holds, where $x(y) \in D$ satisfies $\langle f'(y), x(y) \rangle = \max_{x \in D} \langle f'(y), x \rangle$.

**Proof.** By the definition of $\pi(y)$, we have

$$\pi(y) = \max_{x \in D} \langle f'(y), x - y \rangle = \langle f'(y), x(y) - y \rangle.$$

Since $f$ is convex, we have

$$f(u) - f(v) \geq \langle f'(v), u - v \rangle$$

for all $u, v \in \mathbb{R}^n$. Therefore, the assumption in the lemma implies that

$$f(x(y)) - f(z) = f(x(y)) - f(y) \geq \langle f'(y), x(y) - y \rangle = \pi(y) > 0. \quad \Box$$

**Theorem 3.1** Let $z \in D$ and $f'(z) \neq 0$. If $\theta(z) = 0$ then the point $z$ is a solution to problem (1).
Proof is immediate from the following inequalities:
\[
\langle f'(y), x - y \rangle \leq \pi(y) = \max_{x \in D} \langle f'(y), x - y \rangle \leq \sup_{y \in E_{f(z)}(f)} \max_{x \in D} \langle f'(y), x - y \rangle = \theta(z) = 0
\]
which hold for all \( x \in D \) and \( y \in E_{f(z)}(f) \). □

Lemma 3.2 Let \( f(x) \) is a continuously differentiable convex function on a compact set \( D \in \mathbb{R}^n \). If
\[
\{ \arg \min_{x \in \mathbb{R}^n} f(x) \} \notin D
\]
then there exists a positive number \( \delta \) such that \( |f'(x)| \geq \delta > 0 \) for all \( x \in D \).

Proof. Note that \( f'(x) \neq 0 \) for all \( x \in D \). Write down the following inequality:
\[
|f'(x)| \geq \inf_{x \in D} |f'(x)|, \forall x \in D.
\]
Since \( D \) is compact and \( f \) is continuously differentiable, then we have
\[
|f'(x)| \geq \inf_{x \in D} |f'(x)| = \delta > 0
\]
which proves the lemma. □

Algorithm MAX

Input: A convex function \( f \), and a convex compact set \( D \), and sequence \( \epsilon_k \) such that \( \epsilon_k > 0 \), \( \forall k \) and \( \sum_{k=0}^{\infty} \epsilon_k < \infty \)

Output: A solution \( x \) to problem (1); i.e., a global maximizer of \( f \) over \( D \).

Step 1. Choose a point \( x^0 \in D \) such that \( f'(x^0) \neq 0 \). Set \( k := 0 \).

Step 2. Solve the following problem
\[
\sup \pi(y) \text{ subject to } y \in E_{f(x^k)}(f).
\]
Let \( y^k \) be a solution of this problem, i.e.,
\[
\pi(y^k) \geq \sup_{y \in E_{f(x^k)}(f)} \max_{x \in D} \langle f'(y), x - y \rangle - \epsilon_k.
\]
Let \( \theta(x^k) \leq \pi(y^k) + \epsilon_k \), and let \( x^{k+1} \) be a solution satisfying
\[
\pi(y^k) = \max_{x \in D} \langle f'(y^k), x - y^k \rangle = \langle f'(y^k), x^{k+1} - y^k \rangle.
\]

Step 3. If \( \theta(x^k) = 0 \) then output \( x = x^k \) and terminate. Otherwise set \( k := k + 1 \) and return to Step 2. □

The convergence of this Algorithm is given by the following theorem.
Theorem 3.2 Assume that \( f : \mathbb{R}^n \to \mathbb{R} \) be convex and continuously differentiable. Then the sequence \( \{x^k\}, (k = 0, 1, \ldots) \) generated by Algorithm MAX is a maximizing sequence for problem (1), that is,

\[
\lim_{k \to \infty} f(x^k) = \max_{x \in D} f(x),
\]

and every accumulation point of the sequence \( \{x^k\} \) is a global maximizer of (1).

Proof. Note that Algorithm MAX generates points \( x^k \in D \) and \( f(x^k) \leq f^* \), where \( f^* = f(x^*) = \max_{x \in D} f(x) \). Clearly, for all \( y \in E_{f(x^k)}(f) \) and \( x \in D \) we have

\[
\langle f'(y), x - y \rangle \leq \pi(y) \leq \theta(x^k).
\]

If there exists a \( k \) such that \( \theta(x^k) = 0 \) then, by theorem 3.2, \( x^k \) is a solution to problem (1) and, consequently, the desired result follows. Therefore, without loss of generality, we can suppose that \( \theta(x^k) > 0 \) for all \( k = 0, 1, \ldots \), and prove the theorem by contradiction.

If the assertion is false, that is, \( x^k \) is not a maximizing sequence for problem (2.6), the following inequality holds.

\[
\lim_{k \to \infty} \sup f(x^k) < f^*. \tag{5}
\]

First, we show that the sequence \( \{f(x^k)\} \) is increasing. From the construction of the Algorithm and the definition of \( \theta(x^k) \), we have

\[
\theta(x^k) > \pi(y^k) = \langle f'(y^k), x^{k+1} - y^k \rangle - \epsilon_k,
\]

where \( y^k \in E_{f(x^k)}(f) \). Then this fact and the convexity of \( f \) imply that

\[
f(x^{k+1}) - f(x^k) = f(x^{k+1}) - f(y^k) \geq \langle f'(y^k), x^{k+1} - y^k \rangle.
\]

and

\[
f(x^{k+1}) - f(x^k) + \epsilon_k \geq \langle f'(y^k), x^{k+1} - y^k \rangle + \epsilon_k = \pi(y^k) + \epsilon_k > \theta(x^k) > 0.
\]

Therefore, \( f(x^{k+1}) > f(x^k) - \epsilon_k \) for all \( k \). Furthermore, as the sequence \( \{f(x^k)\} \) is bounded from above by \( f^* \), it has a limit due to [3]:

\[
\lim_{k \to \infty} f(x^k) = A < +\infty,
\]

and hence we have

\[
\lim_{k \to \infty} (f(x^{k+1}) - f(x^k)) = 0.
\]
Then from the above results, we can easily conclude that
\[ \lim_{k \to \infty} \theta(x^k) = 0. \]  

(6)

Now introduce the following sets which are closed and convex.
\[ C_k = \{ x \in \mathbb{R}^n : f(x) \leq f(x^k) \}, \quad k = 0, 1, 2, \ldots \]

It is clear that \( x^* \not\in C_k \) and \( \text{int } C_k \neq \emptyset \). Then take the projection \( u^k \in C_k \) of the point \( x^* \) on \( C_k \) such that
\[ \| u^k - x^* \| = \min_{x \in C_k} \| x - x^* \|. \]  

(7)

Note that
\[ \| u^k - x^* \| > 0 \]  

(8)

holds because \( x^* \not\in C_k \). The optimality condition at the solution \( u^k \) for the convex minimization problem (7) is given as follows [10, 11].
\[ \begin{cases} 
  u^k - x^* + \lambda_k f'(u^k) = 0 \\
  f(u^k) = f(x^k), 
\end{cases} \]  

(9)

where \( \lambda_k \) is the Lagrange multiplier. Hence, we have
\[ \lambda_k = \frac{\| u^k - x^* \|}{\| f'(u^k) \|}. \]  

(10)

Then condition \( f(u^k) = f(x^k) \) of (9) and \( \theta(x^k) \) imply
\[ \theta(x^k) = \sup_{y \in E_{f(x^k)}(f)} \pi(y) \geq \max_{x \in D} \langle f'(u^k), x - u^k \rangle \geq \langle f'(u^k), x^* - u^k \rangle. \]  

(11)

Using (9), (10) and (11), we have
\[ \langle f'(u^k), x^* - u^k \rangle = \| f'(u^k) \||x^* - u^k| \leq \theta(x^k). \]  

(12)

Also by Lemma 3.2, there exists a positive number \( \delta \) which satisfies
\[ \| f'(u^k) \| \geq \delta \]  

(13)

for all \( k \). Therefore, from (12) and (13), we conclude that
\[ 0 \leq \delta \| x^* - u^k \| \leq \theta(x^k). \]

Taking into account (6), we have
\[ \lim_{k \to \infty} u^k = x^*. \]
Hence, by continuity of \( f \) on \( \mathbb{R}^n \),

\[
\lim_{k \to \infty} f(x^k) = \lim_{k \to \infty} f(u^k) = f(x^*), \tag{14}
\]

which yields a contradiction to (5).
Consequently, \( \{x^k\} \subset D \) is a maximizing sequence for problem (1). Since \( D \) is compact, we can always select the convergent subsequences \( \{x^{k_l}\} \) from \( \{x^k\} \) such that

\[
\lim_{l \to \infty} x^{k_l} = \bar{x} \in D.
\]

Then together with (14), we obtain

\[
\lim_{l \to \infty} f(x^{k_l}) = f(\bar{x}) = f^*,
\]

which completes the proof. \(\square\)

4 Conclusion.

A gap function was introduced for constructing an algorithm for solving the convex maximization problem. It has been shown that the proposed algorithm converges globally. A numerical implementation of the algorithm will be provided in next papers.

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References


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