Sumudu Transforms for Boehmians

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Abstract. In this article, the Sumudu transform is extended to the context of Boehmian spaces. The extended Sumudu transform is shown to be an isomorphism from a space of Boehmians onto another space.

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1. Introduction

The Sumudu transform of a function \( f(x) \) of one variable is defined by [12]

\[
M[f(x);u] = M(u) = \int_{\mathbb{R}^+} \frac{1}{u} \exp \left( -\frac{x}{u} \right) f(x) \, dx,
\]

provided the integral exists.

Despite the potential presented by this new operator, only few theoretical investigations have recently appeared in the literature which are still classical not referring the Sumudu transform in the space of generalized functions.

Having scale and unit-preserving properties, the Sumudu transform may solve intricate problems in engineering mathematics and applied sciences without resorting to a new frequency domain. The Sumudu transform is also a theoretical dual of the Laplace transform with which interchange the images of the dirac and Heaviside functions. For further discussion of this transform, we refer to [4;12;13] and references cited therein.

As a probably first time investigation, we extend the Sumudu transform to a certain space of generalized functions, namely, Boehmian spaces. We spread results into three sections. In Section 2 and 3 we construct two spaces \( H_1 \) and
\( H_2 \) of Boehmians with an appropriate extension of the Sumudu transform to a space of distributions of compact support. In Section 4, the Sumudu transform of Boehmians is defined as an isomorphism between the spaces.

2. The Boehmian Space \( H_1 \)

For a general construction of Boehmians, (see [1], [2] and [5-11]). Denote by \( E (\mathbb{R}_+) \) the space of all \( C^\infty (\mathbb{R}_+) \) – functions defined on \( \mathbb{R}_+ \) such that for each compact subset \( K \) of \( \mathbb{R}_+ \),

\[
\xi_k (\phi) = \sup_{x \in K} \left| D^k_x \phi (x) \right| < \infty
\]

for each nonnegative integer \( k \) and, \( D^k_x = \frac{d^k}{dx^k} \).

Let \( \hat{E} (\mathbb{R}_+) \) be the dual space of \( E (\mathbb{R}_+) \) of distributions of compact support [7, 10]. Then it is easy to see that the kernel function \( K (u, x) = \frac{e^{-\frac{x}{u}}}{u} \) of the Sumudu transform is a member of \( E (\mathbb{R}_+) \). Hence, it will be suitable to define the distributional Sumudu transform of \( f (x) \in \hat{E} (\mathbb{R}_+) \) as the adjoint operator

\[
\Psi (u) = \left< f (t), \frac{e^{-\frac{x}{u}}}{u} \right>,
\]

where \( u \in \mathbb{R}_+ \) is arbitrary. Let \( D (\mathbb{R}_+) \) be the space of test functions of compact support on \( \mathbb{R}_+ \), then we proceed as in [9] to extend the new transform to the context of Boehmian spaces. The convolution product of two functions \( \phi \) and \( \varphi \) in \( D (\mathbb{R}_+) \) is defined by

\[
(\phi \ast \varphi) (x) = \int_0^x \varphi (x-t) \varphi (t) \, dt.
\]

Define an operation \( \otimes \) between \( f \in \hat{E} (\mathbb{R}_+) \) and \( \phi \in D (\mathbb{R}_+) \) by

\[
\left< f \otimes \phi, \psi \right> = \left< f, \phi \otimes \psi \right>
\]

for every \( \psi \in E (\mathbb{R}_+) \), where

\[
(\phi \otimes \psi) (x) = \int_{\mathbb{R}_+} \phi (t) \psi (x+t) \, dt.
\]

Let \( \Delta \) be a family of delta sequences \( (\phi_n) \) from \( D (\mathbb{R}_+) \) such that

\[
\Delta_1 \int_{\mathbb{R}_+} \phi_n (x) \, dx = 1;
\]

\[
\Delta_2 \int_{\mathbb{R}_+} |\phi_n (x)| \, dx \leq M, M \text{ is a positive number ;}
\]

\[
\Delta_3 \text{supp} \phi_n \subset (0, \epsilon_n), \text{where } \epsilon_n \to 0 \text{ as } n \to \infty .
\]

**Lemma 2.1** If \( \phi \in D (\mathbb{R}_+) \) and, \( \psi \in E (\mathbb{R}_+) \) then

\[
D^k_x (\phi \otimes \psi) = \phi \otimes D^k_x \psi, k \in \mathbb{N}.
\]
Proof We prove the lemma for $k = 1$. Let $\phi \in D(\mathbb{R}_+)$ such that $\text{supp} \phi = K$. Then, using (2.3) we get

\[
(2.4) \quad \left| (\phi \times \psi)(x) - (\phi \times \psi)(y) \right| = \left| \int_K \phi(t) \frac{\psi(x+t) - \psi(y+t)}{x-y} dt \right|.
\]

Hence, uppon allowing $x \to y$ and considering the supremum over $t \in K$, (2.4) is dominated by $\phi(t) |D^1_x \psi(x)|$. The dominated convergence theorem, therefore, shows that

\[
D^1_x (\phi \times \psi) = \phi \times D^1_x \psi.
\]

Finally, a use of the principle of mathematical induction on $k$, completes the proof of the lemma.

Lemma 2.2 Given $\phi \in D(\mathbb{R}_+), \psi \in E(\mathbb{R}_+)$ then $\phi \times \psi \in E(\mathbb{R}_+)$.  
Proof Let $K$ be a compact subset of $\mathbb{R}_+$. Applying Lemma 2.1 and (2.3) yield

\[
|D^k_x (\phi \times \psi)(x)| \leq \int_{\mathbb{R}_+} |\phi(t)| |D^k_x \psi(x+t)| dt.
\]

With aid of property $\Delta_3$ and upon considering supremum over all $x \in K$, we get

\[
(2.5) \quad \xi_k (\phi \times \psi) \leq \xi_k (\psi).
\]

Since the right hand side of (2.5) is finite our Lemma follows.

Lemma 2.3 Given $f \in \hat{E}(\mathbb{R}_+), \phi \in D(\mathbb{R}_+)$ then

\[
f \otimes \phi \in \hat{E}(\mathbb{R}_+).
\]

Proof Lemma 2.2 implies $\phi \times \psi \in E(\mathbb{R}_+)$ for every $\phi \in D(\mathbb{R}_+)$ and $\psi \in E(\mathbb{R}_+)$. Hence $\langle f \otimes \phi, \psi \rangle = \langle f, \phi \times \psi \rangle$, is meaningful. Next, we show $f \otimes \phi \in \hat{E}(\mathbb{R}_+)$. To establish continuity of $f \otimes \phi$, we apply (2.5) and the topology equipped with $\hat{E}(\mathbb{R}_+)$ to get

\[
|\langle f \otimes \phi, \psi \rangle| = |\langle f, \phi \times \psi \rangle| \leq A_1 A_2 \xi_k (\psi),
\]

where $A_1$ and $A_2$ are certain positive constants. Let $(\psi_n)$ be a sequence in $E(\mathbb{R}_+)$ such that $\psi_n \to 0$ as $n \to \infty$ then $\xi_k (\psi_n) \to 0$ as $n \to \infty$. Hence $|\langle f \otimes \phi, \psi_n \rangle| \to 0$ as $n \to \infty$. Next, let $\alpha_1, \alpha_2$ be arbitrary constants and
\[ \psi_1, \psi_2 \in \mathbf{E} (\mathbb{R}_+) \], then by virtue of the properties of distributions we get
\[
\langle f \otimes \phi, \alpha_1 \psi_1 + \alpha_2 \psi_2 \rangle = \langle f, \phi \times (\alpha_1 \psi_1 + \alpha_2 \psi_2) \rangle \\
= \langle f (x), (\phi \times (\alpha_1 \psi_1 + \alpha_2 \psi_2)) (x + t) \rangle \\
i.e. \\
= \alpha_1 \langle f (x), (\phi (t), \psi_1 (x + t)) \rangle \\
+ \alpha_2 \langle f (x), (\phi (t), \psi_2 (x + t)) \rangle \\
i.e. \\
= \alpha_1 \langle f, \phi \times \psi_1 \rangle + \alpha_2 \langle f, \phi \times \psi_2 \rangle \\
= \alpha_1 \langle f \otimes \phi, \psi_1 \rangle + \alpha_2 \langle f \otimes \phi, \psi_1 \rangle,
\]
for all \( \phi \in \mathbf{D} (\mathbb{R}_+) \). Hence \( f \otimes \phi \in \hat{\mathbf{E}} (\mathbb{R}_+) \). This completes the proof of the lemma.

**Lemma 2.4** Given \( \phi, \varphi \in \mathbf{D} (\mathbb{R}_+) \) and \( \psi \in \mathbf{E} (\mathbb{R}_+) \) then
\[
(\phi \ast \varphi) \times \psi = \phi \times (\varphi \times \psi).
\]

**Proof** The proof of this lemma is a straightforward result of definitions and change of variables, see [9].

**Lemma 2.5** Given \( \phi, \varphi \in \mathbf{D} (\mathbb{R}_+) \) and \( f \in \hat{\mathbf{E}} (\mathbb{R}_+) \) then \( f \otimes (\phi \ast \varphi) = (f \otimes \phi) \otimes \varphi \).

**Proof** Applying Lemma 2.4 yields
\[
\langle f \otimes (\phi \ast \varphi), \psi \rangle = \langle f, (\phi \ast \varphi) \times \psi \rangle \\
= \langle f, \phi \times (\varphi \times \psi) \rangle \\
= \langle (f \otimes \phi) \otimes \varphi, \psi \rangle \in \mathbf{E} (\mathbb{R}_+).
\]
The lemma is completely proved.

**Lemma 2.6** Let \( f_1, f_2 \in \hat{\mathbf{E}} (\mathbb{R}_+), \phi \in \mathbf{D} (\mathbb{R}_+) \) then \( \alpha f \otimes \phi = \alpha (f \otimes \phi) \) and \( (f_1 + f_2) \otimes \phi = f_1 \otimes \phi + f_2 \otimes \phi, \alpha \in \mathbb{C} \).

**Proof** is straightforward.

**Lemma 2.7** Given \( f_n \to f \) in \( \hat{\mathbf{E}} (\mathbb{R}_+) \) and \( \phi \in \mathbf{D} (\mathbb{R}_+) \) then
\[
f_n \otimes \phi \to f \otimes \phi \text{ in } \hat{\mathbf{E}} (\mathbb{R}_+) \text{ as } n \to \infty.
\]

**Proof** If \( \psi \in \mathbf{E} (\mathbb{R}_+) \) then \( \phi \times \psi \in \mathbf{E} (\mathbb{R}_+) \), by Lemma 2.2. Hence
\[
(f_n \otimes \phi - f \otimes \phi) (\psi) = ((f_n - f) \otimes \phi) (\psi) \\
= (f_n - f) \otimes (\phi \times \psi) \quad (2.6)
\]
Allowing \( n \to \infty \), implies \( f_n \otimes \phi \to f \otimes \phi \). Hence the lemma.

**Lemma 2.8** Given \( f \in \hat{\mathbf{E}} (\mathbb{R}_+) \) and \( (\phi_n) \in \Delta \) then \( f \otimes \phi_n \to f \) in \( \hat{\mathbf{E}} (\mathbb{R}_+) \) as \( n \to \infty \).

**Proof** Let \( K \) be a compact subset of \( \mathbb{R}_+ \) and assume \( \text{supp} \phi_n \subset (0, \varepsilon_n) \) for all \( n \in \mathbb{N} \). Let \( \psi \in \mathbf{E} (\mathbb{R}_+) \) then, for \( x \in K \), we have
\[
\left| \mathbf{D}_x^k [(\phi_n \times \psi) - \psi] (x) \right| \leq \int_0^{\varepsilon_n} \left| \phi_n (t) \right| \left| (\mathbf{D}_x^k \psi (x + t) - \mathbf{D}_x^k \psi (x)) \right| \, dt \\
\leq \int_0^{\varepsilon_n} t \left| \phi_n (t) \right| \left| \mathbf{D}_x^{k+1} (x + \xi) \right| \, dt
\]
where $\xi \in (0, t)$. The last inequality follows from the mean value theorem. If $N = \sup_{s \in K} |D^{k+1}\psi(s)|$, then considering supremum over all $x \in K$ suggests

$$\xi_k(\phi_n \times \psi - \psi) \leq \varepsilon_n MN \to 0 \text{ as } n \to \infty,$$

by $\Delta_3$.

Hence

$$\phi_n \times \psi \to \psi \text{ as } n \to \infty \text{ in } E(\mathbb{R}^+)$$

Finally, with aid of (2.7) we get

$$\langle (f \otimes \phi_n)(x), \psi(x) \rangle = \langle f(x), (\phi_n \times \psi)(x) \rangle \to \langle f(x), \psi(x) \rangle.$$

That is

$$f \otimes \phi_n \to f \text{ in } \dot{E}(\mathbb{R}^+).$$

The proof is completed. The desired Boehmian space $H_1$ is constructed.

3. The Boehmian Space $H_2$

Denote by $M(\mathbb{R}^+)$ the space of all functions which are Sumudu transforms of functions in $\dot{E}(\mathbb{R}^+)$ then we say $F_n \to F$ in $M(\mathbb{R}^+)$ as $n \to \infty$ when $f_n \to f$ in $\dot{E}(\mathbb{R}^+)$ as $n \to \infty$ and $f_n = \Psi^{-1}F_n, f = \Psi^{-1}F, \Psi^{-1}$ is the inverse Sumudu transform.

Let $F \in M(\mathbb{R}^+), \phi \in D(\mathbb{R}^+)$. Define a mapping

$$(F \otimes \phi)(u) = \int_{\mathbb{R}^+} F(u + t) \phi(t) dt, u \in \mathbb{R}^+,$$

then the following theorem connects the new Boehmian spaces.

**Lemma 3.1** Given $f \in \dot{E}(\mathbb{R}^+), \phi \in D(\mathbb{R}^+)$ then $\Psi(f \otimes \phi) = \Psi f \otimes \phi$.

**Proof** Let $K = \text{supp } \phi$. Then, Definition 2.1 and the general properties of distributions lead to

$$\Psi(f \otimes \phi)(u) = \left\langle (f \otimes \phi)(x), \frac{e^{-\frac{u}{x}}}{u} \right\rangle$$

$$= \left\langle f(x), \left\langle \phi(t), \frac{e^{-\frac{u}{x+t}}}{u+t} \right\rangle \right\rangle$$

$$= \left\langle f(x), \int_K \phi(t) \frac{e^{-\frac{u}{x+t}}}{u+t} dt \right\rangle$$

It is clear that

$$\int_K \phi(t) \frac{e^{-\frac{u}{x+t}}}{u+t} dt \in E(\mathbb{R}^+).$$
Therefore
\[
\Psi (f \otimes \phi)(u) = \left< f(x), \int K \phi(t) \frac{e^{-\frac{x}{u+t}}}{u+t} dt \right>
\]
\[
= \int_{\mathbb{R}_+} f(x) \frac{e^{-\frac{x}{u+t}}}{u+t} \phi(t) dt
\]
\[
= \int_{\mathbb{R}_+} F(u+t) \phi(t) dt
\]
\[
= \Psi f \odot \phi, \text{ where } F = \Psi f.
\]

This completes the proof of the lemma.

Lemma 3.2 Given \( F \in M(\mathbb{R}_+) \), \( \phi \in D(\mathbb{R}_+) \) then \( F \odot \phi \in M(\mathbb{R}_+) \).

Proof Since \( F \in M(\mathbb{R}_+) \) there is \( f \in \hat{\mathcal{E}}(\mathbb{R}_+) \) such that \( \Psi f = F \). Thus Lemma 3.1 gives
\[
F \odot \phi = \Psi f \odot \phi = \Psi (f \otimes \phi)
\]
which is a member of \( M(\mathbb{R}_+) \) by Lemma 2.3. This completes the proof.

Next are lemmas which are auxilliary for constructing the second Boehmian space. Since the Proofs are similar to that of the corresponding ones proved in Chapter 2, we merely prove the first lemma.

Lemma 3.3 Given \( F \in M(\mathbb{R}_+) \), \( \phi \in D(\mathbb{R}_+) \) then \( \Psi^{-1} (F \odot \phi) = \Psi^{-1} F \odot \phi \).

Proof Let \( f \in \hat{\mathcal{E}}(\mathbb{R}_+) \), such that \( F = \Psi f \). Lemma 3.1 implies \( \Psi^{-1} (F \odot \phi) = \Psi^{-1} (\Psi (f \otimes \phi)) = f \otimes \phi = \Psi^{-1} F \odot \phi \). This completes the proof of the lemma.

Lemma 3.4 Let \( F_1, F_2 \in M(\mathbb{R}_+) \) and \( \phi, \varphi \in D(\mathbb{R}_+) \), then
\[
\begin{align*}
(1) \quad (F_1 + F_2) \odot \phi &= F_1 \odot \phi + F_2 \odot \phi. \\
(2) \quad (aF) \odot \phi &= a (F \odot \phi). \\
(3) \quad F \odot (\phi \ast \varphi) &= (F \odot \phi) \ast \varphi.
\end{align*}
\]

Lemma 3.5 Given \( F_n \to F \) in \( M(\mathbb{R}_+) \) as \( n \to \infty \) and \( (\phi_n) \in \Delta \) then \( F_n \odot \phi_n \to F \) in \( M(\mathbb{R}_+) \) as \( n \to \infty \).

Lemma 3.6 Given \( F_n \to F \) in \( M(\mathbb{R}_+) \) as \( n \to \infty \) and \( \phi \in D(\mathbb{R}_+) \) then \( F_n \odot \phi \to F \odot \phi \) as \( n \to \infty \).

The Boehmian space \( H_2 = H(M(\mathbb{R}_+), D(\mathbb{R}_+), \ast, \odot, \Delta) \) is thus constructed.

4. THE SUMUDU TRANSFORM OF BOEHMIAN

Definition 4.1 In view of Lemma 3.1 we define the Sumudu transform of a Boehmian \([f_n/\phi_n]\) in \( H_1 \) as a Boehmian in \( H_2 \) by the relation
\[
\Lambda [f_n/\phi_n] = [\Lambda f_n/\phi_n]
\]

Lemma 4.2 The extended Sumudu transform \( \Lambda : H_1 \to H_2 \) is well-defined.
Proof Let \([f_n/\phi_n] = [g_n/\psi_n] \in H_1\), then \(f_n \otimes \psi_m = g_m \otimes \phi_n\). Applying the distributional Sumudu transform on both sides, we get \(\Psi f_n \otimes \psi_m = \Psi g_m \otimes \phi_n\). Hence

\[
\Psi f_n/\phi_n \sim \Psi g_n/\psi_m \text{ in } H_2.
\]
Therefore,

\[
[\Psi f_n/\phi_n] = [\Psi g_n/\psi_n].
\]

Equivalently,

\[
\Lambda [f_n/\phi_n] = \Lambda [g_n/\psi_n].
\]

This completes the proof of the Lemma.

**Theorem 4.3** The extended Sumudu transform \(\Lambda : H_1 \to H_2\) is linear.

**Proof** is a straightforward consequence of the linearity of the distributional Sumudu transform.

**Theorem 4.4** The generalized Sumudu transform \(\Lambda : H_1 \to H_2\) is an isomorphism

**Proof** We show that \(\Lambda\) is one-one. Assume \(\Lambda [f_n/\phi_n] = \Lambda [g_n/\psi_n]\), then

\[
\Psi f_n \otimes \psi_m = \Psi (f_n \otimes \psi_m) = \Psi g_m \otimes \phi_n = \Psi (g_m \otimes \phi_n).
\]
Therefore, we get

\[
\Psi (f_n \otimes \psi_m) = \Psi (g_m \otimes \phi_n).
\]

Thus linearity of the distributional Sumudu transform implies

\[
f_n \otimes \psi_m = g_m \otimes \phi_n.
\]

Next, we show that \(\Lambda\) is onto. For, let \([f_n/\phi_n] \in H_2\) be arbitrary, then

\[
F_n \otimes \phi_m = F_m \otimes \phi_n, \forall m, n \in \mathbb{N}.
\]

Let \(f_n \in \hat{E}(\mathbb{R}_+^{+})\) be such that \(\Psi f_n = F_n\), then

\[
\Psi f_n \otimes \phi_m = \Psi f_m \otimes \phi_n.
\]

That is,

\[
\Psi (f_n \otimes \phi_m) = \Psi (f_m \otimes \phi_n).
\]

Now, since \(\Psi\) is one-one we get \(f_n \otimes \phi_m = f_m \otimes \phi_n\). This completes the proof of the lemma.
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