Intuitionistic Fuzzy BCI-Positive Implicative Ideals in BCI-Algebras

M. Touqeer and M. Aslam Malik

Department of Mathematics, University of the Punjab
Quaid-e-Azam Campus, Lahore-54590, Pakistan
touqeer-muavia@yahoo.com
malikpu@math.pu.edu.pk

Abstract

In this paper, we consider the intuitionistic fuzzification of the concept of BCI-positive implicative ideals in BCI-algebras and investigate some of their properties.

Keywords: Intuitionistic fuzzy set, Intuitionistic fuzzy ideal, Intuitionistic fuzzy BCI-positive implicative ideal

1 Introduction and Preliminaries

In 1966, Y. Imai and K. Iseki introduced the notion of BCK-algebra. Until 1966, various algebras were introduced after considering the properties of the union and the intersection. The case of set difference and its properties remained unexplored. The consideration of these operations and its properties by K. Iseki gave birth to the notion of a BCK-algebra. In 1966, K. Iseki introduced another notion called BCI-algebra, which is in fact a generalization of the notion of a BCK-algebra. After the introduction of the concept of fuzzy sets by Zadeh, several researches were conducted on the generalization of the notion of fuzzy sets. The idea of "intuitionistic fuzzy set" was first published by Atanassov, as a generalization of the notion of fuzzy sets. After that many researchers considered the fuzzification of ideals and subalgebras in BCK(BCI)-algebras. In this paper, we introduce the concept of intuitionistic fuzzy BCI-positive implicative ideals in BCI-algebras and investigate some of
Let us recall that an algebra \((X, *, 0)\) of type \((2, 0)\) is called a BCI-algebra if it satisfies the following conditions:

1. \((x * y) * (x * z) * (z * y) = 0\)
2. \((x * (x * y)) * y = 0\)
3. \(x * x = 0\)
4. \(x * y = 0\) and \(y * x = 0\) imply \(x = y\)

In a BCI-algebra, we can define a partial ordering "\(\leq\)" by \(x \leq y\) if and only if \(x * y = 0\). In a BCI-algebra \(X\), the set \(M = \{x \in X \mid 0 * x = 0\}\) is a subalgebra and is called the BCK-part of \(X\). A BCI-algebra \(X\) is called proper if \(X - M \neq \emptyset\). Otherwise it is improper. Moreover, in a BCI-algebra the following conditions hold:

1. \((x * y) * z = (x * z) * y\)
2. \(x * 0 = x\)
3. \(x \leq y\) implies \(x * z \leq y * z\) and \(z * y \leq z * x\)
4. \(0 * (x * y) = (0 * x) * (0 * y)\)
5. \(0 * (0 * (x * y)) = 0 * (y * x)\)
6. \((x * z) * (y * z) \leq x * y\)

A mapping \(f : X \rightarrow Y\) of BCI-algebras is called a homomorphism if \(f(x * y) = f(x) * f(y)\) for all \(x, y \in X\).

By a fuzzy set \(\mu\) in a nonempty set \(X\) we mean a function \(\mu : X \rightarrow [0, 1]\) and the complement of \(\mu\), denoted by \(\bar{\mu}\), is the fuzzy set in \(X\) given by \(\bar{\mu}(x) = 1 - \mu(x)\) for all \(x \in X\).

**Definition 1.1** A non-empty subset \(I\) of a BCI-algebra \(X\) is called an ideal of \(X\) if for any \(x, y \in X\)

1. \(0 \in I\)
2. \(x * y \in I\) and \(y \in I\) implies \(x \in I\)

**Definition 1.2** An ideal \(I\) of a BCI-algebra \(X\) is said to be closed if \(0 * x \in I\), for all \(x \in I\).

**Definition 1.3** A fuzzy subset \(\mu\) of a BCI-algebra \(X\) is called a fuzzy ideal of \(X\) if it satisfies:

1. \(\mu(0) \geq \mu(x)\)
2. \(\mu(x) \geq \min\{\mu(x * y), \mu(y)\}\)

for all \(x, y \in X\).

**Definition 1.4** A fuzzy subset \(\mu\) in a BCI-algebra \(X\) is said to be a fuzzy closed ideal of \(X\) if it satisfies:
1. $\mu(0 * x) \geq \mu(x)$
2. $\mu(x) \geq \min\{\mu(x \ast y), \mu(y)\}$
for all $x, y \in X$.

**Definition 1.5** A non-empty subset $I$ of a BCI-algebra $X$ is said to be BCI-positive implicative ideal of $X$ if it satisfies:
1. $0 \in I$
2. $((x \ast z) \ast z) \ast (y \ast z) \in I$ and $y \in I$ implies $x \ast z \in I$
for all $x, y, z \in X$.

**Definition 1.6** A fuzzy subset $\mu$ in a BCI-algebra $X$ is called a fuzzy BCI-positive implicative ideal of $X$ if it satisfies:
1. $\mu(0) \geq \mu(x)$
2. $\mu(x \ast z) \geq \min\{\mu(((x \ast z) \ast z) \ast (y \ast z)), \mu(y)\}$
for all $x, y \in X$.

Note that a fuzzy subset $\mu$ of $X$ is called a fuzzy closed BCI-positive implicative ideal of $X$ if it satisfies the condition 2 and the condition $\mu(0 * x) \geq \mu(x)$ , for all $x \in X$.

**Intuitionistic fuzzy set**

An intuitionistic fuzzy set $A$ in a non-empty set $X$ is an object having the form $IFSA = \{(x, \alpha_A(x), \beta_A(x)) | x \in X\}$ where the functions $\alpha_A : X \mapsto [0, 1]$ and $\beta_A : X \mapsto [0, 1]$ denote the degree of membership and the degree of non-membership and $0 \leq \alpha_A(x) + \beta_A(x) \leq 1$ for all $x \in X$.

An intuitionistic fuzzy set $IFSA = \{(x, \alpha_A(x), \beta_A(x)) | x \in X\}$ in $X$ can be identified to an ordered pair $(\alpha_A, \beta_A)$ in $I^X \times I^X$. For the sake of simplicity we shall use the symbol IFSA $= (\alpha_A, \beta_A)$ insted of the symbol $IFSA = \{(x, \alpha_A(x), \beta_A(x)) | x \in X\}$.

**Definition 1.7** An IFSA $= (\alpha_A, \beta_A)$ in a BCI-algebra $X$ is called an intuitionistic fuzzy subalgebra of $X$ if it satisfies:
1. $\alpha_A(x \ast y) \geq \min\{\alpha_A(x), \alpha_A(y)\}$
2. $\beta_A(x \ast y) \leq \max\{\beta_A(x), \beta_A(y)\}$
for all $x, y \in X$.

**Example 1.8** Consider a BCI-algebra $X = \{0, a, b, c\}$ with the following cayley table
Let $A = (\alpha_A, \beta_A)$ be an intuitionistic fuzzy set in $X$ defined by

$\alpha_A(0) = \alpha_A(a) = \alpha_A(c) = 0.7 > 0.3 = \alpha_A(b)$

$\beta_A(0) = \beta_A(a) = \beta_A(c) = 0.2 < 0.5 = \beta_A(b)$

Then IFSA = $(\alpha_A, \beta_A)$ is an intuitionistic fuzzy subalgebra of $X$.

**Proposition 1.9** Every intuitionistic fuzzy subalgebra $A = (\alpha_A, \beta_A)$ of $X$ satisfies the inequalities:

$\alpha_A(0) \geq \alpha_A(x)$ and $\beta_A(0) \leq \beta_A(x)$ for all $x \in X$.

**Proof.** $\alpha_A(0) = \alpha_A(x \ast x) \geq \min\{\alpha_A(x), \alpha_A(x)\}$ and

$\beta_A(0) = \beta_A(x \ast x) \leq \min\{\beta_A(x), \beta_A(x)\}$

This completes the proof.

### 2. Intuitionistic Fuzzy Ideals

An IFSA = $(\alpha_A, \beta_A)$ in a BCI-algebra $X$ is called an intuitionistic fuzzy ideal of $X$ if it satisfies the following inequalities:

(IFI-1) $\alpha_A(0) \geq \alpha_A(x)$ and $\beta_A(0) \leq \beta_A(x)$ for all $x \in X$.

(IFI-2) $\alpha_A(x) \geq \min\{\alpha_A(x \ast y), \alpha_A(y)\}$

(IFI-3) $\beta_A(x) \leq \max\{\beta_A(x \ast y), \beta_A(y)\}$

for all $x, y \in X$.

**Definition 2.1** An IFSA = $(\alpha_A, \beta_A)$ in $X$ is called an intuitionistic fuzzy closed ideal of $X$ if it satisfies (IFI-2), (IFI-3) and (IFI-4) $\alpha_A(0 \ast x) \geq \alpha_A(x)$ and $\beta_A(0 \ast x) \leq \beta_A(x)$ for all $x \in X$.

**Lemma 2.2** Let IFSA = $(\alpha_A, \beta_A)$ be an intuitionistic fuzzy ideal of $X$. If the inequality $x \ast y \leq z$ holds in $X$, then $\alpha_A(x) \geq \min\{\alpha_A(y), \alpha_A(z)\}$ and $\beta_A(x) \leq \max\{\beta_A(y), \beta_A(z)\}$

**Lemma 2.3** Let IFSA = $(\alpha_A, \beta_A)$ be an intuitionistic fuzzy ideal of $X$. If the inequality $x \leq y$ holds in $X$, then $\alpha_A(x) \geq \alpha_A(y)$ and $\beta_A(x) \leq \beta_A(y)$, that is $\alpha_A$ is order reversing while $\beta_A$ is order preserving.

### 3. Intuitionistic fuzzy BCI-positive implicative ideal

An IFSA = $(\alpha_A, \beta_A)$ in a BCI-algebra $X$ is called an intuitionistic fuzzy BCI-positive implicative ideal of $X$ if it satisfies:

(IFBCI-PI-1) $\alpha_A(0) \geq \alpha_A(x)$ and $\beta_A(0) \leq \beta_A(x)$

(IFBCI-PI-2) $\alpha_A(x \ast z) \geq \min\{\alpha_A(((x \ast z) \ast z) \ast (y \ast z)), \alpha_A(y)\}$
**Example 3.1** consider the BCI-algebra $X = \{0, 1, 2, 3\}$ with the following cayley table:

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Define an IFSA $= (\alpha_A, \beta_A)$ in $X$ as follows:

$\alpha_A(0) = \alpha_A(3) = 1, \alpha_A(1) = \alpha_A(2) = t$

$\beta_A(0) = \beta_A(3) = 0, \beta_A(1) = \beta_A(2) = s$

where $s, t \in (0, 1)$ and $s + t \leq 1$

By routine calculations it is easy to verify that IFSA $= (\alpha_A, \beta_A)$ is an intuitionistic fuzzy BCI-positive implicative ideal of $X$.

**Theorem 3.2** Every intuitionistic fuzzy BCI-positive implicative ideal of a BCI-algebra $X$ is an intuitionistic fuzzy ideal of $X$ but the converse is not true.

**Proof.** Assume that IFSA $= (\alpha_A, \beta_A)$ is an intuitionistic fuzzy BCI-implicative ideal of $X$. Then by definition we have:

$\alpha_A(x * z) \geq \min\{\alpha_A(((x * z) * z) * (y * z)), \alpha_A(y)\}$

$\beta_A(x * z) \leq \max\{\beta_A(((x * z) * z) * (y * z)), \beta_A(y)\}$

for all $x, y \in X$.

Putting $z=0$ we get

$\alpha_A(x * 0) \geq \min\{\alpha_A(((x * 0) * 0) * (y * 0)), \alpha_A(y)\}$

$\beta_A(x * 0) \leq \max\{\beta_A(((x * 0) * 0) * (y * 0)), \beta_A(y)\}$

imply:

$\alpha_A(x) \geq \min\{\alpha_A(x * y), \alpha_A(y)\}$

$\beta_A(x) \leq \max\{\beta_A(x * y), \beta_A(y)\}$

Hence IFSA $= (\alpha_A, \beta_A)$ is an intuitionistic fuzzy ideal of $X$. Whereas the converse of this theorem may not be true. For this consider the following example:

**Example 3.3** Consider the BCI-algebra $X = \{0, 1, 2, 3, 4\}$ with the following cayley table:
Define an IFSA = (α_A, β_A) in X by:

\[ \alpha_A(0) = \alpha_A(2) = 1, \alpha_A(1) = \alpha_A(3) = \alpha_A(4) = t \]
\[ \beta_A(0) = \beta_A(2) = 0, \beta_A(1) = \beta_A(3) = \beta_A(4) = s \]

where \( s, t \in (0, 1) \) and \( s + t \leq 1 \).

By routine calculations it is easy to verify that IFSA = (α_A, β_A) is an intuitionistic fuzzy ideal of X but it is not an intuitionistic fuzzy BCI-positive implicative ideal of X because:

\[ \alpha_A(4 * 3) = \alpha_A(1) = t < 1 = \alpha_A(0) \min \{\alpha(((4 * 3) * 3) * (0 * 3)), \alpha_A(0)\} \]

**Theorem 3.4** Let IFSA = (α_A, β_A) be an intuitionistic fuzzy ideal of a BCI-algebra X. Then the following conditions are equivalent:

1. IFSA = (α_A, β_A) is an intuitionistic fuzzy BCI-positive implicative ideal of X.

2. \( \alpha_A((x * y) * z) \geq \alpha_A(((x * z) * z) * (y * z)) \)

and

3. \( \beta_A((x * y) * z) \leq \beta_A(((x * z) * z) * (y * z)) \)

2. \( \beta_A((x * y) * z) = \beta_A(((x * z) * z) * (y * z)) \)

4. \( \alpha_A(x * y) = \alpha_A(((x * y) * y) * (0 * y)) \)

and \( \beta_A(x * y) = \beta_A(((x * y) * y) * (0 * y)) \)

5. \( \alpha_A(x * y) \geq \alpha_A(((x * y) * y) * (0 * y)) \)

and \( \beta_A(x * y) \leq \beta_A(((x * y) * y) * (0 * y)) \)

**Proof.** (1 \( \rightarrow \) 2) Let IFSA = (α_A, β_A) be an intuitionistic fuzzy BCI-positive implicative ideal of X. Then by definition we have:

\[ \alpha_A((x * y) * z) \geq \min \{\alpha_A(((x * y) * z) * (0 * z)), \alpha_A(0)\} \]
\[ = \alpha_A(((x * z) * z) * (0 * z)) \] and \( \beta_A((x * y) * z) \leq \max \{\beta_A(((x * y) * z) * z) * (0 * z)), \beta_A(0)\} = \beta_A(((x * y) * z) * (0 * z)) \)

Now \(((x * y) * z) * (0 * z) = (((x * z) * z) * y) * ((y * z) * y) \leq ((x * z) * z) * (y * z) \)
Therefore by using lemma 2.2 we get
\[ \alpha_A(((x \ast y) \ast z) \ast z) \ast (0 \ast z) \geq \alpha_A(((x \ast z) \ast z) \ast (y \ast z)) \]
and
\[ \beta_A(((x \ast y) \ast z) \ast z) \ast (0 \ast z) \leq \beta_A(((x \ast z) \ast z) \ast (y \ast z)) \]
Hence we have
\[ \alpha_A((x \ast y) \ast z) \geq \alpha_A(((x \ast z) \ast z) \ast (y \ast z)) \]
and
\[ \beta_A((x \ast y) \ast z) \leq \beta_A(((x \ast z) \ast z) \ast (y \ast z)) \]
(2 \rightarrow 3) Assume that
\[ \alpha_A((x \ast y) \ast z) \geq \alpha_A(((x \ast z) \ast z) \ast (y \ast z)) \]
and
\[ \beta_A((x \ast y) \ast z) \leq \beta_A(((x \ast z) \ast z) \ast (y \ast z)) \]
Now \((x \ast z) \ast (y \ast z) \leq (x \ast z) \ast y = (x \ast y) \ast z\)
Therefore by using lemma 2.2 we have
\[ \alpha_A(((x \ast z) \ast z) \ast (y \ast z)) \geq \alpha_A((x \ast y) \ast z) \]
and
\[ \beta_A(((x \ast z) \ast z) \ast (y \ast z)) \leq \beta_A((x \ast y) \ast z) \]
So we have
\[ \alpha_A((x \ast y) \ast z) = \alpha_A(((x \ast z) \ast z) \ast (y \ast z)) \]
and
\[ \beta_A((x \ast y) \ast z) = \beta_A(((x \ast z) \ast z) \ast (y \ast z)) \]
(3 \rightarrow 4) Assume that
\[ \alpha_A((x \ast y) \ast z) = \alpha_A(((x \ast z) \ast z) \ast (y \ast z)) \]
and
\[ \beta_A((x \ast y) \ast z) = \beta_A(((x \ast z) \ast z) \ast (y \ast z)) \]
Putting \(z = y\) and \(y = 0\) we get
\[ \alpha_A((x \ast 0) \ast y) = \alpha_A(((x \ast y) \ast y) \ast (0 \ast y)) \]
and
\[ \beta_A((x \ast 0) \ast y) = \beta_A(((x \ast y) \ast y) \ast (0 \ast y)) \]
that is
\[ \alpha_A(x \ast y) = \alpha_A(((x \ast y) \ast y) \ast (0 \ast y)) \]
and
\[ \beta_A(x \ast y) = \beta_A(((x \ast y) \ast y) \ast (0 \ast y)) \]
(4 \rightarrow 5) Obviously.
(5 \rightarrow 1) Assume that
\[ \alpha_A(x \ast y) \geq \alpha_A(((x \ast y) \ast y) \ast (0 \ast y)) \]
and
\[ \beta_A(x \ast y) \leq \beta_A(((x \ast y) \ast y) \ast (0 \ast y)) \]
Now \(((x \ast y) \ast y) \ast (0 \ast y) \ast ((x \ast y) \ast y) \ast (z \ast y) \leq (z \ast y) \ast (0 \ast y) \leq z \ast 0 = z\)
Therefore by lemma 2.2 we have
\[ \alpha_A(((x \ast y) \ast y) \ast (0 \ast y)) \geq \min\{\alpha_A(((x \ast y) \ast y) \ast (z \ast y)), \alpha_A(z)\} \]
and
\[ \beta_A(((x \ast y) \ast y) \ast (0 \ast y)) \leq \max\{\beta_A(((x \ast y) \ast y) \ast (z \ast y)), \beta_A(z)\} \]
that is
\[ \alpha_A(x \ast y) \geq \min\{\alpha_A(((x \ast y) \ast y) \ast (z \ast y)), \alpha_A(z)\} \]
and
\[ \beta_A(x \ast y) \leq \max\{\beta_A(((x \ast y) \ast y) \ast (z \ast y)), \beta_A(z)\} \]
Hence IFSA = \((\alpha_A, \beta_A)\) is an intuitionistic fuzzy BCI-positive implicative ideal of \(X\).

**Lemma 3.5** An IFSA = \((\alpha_A, \beta_A)\) is an intuitionistic fuzzy BCI-positive im-
implicative ideal of a BCI-algebra $X$ if and only if $\alpha_A$ and $\bar{\beta}_A$ are fuzzy BCI-positive implicative ideals of $X$.

**Proof.** Suppose that IFSA = $(\alpha_A, \beta_A)$ is an intuitionistic fuzzy BCI-positive implicative ideal of a BCI-algebra $X$. Then

$\alpha_A(0) \geq \alpha_A(x)$ and $\beta_A(0) \leq \beta_A(x)$

Also

$\alpha_A(x \ast z) \geq \min\{\alpha_A(((x \ast z) \ast z) \ast (y \ast z)), \alpha_A(y)\}$

$\beta_A(x \ast z) \leq \max\{\beta_A(((x \ast z) \ast z) \ast (y \ast z)), \beta_A(y)\}$

for all $x, y \in X$.

Then clearly $\alpha_A$ is a fuzzy BCI-positive implicative ideal of $X$. Now $\beta_A(0) \leq \beta_A(x) \Rightarrow 1 - \bar{\beta}_A(0) \leq 1 - \bar{\beta}_A(x) \Rightarrow \bar{\beta}_A(0) \geq \bar{\beta}_A(x)$. Now

$\beta_A(x \ast z) \leq \max\{\beta_A(((x \ast z) \ast z) \ast (y \ast z)), \beta_A(y)\} \Rightarrow 1 - \bar{\beta}_A(x \ast z) \leq \max\{1 - \beta_A(((x \ast z) \ast z) \ast (y \ast z)), 1 - \beta_A(y)\}$

$\Rightarrow \bar{\beta}_A(x \ast z) \geq \min\{\beta_A(((x \ast z) \ast z) \ast (y \ast z)), \bar{\beta}_A(y)\}$

Hence $\bar{\beta}_A$ is also a fuzzy BCI-positive implicative ideal of $X$.

Conversely suppose that $\alpha_A$ and $\bar{\beta}_A$ are fuzzy BCI-positive implicative ideals of $X$. Then

$\alpha_A(0) \geq \alpha_A(x)$ and $\bar{\beta}_A(0) \geq \bar{\beta}_A(x)$. Also

$\alpha_A(x \ast z) \geq \min\{\alpha_A(((x \ast z) \ast z) \ast (y \ast z)), \alpha_A(y)\}$

$\bar{\beta}_A(x \ast z) \geq \min\{\beta_A(((x \ast z) \ast z) \ast (y \ast z)), \beta_A(y)\}$

Then $\bar{\beta}_A(0) \geq \bar{\beta}_A(x) \Rightarrow 1 - \beta_A(0) \geq 1 - \beta_A(x) \Rightarrow \beta_A(0) \leq \beta_A(x)$

Also $\bar{\beta}_A(x \ast z) \geq \min\{\beta_A(((x \ast z) \ast z) \ast (y \ast z)), \bar{\beta}_A(y)\}$

$\Rightarrow 1 - \beta_A(x \ast z) \leq \min\{1 - \beta_A(((x \ast z) \ast z) \ast (y \ast z)), 1 - \beta_A(y)\}$

$\Rightarrow \beta_A(x \ast z) \leq \max\{\beta_A(((x \ast z) \ast z) \ast (y \ast z)), \beta_A(y)\}$. Hence IFSA = $(\alpha_A, \beta_A)$ an intuitionistic fuzzy BCI-positive implicative ideal of $X$.

**Lemma 3.6** An IFSA = $(\alpha_A, \beta_A)$ is an intuitionistic fuzzy closed BCI-positive implicative ideal of a BCI-algebra $X$ if and only if $\alpha_A$ and $\bar{\beta}_A$ are fuzzy closed BCI-positive implicative ideals of $X$.

**Proof.** Suppose that IFSA = $(\alpha_A, \beta_A)$ is an intuitionistic fuzzy closed BCI-
positive implicative ideal of X. Then it satisfies (IFBCI-PI-2), (IFBCI-PI-3) and (IFBCI-PI-4) that is $\alpha_A(0 \ast x) \geq \alpha_A(x)$ and $\beta_A(0 \ast x) \leq \beta_A(x)$ for all $x \in X$.

Then it is clear that $\alpha_A$ is fuzzy closed BCI-positive implicative ideal of X. For $\beta_A$ it can be easily verified as done earlier in theorem 3.5 that $\bar{\beta}_A(x \ast z) \geq \min\{\bar{\beta}_A(((x \ast z) \ast z) \ast (y \ast z)), \bar{\beta}_A(y)\}$ for all $x, y, z \in X$. It is therefore required to show only $\bar{\beta}(0 \ast x) \geq \bar{\beta}(x)$. Since $\beta(0 \ast x) \leq \beta(x) \Rightarrow 1 - \bar{\beta}_A(0 \ast x) \leq 1 - \bar{\beta}_A(x) \Rightarrow \bar{\beta}_A(0 \ast x) \geq \bar{\beta}_A(x) \Rightarrow \bar{\beta}_A$ is also a fuzzy closed BCI-positive implicative ideal of X.

Conversely suppose that $\alpha_A$ and $\bar{\beta}_A$ are fuzzy closed BCI-positive implicative ideals of X. Then

$\alpha_A(0 \ast x) \geq \alpha_A(x)$ and $\bar{\beta}_A(0 \ast x) \geq \bar{\beta}_A(x)$ for all $x \in X$. Now $\bar{\beta}_A(0 \ast x) \geq \bar{\beta}_A(x) \Rightarrow 1 - \bar{\beta}_A(0) \geq 1 - \bar{\beta}_A(x) \Rightarrow \bar{\beta}_A(0) \leq \bar{\beta}_A(x)$.

Also $\alpha_A(x \ast z) \geq \min\{\alpha_A(((x \ast z) \ast z) \ast (y \ast z)), \alpha_A(y)\}$ and $\bar{\beta}_A(x \ast z) \geq \min\{\bar{\beta}_A(((x \ast z) \ast z) \ast (y \ast z)), \bar{\beta}_A(y)\}$

Now $\bar{\beta}_A(x \ast z) \geq \min\{\bar{\beta}_A(((x \ast z) \ast z) \ast (y \ast z)), \bar{\beta}_A(y)\}$

$\Rightarrow 1 - \bar{\beta}_A(x \ast z) \geq \min\{1 - \bar{\beta}_A(((x \ast z) \ast z) \ast (y \ast z)), 1 - \bar{\beta}_A(y)\}$

$\Rightarrow \bar{\beta}_A(x \ast z) \leq 1 - \min\{1 - \bar{\beta}_A(((x \ast z) \ast z) \ast (y \ast z)), 1 - \bar{\beta}_A(y)\}$

$\Rightarrow \bar{\beta}_A(x \ast z) \leq \max\{\beta_A(((x \ast z) \ast z) \ast (y \ast z)), \beta_A(y)\}$. Hence IFSA = $(\alpha_A, \beta_A)$ is an intuitionistic fuzzy closed BCI-positive implicative ideal of X.

**Theorem 3.7** An IFSA = $(\alpha_A, \beta_A)$ is an intuitionistic fuzzy BCI-positive implicative ideal of a BCI-algebra X if and only if $\Box A = (\alpha_A, \alpha_A)$ and $\circ A = (\bar{\beta}_A, \beta_A)$ are intuitionistic fuzzy BCI-positive implicative ideals of X.

**Proof.** Suppose that IFSA = $(\alpha_A, \beta_A)$ is an intuitionistic fuzzy BCI-positive implicative ideal of a BCI-algebra X. Then by theorem (3.5) $\alpha_A$ and $\bar{\beta}_A$ are fuzzy BCI-positive implicative ideals of X i.e. $\bar{\alpha}_A = \alpha_A$ and $\bar{\beta}_A$ are fuzzy BCI-positive implicative ideals of X. Therefore by theorem 3.5 $\Box A = (\alpha_A, \alpha_A)$ and $\circ A = (\bar{\beta}_A, \beta_A)$ are intuitionistic fuzzy BCI-positive implicative ideals of X.

Conversely suppose that $A = (\alpha_A, \alpha_A)$ and $\circ A = (\bar{\beta}_A, \beta_A)$ are intuitionistic fuzzy BCI-positive implicative ideals of X. Then by theorem 3.5 $\alpha_A$ and $\bar{\beta}_A$ are fuzzy BCI-positive implicative ideals of X. Therefore by theorem (3.5) IFSA = $(\alpha_A, \beta_A)$ is an intuitionistic fuzzy BCI-positive implicative ideal of X.

**Theorem 3.8** An IFSA = $(\alpha_A, \beta_A)$ is an intuitionistic fuzzy closed BCI-
positive implicative ideal of a BCI-algebra $X$ if and only if $\Box A = (\alpha_A, \bar{\alpha}_A)$ and $\Diamond A = (\beta_A, \bar{\beta}_A)$ are intuitionistic fuzzy closed BCI-positive implicative ideals of $X$.

**Proof.** Suppose that IFSA = $(\alpha_A, \beta_A)$ is an intuitionistic fuzzy closed BCI-positive implicative ideal of a BCI-algebra $X$. Then by theorem 3.6 $\alpha_A$ and $\beta_A$ are fuzzy closed BCI-positive implicative ideals of $X$ i.e $\bar{\alpha} = \alpha_A$ and $\bar{\beta} = \beta_A$ are fuzzy closed BCI-positive implicative ideals of $X$. Therefore by theorem 3.6 $\Box A = (\alpha_A, \alpha_A)$ and $\Diamond A = (\beta_A, \beta_A)$ are intuitionistic fuzzy closed BCI-positive implicative ideals of $X$.

Conversely suppose that $\Box A = (\alpha_A, \alpha_A)$ and $\Diamond A = (\beta_A, \beta_A)$ are intuitionistic fuzzy closed BCI-positive implicative ideals of $X$. Then by theorem 3.6 $\alpha_A$ and $\beta_A$ are fuzzy closed BCI-positive implicative ideals of $X$. Then by theorem 3.6 IFSA = $(\alpha_A, \beta_A)$ is an intuitionistic fuzzy closed BCI-positive implicative ideal of $X$.

For any $t, s \in [0, 1]$ and a fuzzy subset $\mu$ in a non-empty set $X$, the set $U(\alpha_A; t) = \{x \in X \mid \alpha_A(x) \geq t\}$ is called an upper $t$-level cut of $\alpha_A$ and the set $L(\beta_A; s) = \{x \in X \mid \beta_A(x) \leq s\}$ is called a lower $s$-level cut of $\beta_A$.

**Theorem 3.9** An IFSA = $(\alpha_A, \beta_A)$ is an intuitionistic fuzzy BCI-positive implicative ideal of a BCI-algebra $X$ if and only if the non-empty upper $t$-level cut $U(\alpha_A; t)$ and the non-empty lower $s$-level cut $L(\beta_A; s)$ are BCI-positive implicative ideals of $X$ for any $s, t \in [0, 1]$.

**Proof.** Suppose that An IFSA = $(\alpha_A, \beta_A)$ is an intuitionistic fuzzy BCI-positive implicative ideal of a BCI-algebra $X$. Since $U(\alpha_A; t) \neq \emptyset$, $L(\beta_A; s) \neq \emptyset$. So for any $x \in U(\alpha_A; t)$ we have $\alpha_A(x) \geq t \Rightarrow \alpha_A(0) \geq \alpha_A(x) \geq t \Rightarrow 0 \in U(\alpha_A; t)$. Now let $(((x \ast z) \ast z) \ast (y \ast z)) \in U(\alpha_A; t)$ and $y \in U(\alpha_A; t)$. Then $\alpha_A(((x \ast z) \ast z) \ast (y \ast z)) \geq t$ and $\alpha_A(y) \geq t$. Since $\alpha_A(x \ast z) \geq \min\{\alpha_A(((x \ast z) \ast z) \ast (y \ast z)), \alpha_A(y)\} \geq t \Rightarrow x \ast z \in U(\alpha_A; t) \Rightarrow U(\alpha_A; t)$ is BCI-positive implicative ideal of $X$. Similarly we can prove that $L(\beta_A; s)$ is a BCI-positive implicative ideal of $X$. Conversely suppose that the non-empty upper $t$-level cut $U(\alpha_A; t)$ and the non-empty lower $s$-level cut $L(\beta_A; s)$ are BCI-positive implicative ideals of $X$ for any $s, t \in [0, 1]$. If possible assume that there exists some $x_0 \in X$ such that $\alpha_A(0) < \alpha_A(x_0)$ and $\beta_A(0) > \beta_A(x_0)$. put $t_0 = 1/2\{\alpha_A(0) + \alpha_A(x_0)\}$ then $\alpha_A(0) < t_0 < \alpha_A(x_0) \Rightarrow x_0 \in U(\alpha_A; t_0)$ and 0 does not belong to $U(\alpha_A; t_0)$ which is a contradiction to the fact that $U(\alpha_A; t_0)$ is a BCI-positive implicative ideal of $X$. Therefore we must have $\alpha_A(0) \geq \alpha_A(x)$ for all $x \in X$. Similarly by putting $s_0 = 1/2\{\beta_A(0) + \beta_A(x_0)\}$ we can prove that $\beta_A(0) \leq \beta_A(x)$ for all $x \in X$. 


If possible assume that there exists some \(x_0, y_0, z_0 \in X\) such that \(p = \alpha_A(x_0 * z_0) < \min \{\alpha_A(((x_0 * z_0) * z_0) * (y_0 * z_0)), \alpha_A(y_0)\} = q\)

Put \(t_0 = 1/2\{p + q\}\) then \(p < t_0 < q\)

\[\Rightarrow ((x_0 * z_0) * (y_0 * z_0)) \in U(\alpha_A; t_0)\text{ and } y_0 \in U(\alpha_A; t_0)\] whereas \(x_0 * z_0\) does not belong to \(U(\alpha_A; t_0)\) which is a contradiction to the fact that \(U(\alpha_A; t_0)\) is a BCI-positive implicative ideal of \(X\). Therefore \(\alpha_A(x * z) \geq \min \{\alpha_A(((x * z) * (y * z))), \alpha_A(y)\}\) for all \(x, y, z \in X\). Similarly we can prove that \(\beta_A(x * z) \geq \min \{\beta_A(((x * z) * (y * z))), \beta_A(y)\}\) for all \(x, y, z \in X\). Hence IFSA = \((\alpha_A, \beta_A)\) is an intuitionistic fuzzy BCI-positive implicative ideal of \(X\).

**Theorem 3.10** An IFSA = \((\alpha_A, \beta_A)\) is an intuitionistic fuzzy closed BCI-positive implicative ideal of a BCI-algebra \(X\) if and only if the non-empty upper \(t\)-level cut \(U(\alpha_A; t)\) and the non-empty lower \(s\)-level cut \(L(\beta_A; s)\) are closed BCI-positive implicative ideals of \(X\) for any \(s, t \in [0, 1]\).

**Proof.** Suppose that An IFSA = \((\alpha_A, \beta_A)\) is an intuitionistic fuzzy closed BCI-positive implicative ideal of a BCI-algebra \(X\). Then \(\alpha_A(0 * x) \geq \alpha_A(x)\) and \(\beta_A(0 * x) \leq \beta_A(x)\) for all \(x \in X\).

Since \(U(\alpha_A; t) \neq \emptyset\), \(L(\beta_A; s) \neq \emptyset\). So for any \(x \in U(\alpha_A; t)\) we have \(\alpha_A(x) \geq t \Rightarrow \alpha_A(0 * x) \geq \alpha_A(x) \geq t \Rightarrow 0 * x \in U(\alpha_A; t)\). Similarly for any \(x \in L(\beta_A; s)\) we have \(\beta_A(x) \leq s \Rightarrow \beta_A(0 * x) \leq \beta_A(x) \leq s \Rightarrow 0 * x \in L(\beta_A; s)\). Hence \(U(\alpha_A; t)\) and \(L(\beta_A; s)\) are closed BCI-positive implicative ideals of \(X\).

Conversely suppose that the non-empty upper \(t\)-level cut \(U(\alpha_A; t)\) and the non-empty lower \(s\)-level cut \(L(\beta_A; s)\) are closed BCI-positive implicative ideals of \(X\) for any \(s, t \in [0, 1]\). We want to show that IFSA = \((\alpha_A, \beta_A)\) is an intuitionistic fuzzy closed BCI-positive implicative ideal of \(X\). It is enough to show that \(\alpha_A(0 * x) \geq \alpha_A(x)\) and \(\beta_A(0 * x) \leq \beta_A(x)\) for all \(x \in X\). If possible assume that there exists some \(x_0 \in X\) such that \(\alpha_A(0 * x_0) < \alpha_A(x_0)\). Take \(t_0 = 1/2\{\alpha_A(0 * x_0) + \alpha_A(x_0)\}\) then \(\alpha_A(0 * x_0) < t_0 < \alpha_A(x_0) \Rightarrow x_0 \in U(\alpha_A; t_0)\) whereas \(0 * x_0\) does not belong to \(U(\alpha_A; t_0)\) which is a contradiction to the fact that \(U(\alpha_A; t_0)\) is a closed BCI-positive implicative ideal of \(X\). Therefore we must have \(\alpha_A(0 * x) \geq \alpha_A(x)\) for all \(x \in X\). Similarly we can prove that \(\beta_A(0 * x) \leq \beta_A(x)\) for all \(x \in X\). Hence IFSA = \((\alpha_A, \beta_A)\) is an intuitionistic fuzzy closed BCI-positive implicative ideal of \(X\).

**Theorem 3.11** Let \(\{I_t \mid t \in \Lambda\}\) be a collection of BCI-positive implicative ideals of \(X\) such that

1. \(X = \bigcup_{t \in \Lambda} I_t\).
2. \(s > t\) if and only if \(I_s \subset I_t\) for all \(s, t \in \Lambda\).

Then an IFSA = \((\alpha_A, \beta_A)\) defined by \(\alpha_A(x) = \text{Sup}\{t \in \Lambda \mid x \in I_t\}\) and
\[ \beta_A(x) = \inf \{ s \in \Lambda \mid x \in I_s \} \text{ for all } x \in X \text{ is an intuitionistic fuzzy BCI-positive implicative ideal of } X. \]

**Proof.** By Theorem (3.9) it is sufficient to prove that \( U(\alpha_A; t) \) and \( L(\beta_A; s) \) are BCI-positive implicative ideals of \( X \). To prove that \( U(\alpha_A; t) \) is a BCI-positive implicative ideal of \( X \), we divide the proof into the following two cases:

1. \( t = \sup \{ q \in \Lambda \mid q < t \} \)
2. \( t \neq \sup \{ q \in \Lambda \mid q < t \} \)

The case (1) implies that \( x \in U(\alpha_A; t) \Leftrightarrow x \in I_q \), for all \( q < t \Leftrightarrow x \in \bigcap_{q< t} I_q \) so that \( U(\alpha_A; t) = \bigcap_{q< t} I_q \) which is a BCI-positive implicative ideal of \( X \).

For the case (2) we claim that \( U(\alpha_A; t) = \bigcup_{q \geq t} I_q \). If \( x \in \bigcup_{q \geq t} I_q \) then \( x \in I_q \) for some \( q \geq t \). It follows that \( \alpha_A(x) \geq q \geq t \), so that \( x \in U(\alpha_A; t) \). This shows that \( \bigcup_{q \geq t} I_q \subseteq U(\alpha_A; t) \). Now assume that \( x \notin \bigcup_{q \geq t} I_q \) then \( x \notin I_q \) for all \( q \geq t \). Since \( t \neq \sup \{ q \in \Lambda \mid q < t \} \), there exists some \( \epsilon > 0 \) such that \( (t - \epsilon, t) \cap \Lambda = \emptyset \). Hence \( x \notin I_q \) for all \( q > t - \epsilon \) which means that \( x \in I_q \) if \( q \leq t - \epsilon < t \). Thus \( \alpha_A(x) \leq t - \epsilon < t \) and so \( x \notin U(\alpha_A; t) \). Therefore \( U(\alpha_A; t) \subseteq \bigcup_{q \geq t} I_q \) and that \( U(\alpha_A; t) = \bigcup_{q \geq t} I_q \) which is a BCI-positive implicative ideal of \( X \).

Next we prove that \( L(\beta_A; s) \) is a BCI-positive implicative ideal of \( X \). For this we divide the proof into the following two cases:

1. \( s = \inf \{ r \in \Lambda \mid s < r \} \)
2. \( s \neq \inf \{ r \in \Lambda \mid s < r \} \)

The case (1) implies that The case (1) implies that \( x \in L(\beta_A; s) \Leftrightarrow x \in I_r \), for all \( s < r \Leftrightarrow x \in \bigcap_{s< r} I_r \) so that \( L(\beta_A; s) = \bigcap_{s< r} I_r \) which is a BCI-positive implicative ideal of \( X \).

For the case (2) we claim that \( L(\beta_A; s) = \bigcup_{r \leq s} I_r \). If \( x \in \bigcup_{r \leq s} I_r \) then \( x \in I_r \) for some \( r \leq s \). It follows that \( \beta_A(x) \leq r \leq s \), so that \( x \in L(\beta_A; s) \). This shows that \( \bigcup_{r \leq s} I_r \subseteq L(\beta_A; s) \). Now assume that \( x \notin \bigcup_{r \leq s} I_r \) then \( x \notin I_r \) for all \( r \leq s \). Since \( s \neq \inf \{ r \in \Lambda \mid s < r \} \), there exists some \( \epsilon > 0 \) such that \( (s, s + \epsilon) \cap \Lambda = \emptyset \). Hence \( x \notin I_r \) for all \( r < s + \epsilon \) which means that \( x \in I_r \) if \( r \geq s + \epsilon > s \). Thus \( \beta_A(x) \geq s + \epsilon > s \) and so \( x \notin L(\beta_A; s) \). Therefore \( L(\beta_A; s) \subseteq \bigcup_{r \leq s} I_r \) and that \( L(\beta_A; s) = \bigcup_{r \leq s} I_r \) which is a BCI-positive implicative ideal of \( X \). This completes the proof.

**Theorem 3.12** If \( \text{IFS} = (\alpha_A; \beta_A) \) is an intuitionistic fuzzy closed BCI-positive implicative ideal of a BCI-algebra \( X \), then the sets \( J= \{ x \in X \mid \alpha_A(x) = \alpha_A(0) \} \) and \( K= \{ x \in X \mid \beta_A(x) = \beta_A(0) \} \) are BCI-positive implicative ideals of \( X \).

**Proof.** Since \( 0 \in X \), \( \alpha_A(0) = \alpha_A(0) \) and \( \beta_A(0) = \beta_A(0) \) implies \( 0 \in J \) and
0 ∈ K, so J ̸= Φ and K ̸= Φ. Now let ((x * z) * z) * (y * z) ∈ J and y ∈ J.
Then α_A(((x * z) * z) * (y * z)) = α_A(0) and α_A(y) = α_A(0). Since α_A(x * z) ≥
min{α_A(((x * z) * z) * (y * z)), α_A(y)} = α_A(0). But α_A(0) ≥ α_A(x * z).
Therefore α_A(x * z) = α_A(0). It follows that that x * z ∈ J for all x, y, z ∈ X. Hence
J is a BCI-positive implicative ideal of X. Similarly we can prove that K is a
BCI-positive implicative ideal of X.

4. Definition Let f be a mapping on a set X and A = (α_A, β_A) be an intu-
itionistic fuzzy BCI-positive implicative ideal of X. If A′ = (u, v) is an intuitionistic
fuzzy BCI-positive implicative ideal of BCI-algebra X′ then the pre-image of A′ = (u, v)
derived from f under the homomorphism f is an intuitionistic fuzzy BCI-positive
implicative ideal of X.

Proof. Let an IFSA = (α_A, β_A) where α_A = uof and β_A = vof be the
pre-image of A′ = (u, v) under f. Since A′ = (u, v) is an intuitionistic fuzzy
BCI-positive implicative ideal of X′, we have u(0′) ≥ u(f(x)) = α_A(x) and
v(0′) ≤ v(f(x)) = β_A(x). On the other hand u(0′) = u(f(0)) = α_A(0) and
v(0′) = v(f(0)) = β_A(0). Therefore α_A(0) ≥ α_A(x) and β_A(0) ≤ β_A(x), for all
x ∈ X. Now we show that α_A(x * z) ≥ min{α_A(((x * z) * z) * (y * z)), α_A(y)}
and β_A(x * z) ≤ max{β_A(((x * z) * z) * (y * z)), β_A(y)} for all x, y, z ∈ X.

Now α_A(x * z) = u(f(x * z)) = u(f(x) * f(z)) ≥ min{u(((f(x) * f(z)) * f(z)) * (y * f(z))), u(y′)}.
Since f is an onto homomorphism, so there exists some y ∈ X such that
f(y) = y′. Thus α_A(x * z) ≥ min{u(((f(x) * f(z)) * f(z)) * (f(y) * f(z))), u(f(y))
min{u(f(((x * z) * z) * (y * z))), u(f(y))} = min{α_A(((x * z) * z) * (y * z)), α_A(y)}. Therefore the result α_A(x * z) ≥ min{α_A(((x * z) * z) * (y * z)), α_A(y)} is true
for all x, y, z ∈ X because y′ is an arbitrary element of X′ and f is an onto mapping.
Similarly we can prove that β_A(x * z) ≤ max{β_A(((x * z) * z) * (y * z)), β_A(y)}
for all x, y, z ∈ X.

Hence the pre-image A = (α_A, β_A) of A′ = (u, v) under f is an intuitionistic fuzzy
BCI-positive implicative ideal of X.

Definition 4.2 Let f : X → Y be a homomorphism of BCI-algebras. For
any IFSA = (α_A, β_A) in X we define a new IFSA f = (α_A^f, β_A^f) in Y by
α_A^f(x) = α_A(f(x)), β_A^f(x) = β_A(f(x)) for all x ∈ X. If f : X → Y is a
homomorphism of BCI-algebras then f(0) = 0.
Theorem 4.3 Let $f : X \rightarrow Y$ be a homomorphism of BCI-algebras. If an IFS $\alpha = (\alpha_A, \beta_A)$ in $Y$ is an intuitionistic fuzzy BCI-positive implicative ideal of $Y$, then the IFS $\alpha = (\alpha^f_A, \beta^f_A)$ in $X$ is an intuitionistic fuzzy BCI-positive implicative ideal of $X$.

Proof. We first have that $\alpha^f_A(x) = \alpha_A(f(x)) \leq \alpha_A(0) = \alpha^f_A(0)$
$$ \Rightarrow \alpha^f_A(x) \leq \alpha^f_A(0)$$
$\beta^f_A(x) = \beta_A(f(x)) \geq \beta_A(0) = \beta^f_A(0)$
$$ \Rightarrow \beta^f_A(x) \geq \beta^f_A(0).$$

Let $x, y, z \in X$. Then
$$ \min\{\alpha^f_A((x*z)*y) \}, \alpha^f_A(y) = \min\{\alpha_A(f((x*z)*y)), \alpha_A(f(y))\} = \min\{\alpha_A(((f(x)*f(z)) * (f(y)*f(z)))\}, \alpha_A(f(y))\} \leq \alpha_A(f(x) * f(z)) = \alpha_A(f(x*z)) = \alpha^f_A(x*z)$$
Similarly $\max\{\beta^f_A((x*z)*y) \}, \beta^f_A(y) = \max\{\beta_A(f((x*z)*y)), \beta_A(f(y))\} \geq \beta_A(f(x) * f(z)) = \beta^f_A(x*z)$
Hence $\alpha = (\alpha^f_A, \beta^f_A)$ in $X$ is an intuitionistic fuzzy BCI-positive implicative ideal of $X$.

Theorem 4.4 Let $f : X \rightarrow Y$ be an epimorphism of BCI-algebras and IFS $\alpha = (\alpha_A, \beta_A)$ be in $Y$. If IFS $\alpha = (\alpha^f_A, \beta^f_A)$ is an intuitionistic fuzzy BCI-positive implicative ideal of $X$, then IFS $\alpha = (\alpha_A, \beta_A)$ is an intuitionistic fuzzy BCI-positive implicative ideal of $Y$.

Proof. For any $x, y, z \in Y$, there exist some $a, b, c \in X$ such that $f(a) = x$, $f(b) = y$, $f(c) = z$. Then
$$ \alpha_A(x) = \alpha_A(f(a)) = \alpha^f_A(a) \leq \alpha^f_A(0) = \alpha_A(0)$$
$$ \Rightarrow \alpha_A(x) \leq \alpha_A(0)$$
$\beta_A(x) = \beta_A(f(a)) = \beta^f_A(a) \geq \beta^f_A(0) = \beta_A(0)$
$$ \Rightarrow \beta_A(x) \geq \beta_A(0)$$

Now $\alpha_A(x*z) = \alpha_A(f(a) * f(c))$
$$ = \alpha_A(f(a*c)) = \alpha^f_A(a*c) \geq \min\{\alpha^f_A(((a*c)*b*c), \alpha^f_A(b)\} = \min\{\alpha_A(f(((a*c)*b*c)), \alpha_A(f(b))\} = \min\{\alpha_A(((f(a)*f(c))*f(c)), (f(b)*f(c)))\}, \alpha_A(f(b))\} = \min\{\alpha_A(((x*z)*y), \alpha_A(y)\}$$
Similarly $\beta_A(x*z) = \beta_A(f(a) * f(c))$
$$ = \beta_A(f(a*c)) = \beta^f_A(a*c) \leq \max\{\beta^f_A(((a*c)*b*c), \beta^f_A(b)\} = \max\{\beta_A(f(((a*c)*b*c)), \beta_A(f(b))\} = \max\{\beta_A(((f(a)*f(c))*f(c)), (f(b)*f(c)))\}, \beta_A(f(b))\} =
max{β_A((x * z) * z) * (y * z)), β_A(y)}

for all x, y, z ∈ Y.

Hence IFSA = (α_A, β_A) is an intuitionistic fuzzy BCI-positive implicative ideal of Y.

**Theorem 4.5** Let IFSA = (α_A, β_A) be an intuitionistic fuzzy ideal of a BCI-algebra X. If for any x, y ∈ X

\[
\alpha_A(x * (x * (y * (y * x)))) \geq \alpha_A((x * (x * y)) * (y * x)) \geq \beta_A((x * (x * (y * (y * x)))) \leq \\
\beta_A((x * (x * y)) * (y * x))
\]

Then IFSA = (α_A, β_A) is an intuitionistic fuzzy BCI-positive implicative ideal of X.

**Proof.** Putting u = x * y we have

(u * (u * x)) * (x * u) = ((x * y) * ((x * y) * x)) * (x * y) = ((x * y) * (0 * y)) * (x * y)

Thus we have

α_A((u * (u * x)) * (x * u)) = α_A(((x * y) * (x * y)) * (0 * y)) and

β_A((u * (u * x)) * (x * u)) = β_A(((x * y) * (x * y)) * (0 * y))

Now

(u * (u * (x * (x * u)))) = (x * y) * ((x * y) * (x * (x * y)))

Thus we have

α_A(u * (u * (x * (x * u)))) = α_A(x * y) and β_A(u * (u * (x * (x * u)))) = β_A(x * y)

By given hypothesis we have

α_A(u * (u * (x * (x * u)))) ≥ α_A((u * (u * x)) * (x * u)) and β_A(u * (u * (x * (x * u)))) ≤ \\
β_A((u * (u * x)) * (x * u))

Which implies:

\[
\alpha_A(x * y) \geq \alpha_A(((x * y) * (x * y)) * (0 * y)) \text{ and } \beta_A(x * y) \leq \beta_A(((x * y) * (x * y)) * (0 * y))
\]

Hence by theorem 3.4 IFSA = (α_A, β_A) is an intuitionistic fuzzy BCI-positive implicative ideal of X.

**Theorem 4.6** Let X be a BCI-algebra and IFSA = (α_A, β_A) is an intuitionistic fuzzy BCI-positive implicative ideal of X such that α_A(x ∧ y) ≤ α_A(y) and

β_A(x ∧ y) ≥ β_A(y). Then IFSA = (α_A, β_A) is an intuitionistic fuzzy H-ideal of X.

**Proof.** Suppose that IFSA = (α_A, β_A) is an intuitionistic fuzzy BCI-positive implicative ideal of X. Then by theorem 3.4(5)

\[
\alpha_A(x * y) \geq \alpha_A(((x * y) * (y * x)) * (0 * y)) \geq \min\{\alpha_A(((x * y) * (y * x)) * (0 * y)) * x), \alpha_A(x)\} = \\
\min\{\alpha_A(0 * y), \alpha_A(x)\} \geq \min\{\alpha_A(x * 0 * y), \alpha_A(x)\} = \alpha_A(x)
\]

⇒ \alpha_A(x * y) ≥ \alpha_A(x)

Similarly by theorem 3.4(5)

\[
\beta_A(x * y) \leq \beta_A(((x * y) * (y * x)) * (0 * y)) \leq \max\{\beta_A(((x * y) * (y * x)) * (0 * y)) * x), \beta_A(x)\} = \\
\beta_A(x * y) \leq \beta_A(x)
\]
max{β_A(0 * y), β_A(x)} \leq \max{β_A(x \land 0 * y), β_A(x)} = β_A(x)
⇒ β_A(x * y) \leq β_A(x)
for all x, y ∈ X.

Since we know the result that if IFSA = (α_A, β_A) is an intuitionistic fuzzy ideal of X such that α_A(x * y) ≥ α_A(x) and β_A(x * y) ≤ β_A(x) for all x, y ∈ X. Then IFSA = (α_A, β_A) is an intuitionistic fuzzy H-ideal of X. Therefore by using this result it is clear that IFSA = (α_A, β_A) is an intuitionistic fuzzy H-ideal of X.

**Definition 4.7** A BCK-algebra X is said to be positive implicative if it satisfies for all x, y, z ∈ X:
(x * z) * (y * z) = (x * y) * z.

**Definition 4.8. Fuzzy ideal extensions:** Let μ be a fuzzy subset of a BCK-algebra X and a ∈ X. Then the fuzzy subset < μ, a >: X → [0, 1] defined by < μ, a > (x) = μ(x * a) is called the extension of μ by a.

**Definition 4.9. Intuitionistic Fuzzy ideal extensions:** Let (α_A, β_A) be an intuitionistic fuzzy set in a BCK-algebra X and a, b ∈ X. Then the intuitionistic fuzzy set < (α_A, β_A), (a, b) > defined by:
< (α_A, β_A), (a, b) >= < (α_A, a >, < β_A, b >) is called the extension of (α_A, β_A) by (a, b). If a = b, then it is denoted by < (α_A, β_A), a >.

**Theorem 4.10** Let an IFSA = (α_A, β_A) be intuitionistic fuzzy BCI-positive implicative ideal of a positive implicative BCK-algebra X and a, b ∈ X. Then the extension < (α_A, β_A), (a, b) > of (α_A, β_A) by (a, b) is also an intuitionistic fuzzy BCI-positive implicative ideal of X.

**Proof.** Suppose that an IFSA = (α_A, β_A) be intuitionistic fuzzy BCI-positive implicative ideal of a positive implicative BCK-algebra X and a, b ∈ X. Let x, y ∈ X. Then we have < α_A, a > (0) = α_A(0 * a) = α_A(0) ≥ α_A(x * a) =< α_A, a > (x) and
< β_A, b > (0) = β_A(0 * b) = β_A(0) ≤ β_A(x * b) =< β_A, b > (x).

Now < α_A, a > (x * z) = α_A((x * z) * a) = α_A((x * a) * (z * a)) ≥ min{α_A(((x * a) * (z * a)) * ((y * a) * (z * a))), α_A((y * a))} = min{α_A(((x * z) * z) * (z * a)), α_A(y * a)} = min{< α_A, a > (((x * z) * z) * (y * z)), < α_A, a > (y)}.

Similarly < β_A, b > (x * z) = β_A((x * z) * b) = β_A((x * b) * (z * b)) ≤ min{β_A(((x * b) * (z * b)) * (z * b)), β_A(y * b))} = min{α_A(((x * z) * z) * (y * z)), < α_A, b > (y)}.

Hence the extension < (α_A, β_A), (a, b) > of (α_A, β_A) by (a, b) is an intuitionistic fuzzy BCI-positive implicative ideal of X.

**Corollary 4.11** Let an IFSA = (α_A, β_A) be intuitionistic fuzzy BCI-positive implicative ideal of a positive implicative BCK-algebra X and a ∈ X. Then
the extension \(<(\alpha_A, \beta_A), a>\) of \((\alpha_A, \beta_A)\) by \(a\) is also an intuitionistic fuzzy BCI-positive implicative ideal of \(X\).

REFERENCES
17. M. Akram, Intuitionistic Fuzzy closed ideals in BCI-algebras, International
Mathematical Forum, 1, 2006, No.9, 445 – 453.

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