Exp-function Method for Solving
Nonlinear Beam Equation

Khalid A. Abdul Zahra and Mudhir A. Abdul Hussain
University of Basrah, College of Education
Department of Mathematics, Basrah, Iraq
khalidabdul72@yahoo.com, mud_abd@yahoo.com

Abstract. The Exp-function method is one of the powerful methods that appear in recently years for finding exact solutions of different types of nonlinear differential equations. In this work the Exp-function method has been used to find travelling wave solutions of nonlinear beam equation. The method allows us to found exact solution of nonlinear beam equation. Also, the exact solution of boundary value problem of fourth order has been found.

Mathematics Subject Classification 2000: 35L20, 35L70

Keywords: Exp-function method, nonlinear beam equation, exact solution.

1. Introduction

Many of nonlinear partial differential equations are widely used to describe many important phenomena in the Engineering, Physics, Biology, etc. More of these equations have been studied to find only approximation solutions by using different methods. Recently, several powerful methods have been proposed to obtain exact solutions of nonlinear PDEs. Investigating of the travelling wave solutions for nonlinear beams equation plays an important role in the study of Engineering and Physical phenomena. The Exp-function method is one of the powerful methods
appear in recently years to find travelling wave solutions of some nonlinear differential equations. The advantage of this method lies in the fact that we can obtain the generalized solitary solution for nonlinear beams equation. There are many studies of the Exp-function method, for example Hajipour and Mahmoudi [1] used the Exp-function to find the exact solution of Fitzhugh-Nagumo equation. Yao, Wang and Zhou [5] used the Exp-function to find the exact solution of a Huxley Equation with Variable Coefficient. Mohyud-din, Noor [8] applied the Exp-function method for solving higher-order boundary value problems. Xin-Wei Zhou [9] used the exp-function method for solving Huxley equation. In this paper we introduced two examples to illustrate the basic idea of the Exp-function method. In the first example we used the Exp-function method to investigate the existence of the travelling wave solutions of the following nonlinear partial differential equation of elastic beam,

\[
\frac{\partial^2 y}{\partial t^2} + \frac{\partial^4 y}{\partial x^4} + \alpha \frac{\partial^2 y}{\partial x^2} + \beta y + y^2 = 0, \quad y \in \mathbb{R}. \quad \ldots (1.1)
\]

where, \( y \) is the deflection of beam and \( \alpha, \beta \) are parameters. In the second example we used the Exp-function method to investigate the existence of solutions of the following boundary value problem,

\[
u^{(4)} + \alpha \nu'' + \beta \nu + \nu^3 = 0, \quad \nu(0) = A, \quad \nu(1) = B, \quad \nu'(0) = C, \quad \nu'(1) = D. \quad \ldots (1.2)
\]

were, \( u = u(x), \quad x \in [0,1], \quad \alpha, \beta \) are parameters and \( A, B, C \) and \( D \) are arbitrary constants. Equations (1.1) and (1.2) have been studied by many authors, for examples equation (1.2) has been studied by Thompson and Stewart [4]. They showed numerically the existence of periodic solutions of equation (1.2) for some values of parameters. Bardin and Furta [3] used the local method of Lyapunov-Schmidt and found the sufficient conditions of existence of periodic waves of equation (1.2). Equation (1.1) also, has been studied by Abdul Hussain [6, 7], he found the bifurcation solutions of equation (1.1) by using local method of Lyapunov-Schmidt.

\section{2. Travelling wave solutions of the Beam equation}

To apply the Exp-function method to the equation (1.1) we making use the travelling wave transformation
**Exp-function method for solving nonlinear beam equation**

\[ \eta = \omega x + vt, \quad y = y(\eta). \]

where \( \omega \) and \( v \) are constants to be determined later. Then equation (1.1) reduces to an ordinary differential equation.

\[ \omega^4 y^{(4)} + (v^2 + \alpha \omega^2) y'' + \beta y + y^2 = 0 \quad \ldots (2.1) \]

The Exp-function method is based on the assumption that traveling wave solutions can be expressed in the following form,

\[ y(\eta) = \sum_{n=-c}^{d} a_n \exp(n \eta) = \frac{a_c \exp(c \eta)}{b_p \exp(p \eta)} + \cdots + \frac{a_{-d} \exp(-d \eta)}{b_{-q} \exp(-q \eta)} \quad \ldots (2.2) \]

where \( c, d, p, \) and \( q \) are positive integers which are unknown to be further determined, \( a_n \) and \( b_m \) are unknown constants. In order to determine the values of \( c \) and \( p \), we balance the linear term of highest order in equation (1.1) with the highest order nonlinear term. By simple calculation, we have

\[ y^{(4)} = \frac{c_1 \exp(c + 15p) \eta + \cdots}{c_2 \exp(16p \eta) + \cdots} \quad \ldots (2.3) \]

and

\[ y^2 = \frac{c_3 \exp(2c + 14p) \eta + \cdots}{c_4 \exp(16p \eta) + \cdots} \quad \ldots (2.4) \]

where \( c_1, c_2, c_3 \) and \( c_4 \) are determined coefficients only for simplicity. Balancing highest order of Exp-function in equations (2.3) and (2.4) we have

\[ c + 15p = 2c + 14p \]

which leads to the result \( p = c \). Similarly to determine the values of \( d \) and \( q \), we balance the linear term of lowest order in (1.1) with the lowest-order nonlinear term.

\[ y^{(4)} = \frac{\cdots + d_1 \exp(-d - 15q) \eta}{\cdots + d_2 \exp(-16q \eta)} \quad \ldots (2.5) \]

and

\[ y^2 = \frac{\cdots + d_3 \exp(-2d - 14q) \eta}{\cdots + d_4 \exp(-16q \eta)} \quad \ldots (2.6) \]
where \( d_1, d_2, d_3 \) and \( d_4 \) are determined coefficients only for simplicity. From (2.5) and (2.6) we have

\[-d - 15q = -2d - 14q\]

and this gives \( d = q \).

The values of \( c \) and \( d \) can be freely chosen, so for simplicity we investigate three cases.

**Case 1:** \( p = c = 1, \ q = d = 1 \). Equation (2.2) becomes,

\[y(\eta) = \frac{a_1 \exp(\eta) + a_0 + a_{-1} \exp(-\eta)}{\exp(\eta) + b_0 + b_{-1} \exp(-\eta)} \quad \ldots (2.7)\]

Substituting equation (2.7) into equation (2.1) and by the help of software Mathematica 6.0, we have,

\[\frac{1}{K} \left[ C_5 e^{\eta} + C_4 e^{\eta} + C_3 e^{\eta} + C_2 e^{\eta} + C_1 + C_0 e^{-\eta} + C_{-1} e^{-2\eta} + C_{-2} e^{-3\eta} + C_{-3} e^{-4\eta} + C_{-4} e^{-5\eta} \right] = 0 \quad \ldots (2.8)\]

where

\[K = \left( e^{\eta} + b_0 + b_{-1} e^{-\eta} \right)^5\]

\[C_5 = \beta a_1 + a_1^2\]

\[C_4 = \beta a_0 + \alpha \omega^2 a_0 + 2a_1 a_0 - v^2 a_1 b_0 + \omega^4 a_0 + 4\beta a_1 b_0 - \omega^4 a_1 b_0 + 3a_1^2 b_0 - \alpha \omega^2 a_1 b_0 + v^2 a_0\]

\[C_3 = 3a_1^2 b_0^2 + 3a_1 b_0 + 4\beta a_0 b_0 + \beta a_1 + 16\omega^4 a_1 b_0 + 11\omega^4 a_1 b_0 - 4\alpha \omega^2 a_1 b_0 - v^2 a_1 b_0^2 + 4v^2 a_1 b_0 + 4\beta a_1 b_0 - 11\omega^4 a_0 b_0 + 6a_1 b_0^2 + 6a_1 a_0 b_0 + \alpha \omega^2 a_1 b_0 - \alpha \omega^2 a_1 b_0^2 + 2a_1 b_0\]

\[C_2 = a_1^2 b_0^3 + 3a_0^2 b_0 + 11v^2 a_1 b_0 - 7v^2 a_1 b_0 b_{-1} - 4v^2 a_0 b_{-1} + 6a_1 a_0 b_{-1} + 6a_0 a_1 b_0^2 + 6a_1 a_0 b_{-1} + 4\beta a_0 b_{-1} + 6a_1 b_0 b_{-1} + 77\omega^4 a_1 b_0 b_{-1} - \alpha \omega^2 a_1 b_0^2 + 11\alpha \omega^2 a_1 b_0 b_{-1} - 4\alpha \omega^2 a_1 b_0 b_{-1} + 12\beta a_1 b_0 b_{-1} - \omega^4 a_1 b_0 b_{-1} - 7\alpha \omega^2 a_1 b_0 b_{-1} + 2a_0 a_{-1} + 11\omega^4 a_1 b_0^2 + 4\beta a_1 b_0^2 - 6\beta a_0 b_0^2 - 76\omega^4 a_1 b_{-1} + \alpha \omega^2 a_1 b_{-1} + 4\beta a_{-1} b_0 + v^2 a_1 b_0^2 - v^2 a_0 b_{-1} + 11\omega^4 a_1 b_0^2\]
\[ C_1 = 11v^2a_{-1}b_0^2 - 176\omega^4a_{-1}b_1 - 58\omega^4a_{-1}b_1 - 4\omega^2a_{-1}b_1 - 4\omega^2a_{-1}b_1 + 12\beta_0a_0b_0b_{-1} + 11\omega a_1b_0^2 + 4\beta_0b_{-1} + 3a_0^2b_0b_{-1} + 12a_0a_1b_0b_{-1} - 13\omega a_0b_0b_{-1} + 6\beta_0b_1^2 + 6a_0a_{-1}b_0 + v^2a_0b_0^3 + \omega a_0b_0^3 + \omega a_0b_0^3 + \beta a_1b_1^1 + 4\beta_0a_0b_0^3 + 2a_0a_0b_0^3 + \alpha^2b_0^8 - \omega^2a_0b_0^8 + 176\omega b_1b_1 + 47\omega a_0b_0b_{-1} + 3a_0b_{-1}b_{-1} + 11\omega a_0b_0^8 + 3a_0b_0^8 + a_1^2 + 6a_0a_{-1}b_0^2 + 6\beta_0b_0^2 - 13v^2a_0b_0b_{-1} - 4v^2a_1b_1^2 + 3a_1b_1^2 + 12\beta_0b_0b_{-1} + 2v^2a_1b_0^2 + 4v^2a_{-1}b_1 + 6a_0a_{-1}b_1 \]

\[ C_0 = \beta a_0b_0^6 + 5\omega^2a_0b_0^5 + 5\omega^4a_0b_0^5 - 10\omega^2a_0b_0^5 + 5v^2a_0b_0^5 - 10\omega^2a_0b_0^5 \]

\[ C_{-1} = -4v^2a_{-1}b_1 + 6a_0a_1b_0b_{-1} - 176\omega a_1b_1 + 4\beta a_1b_1^3 + a_1b_1^3 + 6a_1a_{-1}b_0^2b_{-1} + 47\omega a_0b_0b_{-1} + 176\omega a_1b_1 + 3a_0b_0^2 + 12a_0a_1b_0b_{-1} - 13\omega a_0b_0b_{-1} + 6\beta a_1b_0^2b_{-1} + v^2a_1b_0^2b_{-1} + v^2a_1b_0^2b_{-1} + \omega a_0b_0^2 + 3a_0b_{-1}b_{-1} + 4\omega a_0b_0^2 + \omega a_0b_0^2 - 13v^2a_0b_0b_{-1} + 6\beta a_1b_0^2b_{-1} + 4\omega a_0b_0^2 + 2a_0a_{-1}b_0^2b_{-1} + 11\omega a_1b_0^2b_{-1} + 11\omega a_1b_0^2b_{-1} + 4v^2a_1b_0^2b_{-1} + 2v^2a_{-1}b_0b_{-1} + 3a_1b_1^2 - 58\omega a_1b_0b_{-1} + 6a_1a_{-1}b_1 \]

\[ C_{-2} = a_2^2b_0^2 + 6a_0a_{-1}b_2^2 + 11\omega a_0b_0^2b_{-1} + 11v^2a_1b_1 - 7v^2a_0b_0b_{-1} + 6\beta a_0b_0^2b_{-1} + 4\omega a_0b_0^2b_{-1} - 7v^2a_{-1}b_1 - 4a_0b_{-1}^3 + 6a_1b_0^2b_{-1} + 11\omega a_0b_0^2b_{-1} - 76\omega a_0b_0^2b_{-1} - 11\omega a_0b_0^2b_{-1} + 77\omega a_0b_0^2b_{-1} \]
Equating the coefficients of all powers of \( \exp(n\eta) \) to be zero, we obtain

\[
\begin{align*}
C_{-3} &= a_0^2 b_{-1}^3 + 6b a_{-1} b_0 b_{-1}^2 + 3a^2 a_{-1} b_{-1}^2 + 3a^2 a_{-1} b_{-1}^2 + 4v^2 a_{-1} b_0^2 b_{-1}^2 + \beta a_{-1} b_{-1}^4 \\
&+ 4b a_{-1} b_{-1}^3 + 4\alpha \omega^2 a_{-1} b_{-1}^4 + \alpha \omega^2 a_{-1} b_0 b_{-1}^3 + 2a_{-1} a_{-1} b_{-1}^3 + 16\omega^4 a_{-1} b_{-1}^3 - 4\alpha \omega^2 a_{-1} b_{-1}^3 \\
&+ 6a_{-1} a_{-1} b_0 b_{-1}^3 + 4\beta a_{-1} b_0 b_{-1}^3 - \alpha \omega^2 a_{-1} b_0 b_{-1}^3 - 4v^2 a_{-1} b_{-1}^3 - 11\omega^4 a_{-1} b_0 b_{-1}^3 + 16\omega^4 a_{-1} b_{-1}^4 \\
&+ v^2 a_{-1} b_0 b_{-1}^3 + 11\omega^4 a_{-1} b_0 b_{-1}^3 
\end{align*}
\]

\( C_{-4} = -\omega^4 a_{-1} b_0 b_{-1}^3 + \alpha \omega^2 a_{-1} b_{-1}^4 - \alpha \omega^2 a_{-1} b_0 b_{-1}^3 + 3a^2 b_{-1} b_{-1}^3 - v^2 a_{-1} b_0 b_{-1}^3 + \omega^4 a_{-1} b_{-1}^4 \\
+ \omega^4 a_{-1} b_0 b_{-1}^3 + 4\beta a_{-1} b_0 b_{-1}^3 + \beta a_{-1} b_{-1}^3 + 2a_{-1} a_{-1} b_{-1}^3 \]

\( C_{-5} = a_{-1}^2 b_{-1} + \beta a_{-1} b_{-1}^4 \)

Solving the system, Eqs. (2.9), simultaneously, we get the following solutions

\[
\begin{align*}
a_{1} &= 0 , \\a_{0} &= 1 , \\b_{-1} &= 0 , \\b_{-1} &= \frac{b_{0}^2}{4} , \\
\beta &= \frac{-19b_{0}^5 + 16b_{0}^2}{45b_{0}^3} , \\
\omega &= \pm \frac{1}{2} \sqrt{-3^{\beta} 3^{\beta} b_{0} - 4 a \omega^2} , \\
v &= \pm \frac{1}{2} \sqrt{-3^{\beta} 3^{\beta} b_{0} - 4 a \omega^2} 
\end{align*}
\]

were \( b_0 \) and \( \alpha \) are free. Substitute these solutions in equation (2.7) we get the following nonlinear exact solutions of equation (2.1)

\[
y(x,t) = \frac{1}{e^{\frac{1}{2} \sqrt{3^{\beta} 3^{\beta} b_{0} - 4 a \omega^2} x + \frac{1}{2} \sqrt{3^{\beta} 3^{\beta} b_{0} - 4 a \omega^2} t} + b_0 + b_{-1} e^{\frac{1}{2} \sqrt{3^{\beta} 3^{\beta} b_{0} - 4 a \omega^2} x + \frac{1}{2} \sqrt{3^{\beta} 3^{\beta} b_{0} - 4 a \omega^2} t}} 
\]

The graph of the above solutions is described in figure 1.
Fig.1 Describe the solution of Eq. (2.1) with $b_0 = 0.5, \alpha = 2$

For the cases with $p = c = 2, q = d = 1$ and $p = c = 2, q = d = 2$ we not found nonlinear solution of equation (2.1).

3. Solution of the boundary value problem

In this section we used the Exp-function method to find the exact solution of the BVP,

$$u^{(4)} + \alpha u'' + \beta u + u^3 = 0,$$

$$u(0) = A, \quad u(1) = B, \quad u'(0) = C, \quad u'(1) = D.$$

were $A,B,C$ and $D$ are arbitrary constants and $\alpha, \beta$ are the parameters of the problem. This type of problem has been studied by [2]. According to the Exp-function method, we assume that the solution of problem (3.1) can be expressed in the form
\[ u(x) = \frac{\sum_{n=-c}^{d} a_n \exp(nx)}{\sum_{m=-p}^{q} b_m \exp(mx)} \] … (3.2)

By using the same manner of the Exp-function method used in the previous section we balance the terms \( u^{(4)} \) and \( u^3 \) to obtain in the case of \( p = c = 1, \ q = d = 1 \) the following result,

\[ u(x) = \frac{a_1 \exp(x) + a_0 + a_{-1} \exp(-x)}{\exp(x) + b_0 + b_{-1} \exp(-x)} \] … (3.3)

Substituting equation (3.3) into equation (3.1) and by the help of software Mathematica 6.0, we have,

\[ \frac{1}{K} \left[ C_5 e^{x_5} + C_4 e^{x_4} + C_3 e^{x_3} + C_2 e^{x_2} + C_1 e^{x_1} + C_0 + C_{-1} e^{-x_1} + C_{-2} e^{-x_2} + C_{-3} e^{-x_3} + C_{-4} e^{-x_4} + C_{-5} e^{-x_5} \right] = 0 \]

where \( K = \left( e^{x_5} + b_0 + b_{-1} e^{-x_5} \right)^5 \)

\[ C_5 = \beta a_1 + a_1^3 \]

\[ C_4 = -a_1 b_0 + 3a_1^2 a_0 + \beta a_0 + a_0 + \alpha a_1 b_0 - \alpha a_1 b_0 + 2a_1 b_0 \]

\[ C_3 = 3a_1 a_0^2 + 11a_1 b_0 - 11a_1 b_0 + 4\beta a_1 b_1 - 16a_1 - \beta a_1 - \alpha a_1 b_0 + \alpha a_1 b_0 + 4\alpha a_1 - 4\alpha a_1 b_1 \]

\[ C_2 = -\alpha a_1 b_0 + a_0 - a_1 b_0 + 6a_1 a_0 b_0 + 4\beta a_0 b_1 - 76a_0 b_1 + 6a_1 a_1 b_1 + 6\alpha a_1 b_1 + 2a_1 b_0 b_1 + \alpha a_1 b_3 - 7\alpha a_1 b_0 b_1 + 77a_1 b_0 b_1 + 4\beta a_1 b_0 - 4\alpha a_0 b_1 + 6a_1 a_0 a_1 + 11\alpha a_1 b_0 + 12\beta a_1 b_0 b_1 - 11\alpha a_1 b_0 + 11a_1 b_0^2 + 4\beta a_1 b_0 + 6\alpha a_0 b_0 + 3a_1 a_0 b_0^2 \]

\[ C_1 = a_1^2 b_1 + 3a_1 a_1 b_0 + 3a_1 a_1 + 11\alpha a_1 b_0 + 4\beta a_1 b_1 + 3a_1 a_0 b_0 + a_1 b_0^2 + \alpha a_1 b_0 - 176a_1 b_1 + 176a_1 b_1 + 6a_1 a_1 b_1 + 3a_1 a_1 b_0 + 47a_1 b_1 b_0 + 12\beta a_1 b_1 b_0 + \alpha a_1 b_0 + \alpha a_1 b_3 + 6a_1 a_1 b_1 + 12\beta a_1 b_1 b_1 + 6a_1 a_0 b_1 + 6\beta a_1 b_1 + 4\alpha a_1 b_1 - 58a_1 b_1 b_1 + 4\beta a_{11} b_0 + 2a_1 b_0 + 12a_1 a_0 a_1 b_0 - 4\alpha a_1 b_1 + 2a_1 a_1 b_0^2 - 13\alpha a_1 b_1 b_0 \]
Exp-function method for solving nonlinear beam equation

\[ C_0 = \beta a_0 b_0^4 + 6\beta a_0 b_0^2 + 6a_1 b_0^5 + 230a_0 b_0^2 + 6a_1 b_0^5 + 5a_3 b_0^2 + 3a_4 b_0^2 + 5\alpha a_0 b_0^3 b_0 + 5a_0 b_0^4 b_0 - 10a_1 a_2 b_0^4 - 12a_1 a_0 b_0^3 b_1 - 115a_0 b_0^3 b_1 + 5\alpha a_1 b_0^3 b_1 + 4\beta a_0 b_0^3 b_1 \\
+ 12\beta a_1 b_0^3 b_1 - 10\alpha a_1 b_0^3 b_1 + 12\beta a_0 b_0^3 b_1 + 2a_1^2 b_0^3 b_1 + 5\alpha a_0 b_0^3 b_1 + 6a_2 a_0 b_0^3 b_1 - 115a_1 b_0^3 b_1 + 6a_0 a_2 b_0^3 b_1 + 6a_1 a_2 b_0^3 b_1 + 4\beta a_2 b_0^3 b_1 + 12\beta a_0 b_0^3 b_1 + 5\alpha a_0 b_0^3 b_1 + 3a_1^2 b_1^3 \\
+ 2a_4 b_0^3 b_1 + 11\alpha a_0 b_0^3 b_1 + 2a_2 a_0 b_0^3 b_1 + 6a_2 a_4 b_0^3 b_1 - 4\alpha a_4 b_0^3 b_1 + 6a_0 a_4 b_0^3 b_1 + 176a_1 b_0^3 b_1 + 4\alpha a_2 b_0^3 b_1 \\
\]

\[ C_+ = a_0^3 + a_1 b_0^4 + \beta a_1 b_0 + 3a_0 a_2 b_0^3 + 6a_1 a_3 b_0^2 + 176a_1 b_0^3 b_1 + 3a_2 a_1 b_0^4 - a_0 b_1^3 b_0 + a_1 b_0^4 b_0 + 2a_1 b_0^4 b_0 + 12\beta a_0 b_0^3 b_1 + 4\beta a_4 b_0^3 b_1 + 11a_1 b_0^3 b_1 + 3a_1 a_2 b_0^2 b_1 + 4\beta a_0 b_0^3 b_1 + 11\alpha a_0 b_0^3 b_1 \\
+ 4\beta a_1 b_0^3 b_1 - 58a_1 b_0^3 b_1 + 11a_1^2 b_0^3 b_1 + 3a_2 a_1 b_0^4 b_1 + 4\beta a_0 b_0^3 b_1 + 6a_2 a_0 b_0^3 b_1 + 6a_1 a_0 a_1 b_0^2 b_1 + 11\alpha a_0 b_0^3 b_1 \\
+ 12\beta a_1 b_0^3 b_1 + 6a_1 a_0 b_0^3 b_1 + 2a_1^3 b_0^3 b_1 + 76a_1 a_0 b_0^3 b_1 + 4\beta a_0 b_0^3 b_1 + 2a_3 b_0^2 + 77a_1 a_0 b_0^3 b_1 \\
\]

\[ C_{-2} = a_1^3 b_0^2 + \alpha a_1 b_0 b_1 + 6a_1 a_3 b_0^3 - a_1 b_0^4 - a_0 a_1 b_0^3 b_0 - 11a_0 b_0^3 b_0 - 11a_1 b_0^3 b_0 \\
+ 6a_0 a_2 b_0^3 + 4\beta a_4 b_0^3 b_1 + 4\beta a_0 b_0^3 b_1 + 6a_0 a_0 a_1 b_0^2 b_1 + 6a_1 a_0 a_1 b_0^2 b_1 + 11\alpha a_0 b_0^3 b_1 \\
+ 12\beta a_1 b_0^3 b_1 + 6a_1 a_0 b_0^3 b_1 + 76a_0 a_2 b_0^3 b_0 - 76a_0 a_2 b_0^3 b_0 + 4\beta a_0 b_0^3 b_1 + 2a_1^3 b_0^3 b_1 \\
+ 3a_1 a_2 b_0^3 b_0 - 76a_0 a_2 b_0^3 b_0 \\
\]

\[ C_{-3} = a_1^3 b_0^2 + 2a_1 b_0^4 + \alpha a_0 b_0^3 b_0 + \beta a_1 b_0^4 + 3a_1 a_3 b_0^3 - a_0 a_1 b_0^3 b_0 + 11a_0 b_0^3 b_0 - 11a_1 b_0^3 b_0 \\
+ 4\beta a_0 b_0^3 b_0 + 3a_0 a_1 b_0^3 b_0 - 6a_1 b_0^3 b_0 + 6a_0 a_0 a_1 b_0^2 b_0 + 4\alpha a_4 b_0^3 b_1 + 16a_1 b_0^3 b_1 \\
\]

\[ C_{-4} = a_0 b_0^4 - \alpha a_1 b_0 b_1 + a_0 b_0^4 + \beta a_1 b_0 + a_0 b_0^4 - a_1 b_0 b_1 + 4\beta a_0 b_0^3 b_1 + 3a_0 a_2 b_0^3 + 2a_1 b_0^3 b_1 \\
\]

\[ C_{-5} = a_1^3 b_0^3 + \beta a_1 b_0^4 \\
\]

Equating the coefficients of all powers of \( \exp(nx) \) to be zero, we obtain

\[ \left[ C_{-0}, C_0 = 0, C_1 = 0, C_2 = 0, C_3 = 0, C_0 = 0, C_{-1} = 0, C_{-2} = 0, C_{-3} = 0, C_{-4} = 0, C_{-5} = 0 \right] \quad (3.4) \]

Solving the system, equations (3.4), simultaneously, we get the following solutions

\[ a_0 = \frac{1}{4} \left( -7a_1 b_0 - 2a_1^2 b_0 \pm \sqrt{(7a_1 b_0 + 2a_1^2 b_0)^2 - 8(32b_0 - 8a_1^2 b_0)} \right), \]

\[ a_{-1} = a_1 b_0, \quad \alpha = -1 - 2a_1^2, \quad \beta = -a_1^2. \]

Substitute these solutions in equation (3.3) we get the following nonlinear exact solutions of the problem (3.1)
\[ u(x) = \frac{a_i e^x + \frac{1}{4} (-7 a_i b_0 - 2 a_i^3 b_0 \pm \sqrt{(7 a_i b_0 + 2 a_i^3 b_0)^2 - 8(32 b_1 - 8 a_i^2 b_1)}) + a_i e^{-x})}{e^x + b_0 + b_1 e^{-x}}. \]

With the following boundary conditions

\[ u(0) = u(1) = 2.735, \quad u'(0) = 0.465, \quad u'(1) = -0.465 \]

the graph of the above solution is given in figure 2.

Fig.2 Describe the solution of Eq. (3.1) with

\[ a_i = 2, \quad b_0 = -1, \quad b_1 = 2.718 \]

References


Received: April, 2011