Surrender in Single and Double Decrement

Markov Chain Life Insurance Models

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Abstract

The actuarial pricing of mortality insurance contracts including the withdrawal cause of decrement is revisited. Based on the discrete-time Markov chain model of life insurance, we decompose the stochastic underwriting gain in investment, insurance and expense risk components. In this framework, a premium tariff is called actuarial fair provided the expected value of the gross insurance risk component, defined as the sum of its insurance and expense risk components, vanishes at each discrete time of valuation. We show that the actuarial fair pricing conditions can be achieved in two different ways according to whether the original tariff uses a single or double decrement mortality table. The method allows to recover on the one side the withdrawal benefits in [24] and on the other side it provides a new discrete time justification of the Zillmer reserve as withdrawal benefit.

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1 Introduction

Most life insurance contracts offer surrender options for the policyholder, where usually a surrender charge applies in case of withdrawal (e.g. [6], Section 15.2). Two different approaches for the valuation of surrender rights are distinguished (e.g. [3]). First, surrender can be considered as an exogenously determined event and modeled using statistical double decrement mortality tables.
Second, it may be assumed that surrender options are rationally exercised by policyholders. Then, in a risk-neutral framework, fair market prices of such options can be obtained as American put option prices (e.g. [1], [3], [9], [20], [25]). Unfortunately, only very few publications have so far analyzed the effects of exogenously given surrender decisions in the first approach (e.g. [15], [19], [27]). Moreover, classical deterministic pricing of mortality products including the withdrawal cause of decrement has been seldom studied (e.g. [13]). The present contribution offers a new approach to the actuarial pricing of mortality products including withdrawal as exogenous cause of decrement. A related recent paper applying another method is [24].

The present contribution is organized as follows. The framework for a discrete time dynamic analysis is the Markov chain model of life insurance, which is briefly recalled in Section 2. Then, using the basic characteristics of a life insurance contract, we obtain in Section 3, Theorem 3.1, the decomposition of the random underwriting gain into three main sources of profit and loss, called investment risk, insurance risk and expense risk. In view of the current development of internal models for Solvency II, this main result is of independent interest. This decomposition is the basic tool required in Section 4 to price mortality contracts taking into account an additional withdrawal cause of decrement. Within our dynamic context, the premium tariff of a life insurance mortality contract is called actuarial fair provided the expected value of the gross insurance risk component, defined as the sum of its insurance and expense risk components, vanishes at each discrete time of valuation. The actuarial fair pricing conditions can be achieved in two different ways. It is possible to calculate traditional premiums using the single decrement mortality table and adjust the surrender value for the withdrawal cause of decrement. Alternatively, it is possible to adjust the actuarial premiums and reserves for the withdrawal cause of decrement using the dependent probabilities of death and set the surrender value equal to the adjusted Zillmer reserve. The results in both cases depend in a great part on the non-negative difference in mortality probabilities of the single and double decrement models. A numerical example shows that the Zillmer reserves in these two cases are almost equal with a relative difference of less than 0.1% from the second year on. This justifies the traditional actuarial tariffing rule to set the surrender value equal to the Zillmer reserve irrespectively of the fact that withdrawal is taken into account or not. This practice has been justified previously and rigorously in a continuous time setting. The obtained approximation reconciles somewhat discrete time and continuous time models.

2 The Markov chain approach to life insurance

Benefit payments in life insurance are due if the policyholder dies, get disabled or exercises a so-called implicit or embedded option (e.g. surrender option, free policy option, early retirement option, lump sum option, etc.). This happens if he
changes its state from “alive” to “dead” or “alive” to “withdrawn”, etc. It is of basic importance to model the process, which describes the state of a policyholder and its random evolution over time. We restrict our attention to the discrete time setting with a finite time index set $T = \{0,1,\ldots,n\}$. The state space $S$ is the finite set of states a policyholder can be during the lifetime of a contract.

We assume that the discrete time stochastic process $(X_t)_{t \in T}$ with values in $S$ that describes the state of an individual policyholder over time is a Markov chain. The event ${X_t = s}$ means that the contract at time $t$ is in state $s$. In a Markov chain the state in the next step only depends on the previous one and not on the states before (loss of memory property). The main features of a Markov chain are summarized as follows.

**Representation Theorem 2.1.** Let $(X_t)_{t \in T}$ be a stochastic process with values in $S$ such that $P(X_0 = s) = 0$ or 1 for all $s \in S$. Then $(X_t)_{t \in T}$ is a Markov chain if and only if for all $t_1 < t_2 < \ldots < t_n \in T$ and all $s_1, s_2, \ldots, s_n \in S$ one has

$$P(X_{t_j} = s_j, j = 1, \ldots, n) = P(X_{t_1} = s_1) \prod_{j=1}^{n-1} p_{s_j, s_{j+1}}(t_j, t_{j+1}).$$

The joint distributions of Markov chains allow a representation based on the one-step transition probability matrix $p(t) = (p_{ij}(t))_{i,j \in S}$, $t \in T$, which is defined by

$$p_{ij}(t) := p_{ij}(t,t+1) = P(X_{t+1} = j \mid X_t = i).$$

The Markov chain model is widespread in life insurance mathematics and goes back at least to [26], [17], [10], [22], [2], and [11]. Several monographs are based on the Markov model and include [18], [21], [30].

**Example 2.1.** In a single decrement model a policyholder aged $x$ at time $t = 0$ is allowed to change state according to the following state space diagram

**single decrement mortality product with two states**

\[ \begin{align*}
\text{alive} & \quad \overset{q_{x+t}}{\longrightarrow} \quad \text{dead} \\
\text{a} & \quad \rightarrow d
\end{align*} \]

The possible state change occur with the one-year probability of death $q_{x+t}$. One has
Example 2.2. In a double decrement model a policyholder aged \( x \) at time \( t = 0 \) changes state according to the diagram:

**Double decrement mortality product with three states**

The possible state changes occur with the following probabilities:

- \( q_{x+t}^{d} \): one-year dependent probability of death
- \( q_{x+t}^{w} \): one-year dependent probability of withdrawal

The one-step transition probabilities of this Markov chain are given by

\[
p(t) = \begin{pmatrix}
p_{aw}(t) & p_{aw}(t) & p_{ad}(t) \\
p_{wd}(t) & p_{wd}(t) & p_{wd}(t) \\
p_{dw}(t) & p_{dw}(t) & p_{dd}(t)
\end{pmatrix} = \begin{pmatrix}
p_{x+t}^{(r)} = 1 - q_{x+t}^{d} - q_{x+t}^{w} & q_{x+t}^{w} & q_{x+t}^{d} \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]  

(2.3)

3. Random underwriting gain of a life insurance contract

We consider a life insurance contract with level premium payments subject to the mortality and withdrawal causes of decrement over the time horizon \([0,n]\). At the time points \( t \in \{1,2,\ldots,n\} \) the contract offers to a policyholder the following benefits:

- \( D_{t} \): death benefit paid end of period in case the policyholder dies in time period \((t-1,t]\)
- \( E_{n} \): survival benefit paid in case the policyholder survives the whole period at time \( n \)
Markov chain life insurance models

$SV_t$: surrender value or withdrawal benefit paid end of period in case the policyholder lapses in time period $[t-1, t)$

To be able to describe the random underwriting gain within the classical model of life insurance ([28], [6], [7], [8], [29], [14]), we consider the following deterministic basic actuarial quantities:

- $\pi_t^R$: risk premium for the time period $[t-1, t)$ due at time $t - 1$
- $\pi_t^S$: saving premium for the time period $[t-1, t)$ due at time $t - 1$
- $\pi_t^N = \pi_t^R + \pi_t^S$: net premium for the time period $[t-1, t)$ at time $t - 1$
- $\pi_t^{E,R}$: expense risk premium for the time period $[t-1, t)$ due at time $t - 1$
- $\pi_t^{E,S}$: expense saving premium for the time period $[t-1, t)$ due at time $t - 1$
- $\pi_t^E = \pi_t^{E,R} + \pi_t^{E,S}$: expense premium for the time period $[t-1, t)$ at time $t - 1$
- $\pi = \pi_t^N + \pi_t^E$: gross premium for the time period $[t-1, t)$ at time $t - 1$
- $V_t^N$: the net actuarial reserve required at time $t$ such that $V_t^N = E_t$
- $V_t^E$: the expense reserve required at time $t$ such that $V_t^E = 0$
- $V_t = V_t^N + V_t^E$: the gross actuarial reserve required at time $t$
- $C_t$: the operating cost charge for the time period $[t-1, t)$ at time $t$

The precise model description and mathematical definitions of these quantities are found in the given references. For example, the net actuarial reserve represents the actuarial present value of future net cash-flows (difference between insurance benefits and net premiums). The idea of including the expense reserve into the gross actuarial reserve is due to Zillmer (1831-1893) [31] (e.g. [7], p. 103). The traditional pricing of such a contract does not take into account the withdrawal risk and is based solely on a single decrement model with death as only cause of decrement. In this setting, actuarial pricing is based on a single decrement mortality table with entries

$q_x$: probability a life aged $x$ will die within one year
$p_x = 1 - q_x$: probability a life aged $x$ will survive to age $x + 1$

and some fixed technical interest rate $i$. The technical discount factor is denoted by $v = (1 + i)^{-1}$. The above actuarial quantities satisfy the following relationships:

$$\pi_t^R = v \cdot q_{s+t-1} \cdot (D_t - V_t^N), \quad t = 1, \ldots, n - 1, \quad \pi_t^R = 0 \quad (3.1)$$

$$V_t^N = (1 - v \pi_t^S) \cdot (1 + i) \quad (3.2)$$
The derivation of (3.1) and (3.2) is well-known and found in any classical text on life insurance. The formulas (3.3) and (3.4) are derived in [14], p. 184.

**Example 3.1.** The operating cost charges of a classical endowment contract such that \( D_1 = D_2 = \ldots = D_n = E_n = 1 \), and with an acquisition cost rate \( \alpha \), are given by

\[
C_t = \left( \pi^E_t - \alpha \cdot \pi^N_t \right) \cdot (1 + i) - \alpha \cdot i, \quad t = 1, \ldots, n. \tag{3.5}
\]

Indeed, inserting the expression \( V^E_t = \alpha \cdot (1 - V^N_t) \) (e.g. [7], p. 102) for the expense reserve into the formula (3.4) for the operating cost charge, one gets with the expressions (3.1) and (3.3) for the expense risk premiums and the risk premiums:

\[
C_t = \left( \pi^E_t - \alpha \cdot \pi^N_t \right) \cdot (1 + i) + \alpha \cdot \pi^N_t \\
= \left( \alpha \cdot i + \alpha \cdot (1 + i) - \alpha \cdot \pi^N_t \right) \\
= \left( \alpha \cdot i + \alpha \cdot (1 + i) - \alpha \cdot q_{v+1} \cdot (1 - V^N_t) \right) \\
= \left( \alpha \cdot i + \alpha \cdot (1 + i) + \alpha \cdot \pi^E_t \cdot (1 + i) + \pi^R \cdot (1 + i) \right) \\
= \left( \pi^E_t - \alpha \cdot \pi^N_t \right) \cdot (1 + i) - \alpha \cdot i.
\]

The random underwriting gain \( G(t) \) of an individual contract at the discrete time points \( t \in \{1, 2, \ldots, n\} \) is split into three components according to its main sources of profit and loss, namely an investment risk component \( G^{\text{inv}}(t) \), an insurance risk component \( G^{\text{ins}}(t) \) and an expense risk component \( G^{\text{exp}}(t) \) such that

\[
G(t) = G^{\text{inv}}(t) + G^{\text{ins}}(t) + G^{\text{exp}}(t). \tag{3.6}
\]

To present formulas for these components, we consider below several time and state dependent random quantities, which are part of the balance sheet of a contract at any time \( t \). Random premium and reserve components are obtained from the corresponding deterministic actuarial quantities very simply. For this, one observes that premium components generate income, resp. reserve components are obligations, only in case a policyholder is in the alive status at the beginning, resp. end of a time period \( [t-1, t] \). Similarly, expenses are only allocated to policyholders in the alive status at the beginning of a time period. On the other side, in case of death, survival or surrender, insurance benefits are due...
and generate random insurance claims. Using the indicator function \( I(s) \) the relevant quantities are given by

\[
\begin{align*}
P^R(t) &= I(X_{t-1} = a) \cdot \pi_t^R \quad : \text{random risk premium over } (t-1, t] \text{ at time } t-1 \\
P^S(t) &= I(X_{t-1} = a) \cdot \pi_t^S \quad : \text{random saving premium over } (t-1, t] \text{ at } t-1 \\
P^N(t) &= P^R(t) + P^S(t) \quad : \text{random net premium over } (t-1, t] \text{ at } t-1 \\
P^{E,R}(t) &= I(X_{t-1} = a) \cdot \pi_{t}^{E,R} \quad : \text{random expense risk premium} \\
P^{E,S}(t) &= I(X_{t-1} = a) \cdot \pi_{t}^{E,S} \quad : \text{random expense saving premium} \\
P^E(t) &= P^{E,R}(t) + P^{E,S}(t) \quad : \text{random gross premium} \\
R^N(t) &= \left[ I(X_t = a) - I(X_n = a) \right] V^N \quad : \text{random net actuarial reserve at time } t \\
R^E(t) &= I(X_t = a) V^E \quad : \text{random expense reserve at time } t \\
R(t) &= R^N(t) + R^E(t) \quad : \text{random gross actuarial reserve at time } t \\
S(t) &= I(X_{t-1} = a) \left[ I(X_t = d) \cdot D_t + I(X_t = w) \cdot SV_t + I(X_n = a) \cdot V^N \right] \quad : \text{random claims due to death, withdrawal or survival at } t \\
C(t) &= I(X_{t-1} = a) \cdot C_t^e \quad : \text{random expenses or effective random costs over } (t-1, t] \text{ at time } t, \text{ where } C_t^e \text{ represents the effective operating costs allocated to a policyholder in period } (t-1, t] \text{ provided he is alive at time } t \\
i \quad : \text{the technical interest rate} \\
I_t \quad : \text{the random rate of return in time period } (t-1, t] \text{ observed at time } t \\
A(t) \quad : \text{invested random assets of a contract at time } t
\end{align*}
\]

We show the following main result, which is of independent interest.

**Theorem 3.1. (Underwriting gain decomposition)** The random underwriting gain of a life insurance contract subject to the mortality and withdrawal causes of decrement can be decomposed into three risk components as follows:

\[
\begin{align*}
G^{inv}(t) &= A(t-1) + P(t) \cdot I_t - \left[ R(t-1) + P(t) \right] \cdot i \\
G^{ini}(t) &= I(X_{t-1} = a) \cdot \left\{ \pi_t^R \cdot (1+i) - I(X_t = d) \cdot \left[ D_t - V^N \right] \right\} \\
G^{exp}(t) &= I(X_{t-1} = a) \cdot \left\{ \pi_t^{E,R} \cdot (1+i) + C_t - C_t^e \right\} \\
& \quad \quad \quad + \left[ I(X_t = d) + I(X_t = w) \right] V^E \\
\end{align*}
\]  

**Proof.** From basic accounting (e.g. [6], chap. 14, Table 14.6), it is well-known that the change in asset values satisfies the following two alternate expressions:
change in asset values = underwriting gain + change in actuarial reserves
= investment income + premiums – insurance claims – expenses

Therefore one must have the identity

\[ G(t) + R(t) - R(t-1) = [A(t-1) + P(t)] \cdot I_t + P(t) - S(t) - C(t) \]  (3.10)

The formula (3.7) for the investment risk in time period \((t-1,t]\) is obtained as difference between investment income (on assets and premiums) and guaranteed technical interest (on actuarial reserves and premiums). Taking (3.7) into account, the identity (3.10) is fulfilled provided one has

\[ G^{ins}(t) = P^N(t) \cdot (1+i) - S(t) - R^N(t) + R^N(t-1) \cdot (1+i) \]  (3.11)

\[ G^{exp}(t) = P^E(t) \cdot (1+i) - C(t) - R^E(t) + R^E(t-1) \cdot (1+i) \]  (3.12)

Inserting the expressions, which define the random quantities appearing in (3.11) and (3.12), and taking into account the relationships (3.1) to (3.4) between the basic actuarial quantities, one obtains successively

\[ G^{ins}(t) = I(X_{t-1} = a) \cdot \left\{ \left[ \pi^r_t + \pi^s_t \right] \cdot (1+i) - I(X_t = d) \cdot D_t - I(X_t = w) \cdot SV_t \right\} \]  (3.13)

\[ = I(X_{t-1} = a) \cdot \left\{ \pi^r_t \cdot (1+i) - I(X_t = d) \cdot D_t \right\} \]

\[ G^{exp}(t) = I(X_{t-1} = a) \cdot \left\{ \left[ \pi^E_t + \pi^E_t \right] \cdot (1+i) - C_t \right\} \]

\[ = I(X_{t-1} = a) \cdot \left\{ \pi^E_t \cdot (1+i) + [1-I(X_t = a)] \cdot V^E + C_t - C_t \right\} \]  (3.14)

Making use of the indicator function relationship between the alive, dead and withdrawal status

\[ 1 - I(X_t = a) = I(X_t = d) + I(X_t = w), \]  (3.15)

one obtains immediately the expressions (3.8) and (3.9), which shows the result.

\[ \Box \]

4. Actuarial fair pricing including withdrawal decrement

For the actuarial pricing including expenses, we consider the gross insurance risk component defined by \( G^{ins,g}(t) = G^{ins}(t) + G^{exp}(t) \). By Theorem 3.1 it can be rewritten as
As a first important remark, we observe that the gross insurance risk component is independent of the withdrawal cause of decrement if, and only if, the surrender value is set equal to the gross actuarial reserve, that is

\[ SV_t = V_t - V_t^N + V_t^E, \quad t \in \{1, \ldots, n\}. \]  

(4.2)

Net of the expense risk component, this result provides a justification in a discrete time setting of the traditional actuarial practice, which consists to put aside the net actuarial reserve as a withdrawal benefit by neglecting the effect of withdrawals on actuarial premiums. In fact, in a continuous time setting such a result has been established as a mathematical result (e.g. [5], [6], Section 15.1, [12], Section 6, [23] and [24], Theorem 2.1). In [23] it is also shown that in a discrete time setting the withdrawal cause of decrement has a non-trivial effect on actuarial premiums and reserves when working in a double decrement environment. This effect can be described easily using the expected value of the gross insurance risk component

\[ E[G^{ins,g}(t)] = \sum_{t=1}^{n} p_{x+t-1}^{(e)} \left[ \pi_t R \cdot (1 + i) - q_{x+t-1}^d \cdot \left( D_t - V_t^N \right) + \pi_t E R \cdot (1 + i) + q_{x+t-1}^d \cdot V_t^E - q_{x+t-1}^w \cdot \left( SV_t - V_t^N - V_t^E \right) + C_t - C_t^e \right] \]  

(4.3)

Suppose that the premium tariff of a life insurance mortality product is actuarial fair provided the expected value (4.3) vanishes at each discrete time \( t \in \{1, \ldots, n\} \).

In our dynamic model, the actuarial fair pricing conditions can be achieved in two different ways. It is possible to use traditional premiums using the single decrement mortality table and adjust the surrender value for the withdrawal cause of decrement. Alternatively, it is possible to adjust the actuarial premiums and reserves for the withdrawal cause of decrement using the dependent probabilities of death \( q_{x+t-1}^d, \ t \in \{1, \ldots, n\} \), and set the surrender value equal to the adjusted Zillmer reserve similarly to (4.2). In both cases, the obtained results depend in a great part on the non-negative difference in mortality probabilities of the single and double decrement models, that is on quantities of the type (e.g. [6], chap. 9)

\[ \Delta q_{x+t} = q_{x+t} - q_{x+t}^d = \int_0^t \left( p_{x+t} - p_{x+t}^{(e)} \right) d \mu_{x+t}^d > 0, \]  

(4.4)

and are summarized as follows. To distinguish actuarial pricing in the double decrement model from the one in the single decrement model, we add the superscript \( w \) when withdrawal is added as cause of decrement.
**Theorem 4.1.** (Actuarial fair pricing including withdrawal decrement) A life insurance mortality product in a double decrement model with death and withdrawal as causes of decrement is actuarial fair, that is \( E[G^{m,d}(i)] = 0 \) for all \( t \in \{1, \ldots, n\} \), provided one of the following cases is fulfilled:

**Case 1:** tariff with single decrement mortality table

\[
SV_t^w = V + \frac{\Delta q_{x+t-1}}{q_{x+t-1}^w} \cdot (D_t - V), \quad C_t^e = C_t, \quad t \in \{1, \ldots, n\} \tag{4.5}
\]

**Case 2:** tariff with double decrement mortality table

\[
SV_t^w = V^w = V^{w,N} + V^{w,E}, \quad C_t^{w,e} = C_t^w, \quad t \in \{1, \ldots, n\}, \tag{4.6}
\]

where all actuarial quantities are calculated using the double decrement life table.

**Proof.** Apply equation (4.3) to the single and double decrement models. ◊

**Remark 4.1.** The surrender values in (4.5) are equal to the equitable withdrawal benefits (16) of Theorem 3.2 in [24]. We note that Case 2 is a new result with practical relevance. Often different patterns of surrender values are used in the insurance industry. For example, the Switzerland based Patria life insurance company offers surrender values from the third year on according to the formula \( SV_t^w = \max(\frac{1}{2}, V^{w,N}, V^w) \) (e.g. [16], p.16) and in most cases one has in fact \( SV_t^w = V^w \). The next numerical example argues in favor of this tariffing rule.

**Example 4.1.** We illustrate the numerical impact of Theorem 4.1 with a 10-year endowment contract for a life aged \( x = 40 \) with insurance benefits \( D_1 = D_2 = \ldots = D_{10} = E_{10} = 1000 \). For comparison purposes we use (up to the expense charge parameters) the same assumptions as in [23], [24]. The technical interest rate is chosen at \( i = 6\% \), the acquisition cost rate at \( \alpha = 4\% \) and the operating cost rate (in percentage of the gross premium) at \( \beta = 5\% \). The single and double decrement life tables are derived from the illustrative life table in [6], p. 560. The notations for the probabilities in Table 4.2 are taken from [6], Chap. 9. The dependent probabilities of death and withdrawal are calculated from the independent probabilities with the formulas (e.g. [6], Section 9.6)

\[
q_{x+t-1}^d = q_{x+t-1}^{(d)} \cdot \left(1 - \frac{1}{2} q_{x+t-1}^{(w)}\right), \quad q_{x+t-1}^w = q_{x+t-1}^{(d)} - q_{x+t-1}^{(w)}, \quad q_{x+t-1}^{(r)} = 1 - (1 - q_{x+t-1}^{(d)})(1 - q_{x+t-1}^{(w)}).
\]
The premiums and technical values for the single and double decrement tariff as in Case 1 and Case 2 of Theorem 4.1 are compared in the next illustrative tables.

Table 4.1: single decrement life table

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<tr>
<th>year</th>
<th>probability of death $q_{x+t}^{(d)}$</th>
<th>probability of survival $t^{-1}p_x$</th>
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<tr>
<td>1</td>
<td>0.0027812</td>
<td>1.00000</td>
</tr>
<tr>
<td>2</td>
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<td>0.99722</td>
</tr>
<tr>
<td>3</td>
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<td>0.99425</td>
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Table 4.2: double decrement life table

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<th>$q_{x+t}^{(w)}$</th>
<th>$q_{x+t}^{(r)}$</th>
<th>$q_{x+t}^{(d)}$</th>
<th>$q_{x+t}^{(w)}$</th>
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Table 4.3: comparison of premiums

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<td>73.248</td>
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Table 4.4: comparison of premium components

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<td>( \pi^S_t )</td>
<td>( \pi^D_t )</td>
<td>( \pi^{E,S}_t )</td>
<td>( \pi^{E,R}_t )</td>
<td>( C_t )</td>
<td>( \pi^{w,S}_t )</td>
<td>( \pi^{w,D}_t )</td>
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One observes that the premiums and reserves in the single and double decrement tariff differ only slightly. The minimum reduction of premium in the double decrement tariff results in a slightly more important decrease in surrender values provided an adjustment of the Zillmer reserve for withdrawal is made in the single decrement tariff. It is remarkable that the Zillmer reserves are almost equal with a relative difference of less than 0.1% from the second year on. This justifies the traditional tariffing rule to set the surrender value equal to the Zillmer reserve irrespectively of the fact that withdrawal is taken into account or not. This

Table 4.5: comparison of reserves and surrender values

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<th></th>
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<td>surrender value</td>
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</table>

One observes that the premiums and reserves in the single and double decrement tariff differ only slightly. The minimum reduction of premium in the double decrement tariff results in a slightly more important decrease in surrender values provided an adjustment of the Zillmer reserve for withdrawal is made in the single decrement tariff. It is remarkable that the Zillmer reserves are almost equal with a relative difference of less than 0.1% from the second year on. This justifies the traditional tariffing rule to set the surrender value equal to the Zillmer reserve irrespectively of the fact that withdrawal is taken into account or not. This
justification is rigorous in a continuous time setting as already stated at the beginning of Section 4. Note that our net premium in the double decrement tariff differ slightly from the one in [23] who apply a different actuarial pricing method.

References


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