On Certain Generalizations of Fuzzy Boundary

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Abstract

Two generalized forms of fuzzy boundaries namely, fuzzy w-semi-boundary and fuzzy c-semi-boundary have been presented in this paper. Several important set theoretic identities for the existing notion of fuzzy semi-boundary as well as for the newly introduced notions have been established. Counter examples have been provided to support the inequalities.

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1 Introduction

Boundary generally marks the division of two contiguous properties. In topology, boundary of a set is defined as the set of points that belongs to both closure of the set and closure of the complement of the set. In many real-life situations, boundary of a spatial object is not well defined due to the inherent fuzziness. The concept of fuzzy boundary naturally came as a sequel. Topologically, fuzzy boundary was defined by Warren [10] in 1977. Later, Pu and Liu [9], gave another definition of fuzzy boundary based on the intersection of closure of the set and closure of the complement of the set. Later, Cuchillo-Ibanez and Tarres [5], provided a new definition of boundary. A comparative analysis of the three fuzzy boundaries were presented in our earlier work [7, 6]. To generalize the concept of fuzzy boundary, Ahmed and Athar [1], proposed concept of fuzzy semi-boundary generalizing the notion of Pu and Liu. In [7] we introduced w-semi-boundary and c-semi-boundary as two generalized forms of fuzzy boundaries. In this paper, we establish some important identities for
these three generalized boundaries, compare them and provide counter examples to support the inequalities, whenever they exist.

2 Preliminaries

A fuzzy set in $X$ is a function from $X$ into the closed unit interval $[0, 1]$. The basic operations are taken as usual.

Definition 2.1. \cite{12} Fuzzy difference between two sets $A$ and $B$ respectively is given by $(A - B)(x) = (A \cap B^c)(x) = \min \{A(x), B^c(x)\}$, for all $x \in X$.

Definition 2.2. If $A$ and $B$ are two fuzzy sets then the simple difference between $A$ and $B$ is defined as $(A - B)(x) = \{A(x) - B(x) : x \in X\}$.

Definition 2.3. \cite{4} A fuzzy topology is a family $T$ of fuzzy sets in $X$ which satisfies the following conditions

i) $0_X, 1_X \in T$

ii) If $A, B \in T$ then $A \cap B \in T$

iii) If $\{A_i : i \in J\} \subset T$, where $J$ is an index set, then $\bigcup_{i \in J} A_i \in T$.

Here, $0_X$ and $1_X$ indicates the empty and the whole set respectively.

The pair $(X, T)$ is called a fuzzy topological space (fts). The members in $T$ are the open sets and their complements are the closed sets.

Definition 2.4. \cite{9} Let $A$ be a fuzzy set in $(X, T)$ then the union of all the open sets contained in $A$ is called the interior of $A$, denoted by $A^\circ$.

Definition 2.5. \cite{9} The intersection of all the closed sets containing $A$ is called the closure of $A$, denoted by $\overline{A}$.

Definition 2.6. \cite{3} A fuzzy set $A$ in $(X, T)$ is called

(i) fuzzy semi-open if there exists a fuzzy open set $B$ such that $B \subseteq A \subseteq \overline{B}$

(ii) fuzzy semi-closed if there exist a fuzzy open set $C$ such that $C^\circ \subseteq A \subseteq C$.

Every fuzzy open (closed) set is a fuzzy semi-open (respectively, semi-closed) set but the converse is not true.

Definition 2.7. \cite{11} Fuzzy semi-closure and fuzzy semi-interior of a fuzzy set $A$ in $X$, are respectively defined by

$A = \bigcap \{B : A \subseteq B, B \text{ is fuzzy semi-closed}\}$

$A_\circ = \bigcup \{B : B \subseteq A, B \text{ is fuzzy semi-open}\}$

Obviously, $A \supset A_\circ \supset \overline{A}$.
Theorem 2.8. [3] Let \((X, T)\) be a fuzzy topological space and \(A\) be a fuzzy set in \(X\) then following are equivalent:
(a) \(A\) is a fuzzy semi-closed set.
(b) \(A^c\) is a fuzzy semi-open set.
(c) \((A^c)^c \leq A\).
(d) \((A^c)^c \geq A^c\).

Obviously, any union (intersection) of fuzzy semi-open (resp. fuzzy semi-closed) sets is a fuzzy semi-open (resp. semi-closed) set.

Theorem 2.9. [1] For fuzzy sets \(A\) and \(B\) in \(X\),
(i) \((A \cup B)_\circ \geq A_\circ \cup B_\circ\) and \((A \cap B)_\circ = A_\circ \cap B_\circ\).
(ii) \(A \cup B = A \cup B\) and \(A \cap B \leq A \cap B\).
(iii) \((A)_\circ = A\) and \((A^c)_\circ = (A^c)\).
(iv) \(A \leq B \Rightarrow A \leq B\) and \(A_\circ \leq B_\circ\).
(vi) If \(A\) is fuzzy semi-closed (resp. fuzzy semi-open) \(\Leftrightarrow A = A\) (resp. \(A_\circ = A\)).

Theorem 2.10. [8] For any fuzzy set \(A\), \(A_\circ = 1_X - 1_X - A\).

Note: If \(A\) is a fuzzy set in \(X\) and \(Y\) is a subspace of \(X\), then \(A|_Y \geq A|_Y\) and \((A|_Y)_\circ \geq (A_\circ)|_Y\).

Definition 2.11. [11] A function \(f : X \rightarrow Y\) is said to be fuzzy irresolute, if \(f^{-1}(A)\) is fuzzy semi-open set of \(X\), for each semi-open set \(A\) in \(Y\).

Let \(A\) be a fuzzy set in a fuzzy topological space \((X, T)\). The three different fuzzy boundaries of \(A\) are defined as follows:

Definition 2.12. [10, Warren] Fuzzy boundary of \(A\) is the infimum of all closed fuzzy sets \(D\) in \(X\) with the property \(D(x) \geq \overline{A(x)}\) for all \(x \in X\) for which \((\overline{A} \land \overline{A^c})(x) > 0\) or \(A_\circ(x) \neq 1\).

Definition 2.13. [9, Pu and Liu] Fuzzy boundary of \(A\) is defined as \(\overline{A} \cap \overline{A^c}\).

Definition 2.14. [5, Cuchillo-Ibanez and Tarres] Fuzzy boundary of \(A\) is the infimum of all closed fuzzy sets \(D\) in \(X\) with the property \(D(x) \geq A(x)\) for all \(x \in X\) for which \((\overline{A} - A_\circ)(x) \geq 0\).

For convenience we shall denote these boundaries by \(\partial_1 A\), \(\partial_2 A\) and \(\partial_3 A\) respectively.
2.1 Generalization of fuzzy boundaries

In order to generalize the concept of fuzzy boundary, Ahmed and Athar proposed the definition of semi-boundary generalizing the notion of Pu and Liu.

Definition 2.15. [1]: Semiboundary of a fuzzy set $A$ in a fts $X$ is defined as $s\partial_2 A = A \cap A^c$. We shall denote it by $s\partial_2 A$.

We now introduce two new definitions of generalized fuzzy boundary:

Definition 2.16. The fuzzy w-semi-boundary of a fuzzy set $A$ in a fts $X$ denoted by $s\partial_1 A$ is defined as the infimum of all fuzzy semi-closed sets $D$ in $X$ with the property $D(x) \geq A(x)$ for all $x \in X$ for which $(A \cap A^c)(x) > 0$.

Definition 2.17. The fuzzy c-semi-boundary of fuzzy set $A$ in a fts $(X, T)$ is defined as the infimum of all semi-closed fuzzy sets $D$ in $X$ with property $D(x) \geq A(x)$ for all $x \in X$ for which $(A - A_\circ)(x) > 0$. We shall represent it by $s\partial_3 A$.

It is obvious that the notion of w-semi-boundary and c-semi-boundary generalizes the notions of fuzzy boundary due to Warren and Cuchillo-Ibanez respectively. Following properties are trivial for all three forms of fuzzy semi-boundaries:

(i) Semi-boundary of a empty space and whole space is empty.
(ii) Semi-boundary of a fuzzy set is always a fuzzy semi closed set.
(iii) Semi-boundary of a fuzzy set is subset of the semi-closure of the set (i.e. $s\partial A \subseteq A$).
(iv) If a fuzzy set is semi-closed then semi-boundary of the set is contained in the set. (If $A$ is fuzzy semi-closed than $s\partial A \subseteq A$).

Further, the following properties hold for respective fuzzy semi-boundaries:

(v) If $(A \cap A^c)(x) > 0$ then $s\partial_1 A(x) = A(x)$ and if $(A \cap A^c)(x) = 0$ then $s\partial_1 A(x) = 0$.
(vi) If $(A - A_\circ)(x) > 0$ then $s\partial_3 A(x) = A(x)$ and if $(A - A_\circ)(x) > 0$ then $s\partial_3 A(x) = 0$.

2.1.1 Interrelationship among generalized boundaries

In general, these three fuzzy semi-boundaries satisfy the following interrelationships.

(i) $A \geq s\partial_1 A \geq s\partial_3 A$ and (ii) $s\partial_1 A \geq s\partial_2 A$

i.e. $s\partial_1 A$ contains the other two semi-boundaries, which in turn are contained in the semi-closure of the set.

Remark 2.18. Fuzzy semi-boundary and fuzzy c-semi-boundary are independent of each other.
Example 2.19. Let $X = \{a, b\}$ be a set with the fuzzy topology $\tau = \{0_X, \{a, b\}, \{a, b, b\}, \{a, b, b, b\}\}$. Then, $s\partial_1 A = \{a, b\}$, $s\partial_2 A = \{a, b, b\}$, $s\partial_3 A = 0_X$ and $s\partial B = \{a, b, b, b\}$.

Hence $s\partial_3 A \not\subseteq s\partial_2 A$ and $s\partial_2 B \not\subseteq s\partial_3 B$.

3 Comparative analysis of properties

Here, we examine various set theoretic properties of the fuzzy semi-boundaries.

Theorem 3.1. (i) $\overline{A} = A \cup s\partial_1 A = A_0 \cup s\partial_1 A$ (i=1,3).

(ii) $\overline{A} \geq A \cup s\partial_2 A$ and $\overline{A} \geq A_0 \cup s\partial_2 A$.

Proof. (i) If $(A \cap A^c)(x) > 0$ or $(A)(x) = 0$ then $s\partial_1 A(x) = A(x)$

If $(\overline{A} \cap \overline{A^c})(x) = 0$ or $(\overline{A})(x) > 0$ then $(\overline{A^c})(x) = 0$

Therefore, $(A^c)(x) = 0$ and hence $A(x) = 1$. This gives $A_0(x) = 1 = A(x)$.

Thus, $A_0 \cup s\partial_1 A = A$. As $A_0 \leq A \leq \overline{A}$, we have $A = A_0 \cup s\partial_1 A$.

Next, since $s\partial_3 A \leq \overline{A}$ and $A_0 \leq \overline{A}$ so, $A_0 \cup s\partial_3 A \leq \overline{A}$.

Now, whenever $(\overline{A} - A_0)(x) > 0$, we have $\overline{A}(x) = s\partial_3 A(x)$

If $(\overline{A} - A_0)(x) = 0$ then $\overline{A}(x) = A_0(x)$ and thus, $\overline{A} = A_0 \cup s\partial_3 A \leq A \cup s\partial_3 A \leq \overline{A}$.

(ii) We have $A_0 \cup s\partial_2 A = A_0 \cup (\overline{A} \cap A^c) = (A_0 \cup \overline{A}) \cap A^c = \overline{A} \cap A^c \leq \overline{A}$

Similarly, it can be shown that $A \cup s\partial_2 A \leq \overline{A}$.

In [2], it was shown that if $A$ is fuzzy semi-open, then $s\partial_2 A \leq A^c$. However similar result for other two forms of semi-boundaries does not hold.

Example 3.2. In Example 2.19, let $A = \{a, b\}$ which is fuzzy semi-open. Then, $s\partial_1 A = \{a, b\} \not\subseteq A^c$, $s\partial_3 A = \{a, b\} \not\subseteq A^c$.

If $A$ is a fuzzy semi-closed set then $s\partial_2 A \leq A$. We further prove the following:

Theorem 3.3. $A$ be a fuzzy semi closed set iff $s\partial_1 A \leq A$, (i=1,3).

Proof. (i) We have $s\partial_1 A(x) \leq A(x)$, i.e. $s\partial_1 A \leq A$.

Next, let $s\partial_1 A \leq A$, then $A = A \cup s\partial_1 A$, as $s\partial_1 A \leq A$.

So, $\overline{A} = A$ \Rightarrow $A$ is fuzzy semi closed.

(ii) $s\partial_2 A \leq A$ is obvious.

Next if $s\partial_3 A \leq A$, then, $A = A \cup s\partial_3 A = A$. Hence $A$ is fuzzy semi closed set.

Theorem 3.4. If $(A \cap A^c) > 0_X$ then (i) $s\partial_1 A \geq A$ and (ii) $s\partial_2 A \leq A$.

Proof. Proof is trivial.
In [2], it was shown that \( s\partial_2(s\partial_2 A) \leq s\partial_2 A \) and \( s\partial_2(s\partial_2(s\partial_2 A)) \leq s\partial_2(s\partial_2 A) \). We now prove the following:

**Theorem 3.5.** For any fuzzy set \( A \) in a fuzzy topological space

(a) \( s\partial_i(s\partial_i A) \leq s\partial_i A \) \( (i=1,3) \)

(b) \( s\partial_i(s\partial_i(s\partial_i A)) \leq s\partial_i(s\partial_i A) \) \( (i=1,3) \).

**Proof.** (a) Since \( s\partial_i A \) \( (i=1,3) \) is fuzzy semi-closed, the result follows.

(b) We take \( s\partial_i A = B \), which is fuzzy semi closed. Therefore, \( s\partial_i(s\partial_i(s\partial_i A)) = (s\partial_i(s\partial_i B)) \leq s\partial_i B = s\partial_i(s\partial_i A) \).

Similarly, \( s\partial_3(s\partial_3(s\partial_3 A)) \leq s\partial_3(s\partial_3 A) \). \( \Box \)

**Theorem 3.6.** \( s\partial_i A_o \leq s\partial_i A \), \( (i=1,2,3) \).

**Proof.** (i) Let \( d = A_o \cap 1_X - A_o \). It follows that \( d \leq A \cap 1_X - A \). So when \( d(x) > 0 \), then \( s\partial_1 \overline{A}(x) = \overline{A}(x) \geq A_o(x) \).

Also, as \( s\partial_1 A \) is fuzzy semi-closed and \( s\partial_1 A_o \leq A_o \), we have \( s\partial_1 A_o \leq s\partial_1 A \).

(ii) Here, \( s\partial_2 A_o = A_o \cap (A^c) = A_o \cap (A^c) = A_o \cap A^c \leq A \cap A^c = s\partial_2 A \).

(iii) Let \( \sigma \) be a semi-closed fuzzy set such that \( \sigma(x) \geq \overline{A}(x) \) for each \( x \) satisfying \( (A - A_o)(x) > 0 \). Then for each point \( x_o \) with \( A_o(x_o) \geq A_o(x_o) > 0 \), we have \( 0 < A_o(x_o) - A_o(x_o) \leq A_o(x_o) - A_o(x_o) \).

It now follows that \( \sigma(x_o) \geq \overline{A}(x_o) \geq A_o(x_o) \) and \( s\partial_3 A_o \leq s\partial_3 A \). \( \Box \)

**Theorem 3.7.** \( s\partial_i A \leq s\partial_i A \), \( (i=1,2,3) \).

**Proof.** (i) Since \( A \cap 1_X - A \leq A \cap 1_X - A \). So, as above \( s\partial_1 A = s\partial_1 A \).

(ii) \( A \leq A \Rightarrow 1_X - A \geq 1_X - A \Rightarrow (A)^c \geq (A)^c \Rightarrow (A)^c \geq (A)^c \Rightarrow A \cap A^c \leq s\partial_2 A \).

(iii) Let \( \sigma \) be a fuzzy semi-closed set such that \( \sigma(x) \geq \overline{A}(x) \) for each \( x \) satisfying \( (A - A_o)(x) > 0 \).

For \( x_1 \) such that \( A(x_1) - A_o(x_1) > 0 \), \( 0 < A(x_1) - A(x_1) \leq A(x_1) - A_o(x_1) \).

Hence, \( \sigma(x_1) \geq \overline{A}(x_1) \) and \( s\partial_i A \leq s\partial_3 A \). \( \Box \)

**Theorem 3.8.** \( s\partial_i A_o \cup s\partial_i A \leq s\partial_i A \), \( (i=1,2,3) \).

**Proof.** We know that \( s\partial_i A_o \leq s\partial_i A \) and \( s\partial_i A \leq s\partial_i A \) \( (i = 1, 2, 3) \).

Hence, \( s\partial_i A_o \cup s\partial_i A \leq s\partial_i A \), \( (i = 1, 2, 3) \). \( \Box \)

In [2], it was shown that \( s\partial_2 A \leq \partial_2 A \). We now prove the following:

**Theorem 3.9.** \( s\partial_i A \leq \partial_i A \), \( (i=1,3) \)

**Proof.** (i) Let \( d = A \cap 1_X - A \leq A \cap 1_X - A \). So, when \( d(x) > 0 \), \( A(x) \leq \overline{A}(x) = \partial_1 A \).

Since \( \partial_1 A \) is fuzzy semi closed, so \( \partial_1 A \leq \partial_1 A \).

(ii) Let \( \sigma \) be fuzzy semi closed set such that \( \sigma(x) \geq \overline{A}(x) \) for each \( x \in X \) satisfying \( (A - A_o)(x) > 0 \). If \( x \in X \) satisfies \( (A - A_o)(x) > 0 \) then \( 0 < A_o(x_1) - A_o(x_1) \leq A(x_1) - A^c(x_1) \leq \overline{A}(x_1) - A^c(x_1) \).

So, \( \sigma(x) \geq \overline{A}(x_1) \geq \overline{A}(x_1) \). Hence \( s\partial_3 A \leq \partial_3 A \). \( \Box \)
In [2], it was shown that \( s\partial_2 A \leq \partial_2 A \). We shall now show the following identities:

**Theorem 3.10.** \( s\partial_i A \leq \partial_i A \) (i=1,3).

*Proof.* Here, \( s\partial_i A \leq \partial_i A \) and \( \partial_i A \) is fuzzy semi closed. So, \( \overline{s\partial_i A} = s\partial_i A \) and hence, \( s\partial_i A \leq \partial_i A \). \( \square \)

**Theorem 3.11.** \( A \cup B \cup s\partial_i (A \cup B) \leq A \cup B \cup s\partial_i A \cup s\partial_i B \) (i=1,3).

*Proof.* (i) \( A \cup B \cup s\partial_1 (A \cup B) = \overline{A \cup B} \leq A \cup B = A \cup s\partial_1 A \cup B \cup s\partial_1 B = A \cup B \cup s\partial_1 A \cup s\partial_1 B \).

(ii) \( A \cup B \cup s\partial_3 (A \cup B) = \overline{A \cup B} \leq A \cup B = A \cup s\partial_3 A \cup B \cup s\partial_3 B = A \cup B \cup s\partial_3 A \cup s\partial_3 B \). \( \square \)

**Remark 3.12.** Though \( s\partial_2 A = s\partial_2 A^c \) as shown in [2], the result does not hold for the other two forms of semi-boundaries.

**Example 3.13.** Take \( A = \{a_4, b_7\} \) in Example 2.19, then \( s\partial_1 A = \{a_5, b_8\} \), \( s\partial_3 A = \{a_5, b_8\} \), \( s\partial_1 A^c = \{a_6, b_3\} \) and \( s\partial_3 A^c = \{a_6, b_3\} \).

Thus \( s\partial_1 A \neq s\partial_1 A^c \) and \( s\partial_3 A \neq s\partial_3 A^c \).

The result \( s\partial_2 A = A - A_o \) (which holds in classical case) does not hold for any fuzzy semi-boundaries \( s\partial_i A \), \( i = 1, 2, 3 \). However, this result is true for \( i = 2 \) if fuzzy difference is considered [2]. The same, however, is not true for the other two forms of semi-boundaries even with fuzzy difference.

**Example 3.14.** In Example 2.19, take \( A = \{a_4, b_7\} \) then, \( A_o = \{a_5, b_8\} \), \( A_o = \{a_1, b_1\} \) and \( A - A_o = \{a_4, b_7\} \).

Thus, \( s\partial_1 A \neq A - A_o \), \( s\partial_2 A = \{a_5, b_8\} \), \( s\partial_3 A = \{a_5, b_8\} \), \( A \neq A - A_o \) and \( s\partial_3 A \neq A - A_o \).

**Theorem 3.15.** \( s\partial_i A \geq \overline{A - A_o} \) (i=1,2,3).

*Proof.* (i) If \( (A \cap A^c)(x) > 0 \) then \( s\partial_1 A = A \) and if for all \( x \in X \), \( (A \cap A^c)(x) = 0 \) then \( s\partial_1 A = 0_X \), since \( A \geq A - A_o \). Hence, \( s\partial_1 A \geq A - A_o \).

(ii) We know that \( s\partial_2 A = A \cap A^c \geq A - A_o \).

(iii) If \( (A - A_o)(x) > 0 \) then \( s\partial_3 A = A \geq A - A_o \).

On the other hand if \( (A - A_o)(x) = 0 \Rightarrow A(x) - A_o(x) = 0 \Rightarrow A(x) = A_o(x) \).

So, \( s\partial_3 A = 0_X \) then \( s\partial_3 A \geq A - A_o = 0_X \). \( \square \)

In [2], it was shown that \( (s\partial_2 A)^c = A_o \cup (A^c)_o \) but the result does not hold in case of other two definitions:

**Example 3.16.** In Example 2.19, let \( A = \{a_5, b_9\} \).

Then, \( s\partial_1 A = \{a_1, b_1\} \), \( s\partial_3 A = \{a_1, b_1\} \), \( (s\partial_2 A)^c = \{a_0, b_0\} \), \( A_o \cup (A^c)_o = \{a_5, b_8\} \).

Thus, \( (s\partial_1 A)^c \neq A_o \cup (A^c)_o \), \( i = 1, 3 \).
The classical result \( A_0 \leq A - s\partial A \) does not hold in the fuzzy setting, but the same is shown by Ahmed\cite{2} for Pu-Liu semi-boundary using fuzzy difference.

**Example 3.17.** Choose \( A = \{a_7, b_8\} \) in Example 2.19. Then, \( s\partial_1 A = \{a_7, b_8\} \), \( s\partial_2 A = \{a_4, b_2\} \), \( s\partial_3 A = \{a_7, b_8\} \) and \( A - s\partial_1 A = 0_X \), \( A - s\partial_2 A = \{a_3, b_6\} \), \( A - s\partial_3 A = 0_X \). Thus, \( A_0 \not\subseteq A - s\partial_i A \) for \( i=1,2,3 \).

Let \( A \leq B \) and \( B \in FSC(X) \). Then as shown in \[2\], \( s\partial_2 A \leq B \) where \( FSC(X) \) denotes the class of fuzzy semi-closed sets in \( X \). We now prove the following:

**Theorem 3.18.** Let \( A \leq B \) and \( B \in FSC(X) \). Then \( s\partial_i A \leq B \), \( (i=1,3) \), where \( FSC(X) \) denotes the class of fuzzy semi-closed sets in \( X \).

**Proof.** As \( s\partial_i A \leq A \) and \( A \leq B \). Therefore, \( s\partial_i A \leq A \leq B = B \) for \( (i=1,3) \).

In \[2\], it was shown that if \( f : X \rightarrow Y \) be a fuzzy irresolute function, then \( s\partial f^{-1}(A) \leq f^{-1}(s\partial A) \) for any fuzzy set \( A \) in \( Y \). We now have the following result:

**Theorem 3.19.** Let \( f : X \rightarrow Y \) be a fuzzy irresolute function. Then for \( (i=1,3) \) \( s\partial_i f^{-1}(A) \leq f^{-1}(s\partial_i A) \), for each fuzzy set \( A \) in \( Y \).

**Proof.** Let \( A \) be a fuzzy semi closed set in \( Y \). Then \( s\partial_i A \) is fuzzy semi closed set in \( Y \). Since \( f \) is fuzzy irresolute so, \( f^{-1}(s\partial_i A) \) is fuzzy semi closed in \( X \). So, \( s\partial_i f^{-1}(A) \leq f^{-1}(s\partial_i A) = f^{-1}(s\partial_i A) \).

The case for \( i=3 \) is similar.

**Theorem 3.20.** If \( A \leq B \) then \( s\partial_i A \leq B \cup s\partial_i B \) \( (i=1,3) \).

**Proof.** As \( s\partial_i A \leq A \) and \( B = B \cup s\partial_i B \). Hence, \( s\partial_i A \leq B \cup s\partial_i A \) \( (i=1,3) \).

**Theorem 3.21.** If \( A_o = (\overline{A})_o \) then \( s\partial_i A = s\partial_i A \) \( (i=1,2,3) \).

**Proof.** (i) It suffices to show that \( s\partial_i A \leq s\partial_i A \).

We know that if \( A \leq B \) then \( s\partial_i A \leq B \cup s\partial_i B \). Therefore, \( s\partial_i A \leq A \cup s\partial_i A \). Since \( (\overline{1 - A})(x) > 0 \) then \( s\partial_i A = A \). So, \( s\partial_i A \leq s\partial_i A \).

(ii) Here, \( s\partial_2 A = A \cap A^c = A \cap (A_o)^c = A \cap ((A)_o)^c \)
\( = B \cap (B_o)^c \) (putting \( B = A \))
\( \leq B \cap B^c = s\partial_2 B = s\partial_2 A \). Now, as \( s\partial_2 A \leq s\partial_2 A \), hence \( s\partial_2 A = s\partial_2 A \).

(iii) If \( A_o = (\overline{A})_o \) then \( A(x) - A_o(x) > 0 \) iff \( A(x) - A_o(x) > 0 \).

The result follows.

**Theorem 3.22.** If \( A = B \) and \( A_o = B_o \) then \( s\partial_i A = s\partial_i B \) \( (i=1,2,3) \).
Proof. (i) For \( i = 1, \) we have \( s \partial_1 A \leq A = B. \) If \( (B \cap B^c)(x) > 0 \) then \( s \partial_1 B = B. \)

So, \( s \partial_1 A \leq s \partial_1 B. \) Likewise, \( s \partial_1 B \leq s \partial_1 A. \) Hence the equality.

(ii) For \( i = 2, \) \( s \partial_2 A = A \cap A^c = B \cap (A_0)^c = B \cap (B_0)^c = B \cap B^c = s \partial_2 B. \)

(iii) For \( i = 3, \) if \( (A - A_0)_0 < 0 \) then \( s \partial_3 A = A \) and if \( (B - B_0)_0 < 0 \) then \( s \partial_3 B = B. \) Since \( A = B, \) therefore, \( s \partial_3 A = s \partial_3 B. \)

\[ \square \]

**Theorem 3.23.** (i) If \( A = A_0 \) iff \( s \partial_1 A \leq A \) (i=1,3).

(ii) If \( A = A_0 \) then \( s \partial_2 A \leq A \).

Proof. (i) First let \( A = A_0. \) Then, \( s \partial_1 A \leq A = A_0. \)

Conversely, let \( s \partial_1 A \leq A_0 \) then \( A = A_0 \cup s \partial_1 A \leq A_0 \cup A_0 = A_0. \) Hence \( A = A_0. \)

Now, if \( A = A_0 \) then \( s \partial_3 A \leq A = A_0. \)

Next, let \( s \partial_3 A \leq A_0 \), then \( A = A_0 \cup s \partial_3 A \leq A_0 \cup A_0 = A_0 \leq A. \)

(ii) \( s \partial_2 A = A \cap A^c = A_0 \cap A^c \leq A_0. \)

\[ \square \]

**Theorem 3.24.** (i) If \( s \partial_1 A \cap A = 0_X \) iff \( A \) is open and crisp.

(ii) \( s \partial_1 A \cap A = 0_X \) (i=2,3) if \( A \) is open and crisp.

Proof. (i) If \( s \partial_1 A \cap A = 0_X \) then we shall show that \( A = 1_X - 1_X - A. \) Since \( s \partial_1 A \cap A = 0_X, \) so when \( A(x) = 0 \) then \( 1 - A(x) = 1 \) and when \( A(x) > 0 \) then \( s \partial_1 A = 0 < A(x). \) Therefore, \( (A \cap 1 - A) \) = 0. Thus, \( 1 - A(x) = 0. \)

Hence \( A(x) = 1. \)

Assume now that \( A \) is open and crisp. If \( (A \cap 1 - A) > 0 \) then \( 1 - A(x) = 1 > A(x). \) So, \( s \partial_1 A \leq 1_X - A. \) Hence when \( A = 1_X \) then \( s \partial_1 A = 0_X. \)

(ii) \( A \cap s \partial_2 A = A \cap A \cap A^c = 0 \) (since \( A \) is crisp).

(iii) Given \( A \) is open, so \( 1_X - A \) is a semi-closed set such that for all \( x \in X \) with \( \frac{A(x)}{A_0(x)} > 0, \) we have \( (1 - A)(x) \geq \frac{A(x)}{A_0(x)} \) because when \( \frac{A(x)}{A_0(x)} > 0, A_0(x) \neq 1 \) and as \( A \) is open and crisp, \( A(x) = A_0(x) = 0 \) that is how if \( A(x) \neq 0 \) then \( s \partial_3 A(x) = 0. \)

\[ \square \]

**Theorem 3.25.** \( s \partial_i A = 0_X, \) (i=1,2,3) iff \( A \) is open, closed and crisp.

Proof. Follows directly from Theorems 3.3 and 3.24.

\[ \square \]

**Theorem 3.26.** (i) \( (A \cap A^c)(x) = 0 \) iff \( s \partial_1 A(x) = 0.(i=1,2). \)

(ii) If \( (A \cap A^c)(x) = 0 \) then \( s \partial_3 A(x) = 0. \)

Proof. (i) If \( s \partial_1 A = 0_X, \) we know that \( s \partial_1 A = A \geq A \cap A^c \Rightarrow A \cap A^c = 0_X. \)

Next, let \( (A \cap A^c) = 0_X \) then \( s \partial_1 A = 0_X. \)

(ii) We know that \( s \partial_2 A = A \cap A^c. \) Thus \( s \partial_2 A = 0 \) iff \( A \cap A^c = 0. \)

(iii) We have \( s \partial_3 A = 0_X \) if \( (A - A_0) = 0_X \)

Since \( A - A_0 \leq A \cap A^c, \) so \( A \cap A^c = 0_X. \) Thus, \( s \partial_3 A = 0_X. \)

\[ \square \]

**Theorem 3.27.** Let \( Y \) be a fuzzy subspace of \( X \) and \( A \) a fuzzy set in \( X. \) Then for any fuzzy set \( A, \) \( s \partial_i (A|_Y) \leq (s \partial_i A)|_Y, \) (i=1,2,3).
Proof. (i) If \((A \mid Y) \cap (A \mid Y)^c(x) > 0\) then \(s \partial_1(A \mid Y) = (A \mid Y)\) otherwise \(s \partial_1(A \mid Y) = 0\).

Similarly, \((s \partial_1 A) \mid Y = A \mid Y\), Since \((A \mid Y) \leq A \mid Y\), Hence \(s \partial_1(A \mid Y) \leq (s \partial_1 A) \mid Y\).

(ii) \(s \partial_2(A \mid Y) = A \mid Y \cap (A \mid Y)^c \leq A \mid Y \cap A^c \mid Y = (s \partial_2 A) \mid Y\). Hence the proof.

(iii) We know that \(A \mid Y \geq A \mid Y\) and \((A \mid Y)^c \geq A^c \mid Y\). Let \(\sigma\) be a semi-closed fuzzy set such that \(\sigma(x) \geq A(x)\) for all \(x \in X\) satisfying \(A(x) - A^c(x) > 0\). Now, \(\sigma|_Y\), which is a semi-closed fuzzy set in \(Y\), satisfies \((\sigma|_Y)(y) \geq (A|_Y)(y)\) for all \(y \in Y\) such that \((A|_Y - (A|_Y)^c)(y) > 0\).

In fact, whenever \(0 < (A|_Y - (A|_Y)^c)(y) \leq A(y) - A^c(y)\), we have \(\sigma(y) \geq A(y) \geq (A|_Y)(y)\). Hence, \(s \partial_3(A \mid Y) \leq (s \partial_3 A) \mid Y\).

Remark 3.28. If the intersection of the semi-closure of the set and semi-closure of the complement of the set is empty then value of all the three forms of semi-boundaries are equal.

4 Conclusion

We have established various fundamental identities for the three variations of generalized fuzzy boundary. Several counter-examples have also been provided to support the strictness of inequalities, wherever they exist. Since the generalized forms of fuzzy boundaries carry wider applicability than fuzzy boundaries per se, their comparative analysis is expected to throw light on their context based scope of applicability.

References


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