Generalized Numerical Radius Inequalities for Operator Matrices

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Abstract
We prove a generalized numerical radius inequality for operator matrices, which improves and generalized an earlier inequality due to Bani-Domi and Kittaneh.

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1 Introduction.

Let $B(H)$ denote the $C^*$–algebra of all bounded linear operators on a complex Hilbert space $H$ with inner product $<.,.>$. For $A \in B(H)$, let

$\omega(A) = \sup\{|\langle Ax, x \rangle| : x \in H, \|x\| = 1\}$,

$\|A\| = \sup\{|\|Ax\| : x \in H, \|x\| = 1\}$, where $\|x\|^2 = \langle x, x \rangle$,

$r(A) = \sup\{|\lambda| : \lambda \in \sigma(A)\}$, where $\sigma(A)$ is the spectrum of $A$,

$|A| = (A^*A)^{\frac{1}{2}}$

denote the numerical radius of $A$, the usual operator norm of $A$, the spectral radius of $A$, and the absolute value of $A$. It is well known that $\omega(.)$ is a norm on $B(H)$, which is equivalent to the usual operator norm $\|\.\|$. In fact, for every $A \in B(H)$,

$\frac{1}{2} \|A\| \leq \omega(A) \leq \|A\|$ \hspace{1cm} (1)

Several numerical radius inequalities improving the inequalities in (1) have been recently given in [3], [13], [15], and [16].

Let $H_1, H_2, ... , H_n$ be complex Hilbert spaces. Then every operator $A \in B\left(\bigoplus_{i=1}^{n} H_i\right)$ can be represented as an $n \times n$ operator matrix of the form $A =$
$[A_{ij}]$ with $A_{ij}$ is a bounded linear operator from $H_j$ into $H_i$. Note that for every vector $x = [x_1 \ x_2 \ ... \ x_n]^T \in \left( \bigoplus_{i=1}^n H_i \right)$, $Ax = \left[ \sum_{j=1}^n A_{1j}x_j \ \sum_{j=1}^n A_{2j}x_j \ ... \ \sum_{j=1}^n A_{nj}x_j \right]^T$.

In this paper, we are interested in estimating a general numerical radii of operator matrices. Such estimates have been very useful in obtaining bounds for the zeros of polynomials (see [2, 14]).

Since the numerical radius norm is weakly unitarily invariant, in the sense that $\omega(U^*AU) = \omega(A)$ for all $A \in B(H)$ and for all unitary operators $U \in B(H)$, it follows that this norm satisfies a pinching inequality. More precisely, if $A = [A_{ij}]$ is an operator matrix in $B\left( \bigoplus_{i=1}^n H_i \right)$, then

$$\omega\left( \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix} \right) \leq \omega(A)$$,

that is,

$$\max\{\omega(A_{ii}) : i = 1, 2, ..., n\} \leq \omega(A). \tag{2}$$

Related pinching type inequalities for the general class of weakly unitarily invariant norms have been recently given in [1]. A very useful numerical radius inequality for operator matrices, due to Hou and Du [8], asserts that if $A = [A_{ij}]$ is an operator matrix in $B\left( \bigoplus_{i=1}^n H_i \right)$, and if $\tilde{A} = [\|A_{ij}\|]$ is the $n \times n$ matrix whose $(i, j)$ entry is $\|A_{ij}\|$, then

$$\omega(A) \leq \omega(\tilde{A}) \tag{3}.$$

It should be mentioned here that similar inequalities hold for the usual operator norm and the spectral radius (see [8]).

Recently, Bani-Domi and Kittaneh [2] improves the inequality in (3), which asserts that if $A = [A_{ij}]$ is an operator matrix in $B\left( \bigoplus_{i=1}^n H_i \right)$, then

$$\omega(A) \leq \omega([a_{ij}]), \tag{4}$$

where,

$$a_{ij} = \begin{cases} \frac{1}{2} \left( \|A_{ii}\| + \|A_{ii}\|^{1/2} \right) & \text{if } i = j \\ \|A_{ij}\| & \text{if } i \neq j \end{cases}.$$

In the next section of this paper, we prove several generalized numerical radii inequalities for operator matrices.
2 Main results.

In order to state our results, we shall need the following well-known lemmas.

The first lemma is a generalized mixed Schwarz inequality (see, e.g., [5, pp. 75-76], or [10]).

**Lemma 1** Let $A \in B(H)$. Then for all $x, y \in H$,

$$|\langle Ax, y \rangle|^2 \leq \langle |A|x, x \rangle \langle |A^*|y, y \rangle.$$  

The second lemma contains a special case of a more general norm inequality that is equivalent to some Löwner-Heinz type inequalities (see, e.g., [4] or [11]).

**Lemma 2** Let $A, B \in B(H)$ be positive operators. Then

$$\left\|A^{\frac{1}{2}}B^{\frac{1}{2}} \right\| \leq \|AB\|^{\frac{1}{2}}.$$  

The third lemma is a generalized form of the mixed Schwarz inequality which has been proved by Kittaneh [10].

**Lemma 3** Let $T$ be an operator in $B(H)$ and let $f$ and $g$ be nonnegative functions on $[0, \infty)$ which are continuous and satisfying the relation $f(t)g(t) = t$ for all $t \in [0, \infty)$. Then

$$|\langle Tx, y \rangle| \leq \|f(|T|x)\| \|g(|T^*|y)\| \text{ for all } x, y \in H.$$  

The fourth lemma is very useful in computing the numerical radius for matrices (see [7]).

**Lemma 4** If $A = [a_{ij}] \in M_n(\mathbb{C})$, then

$$\omega(A) \leq \omega(|a_{ij}|) = \frac{1}{2}r(|a_{ij}| + |a_{ji}|), \text{ where } r(B) \text{ is the spectral radius of } B.$$  

The fifth lemma contains a recent norm inequality for sums of positive operators that is sharper than the triangle inequality (see [12]).

**Lemma 5** Let $A, B \in B(H)$ be positive operators. Then

$$\|A + B\| \leq \frac{1}{2} \left( \|A\| + \|B\| + \sqrt{\|A\|^2 + \|B\|^2 + 4\left\|A^{\frac{1}{2}}B^{\frac{1}{2}}\right\|} \right).$$  

The sixth lemma which is based on the Geršgorin theorem for location of eigenvalues (see [6]), enables us to obtain a resonable estimate for the spectral radius.

**Lemma 6** If $B = [b_{ij}] \in M_n(\mathbb{C})$, then

$$r(B) \leq \min \left( \max_{1 \leq i \leq n} \sum_{j=1}^{n} |b_{ij}|, \max_{1 \leq j \leq n} \sum_{i=1}^{n} |b_{ij}| \right).$$  

Our first result is a generalization of the inequality in (4).
Theorem 7 Let \( A = [a_{ij}] \) be an operator matrix in \( B \left( \bigoplus_{i=1}^{n} H_i \right) \), and let \( f \) and \( g \) be nonnegative functions on \([0, \infty)\) which are continuous and satisfying the relation \( f(t)g(t) = t \) for all \( t \in [0, \infty) \). Then

\[
\omega(A) \leq \omega([a_{ij}]), \tag{5}
\]

where,

\[
a_{ij} = \begin{cases} \\
\frac{1}{2} (\| f^2(|A_{ii}|) + g^2(|A^*_{ii}|) \|) & \text{if } i = j \\
\| f(|A_{ij}|) \| \| g(|A^*_{ij}|) \| & \text{if } i \neq j
\end{cases}
\]

Proof. Let \( x = [x_1 \ x_2 \ ... \ x_n]^T \in \left( \bigoplus_{i=1}^{n} H_i \right) \), with \( \| x \| = 1 \). Then we have

\[
|\langle Ax, x \rangle| = \left| \sum_{i,j=1}^{n} \langle A_{ij}x_j, x_i \rangle \right|
\]

\[
\leq \sum_{i,j=1}^{n} |\langle A_{ij}x_j, x_i \rangle|
\]

\[
= \sum_{i=1}^{n} |\langle A_{ii}x_i, x_i \rangle| + \sum_{i \neq j}^{n} |\langle A_{ij}x_j, x_i \rangle|
\]

\[
\leq \sum_{i=1}^{n} \langle f^2(|A_{ii}|) x_i, x_i \rangle^{\frac{1}{2}} \langle g^2(|A^*_{ii}|) x_i, x_i \rangle^{\frac{1}{2}} + \sum_{i \neq j}^{n} \| f(|A_{ij}|) x_j \| \| g(|A^*_{ij}|) x_i \| \quad \text{(by lemma 3)}
\]

\[
\leq \frac{1}{2} \sum_{i=1}^{n} \langle f^2(|A_{ii}|) + g^2(|A^*_{ii}|) \rangle x_i, x_i \rangle + \sum_{i \neq j}^{n} \| f(|A_{ij}|) \| \| g(|A^*_{ij}|) \| \| x_i \| \| x_j \| \quad \text{(by the arithmetic-geometric mean inequality)}
\]

\[
\leq \frac{1}{2} \sum_{i=1}^{n} \| f^2(|A_{ii}|) + g^2(|A^*_{ii}|) \| \| x_i \|^2 + \sum_{i \neq j}^{n} \| f(|A_{ij}|) \| \| g(|A^*_{ij}|) \| \| x_i \| \| x_j \| \n\]

\[
= \langle [a_{ij}] \bar{x}, \bar{x} \rangle, \text{ where } \bar{x} = [||x_1|| \ ||x_2|| \ ... \ ||x_n||]^T.
\]

Now the result follows by taking the supremum over all unit vectors in \( \bigoplus_{i=1}^{n} H_i \).

Inequality (5) includes several numerical radius inequalities for operator matrices. Samples of inequalities are demonstrated in what follows.

For \( f(t) = t^\alpha \) and \( g(t) = t^{1-\alpha}, \alpha \in (0, 1) \), in inequality (5), we get the following inequality that generalizes (4).
Corollary 8 Let $A = [A_{ij}]$ be an operator matrix in $B \left( \bigoplus_{i=1}^{n} H_i \right)$, and $\alpha \in (0, 1)$. Then
\[
\omega(A) \leq \omega([a_{ij}]),
\]
where,
\[
a_{ij} = \begin{cases} 
\frac{1}{2} \left( \| |A_{ii}|^{2\alpha} + |A_{i}^*|^{2(1-\alpha)} \| \right) & \text{if } i = j \\
\| |A_{ij}|^{\alpha} \| \| |A_{ij}|^{(1-\alpha)} \| & \text{if } i \neq j
\end{cases}
\]

Remark. It follows as a special case of the previous corollary that if $\alpha = \frac{1}{2}$ and by Lemmas 2 and 5 we get the inequality (4).

\[\square\]

Now using Lemma 4, one can formulate the inequality (5) in terms of the spectral radius as follows.

Theorem 9 Let $A = [A_{ij}]$ be an operator matrix in $B \left( \bigoplus_{i=1}^{n} H_i \right)$, and let $f$ and $g$ be nonnegative functions on $[0, \infty)$ which are continuous and satisfying the relation $f(t)g(t) = t$ for all $t \in [0, \infty)$. Then
\[
\omega(A) \leq r([b_{ij}]),
\]
where,
\[
b_{ij} = \begin{cases} 
\frac{1}{2} \left( \| f^2(|A_{ii}|) + g^2(|A_{i}^*|) \| \right) & \text{if } i = j \\
\frac{1}{2} \left( \| f(|A_{ij}|) \| \| g(|A_{ij}|) \| + \| f(|A_{ji}|) \| \| g(|A_{ji}|) \| \right) & \text{if } i \neq j
\end{cases}
\]

When $n = 2$, we have the following sharp generalized numerical radius estimate for operator matrices.

Corollary 10 Let $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ be an operator matrix in $B \left( H_1 \bigoplus H_2 \right)$, and let $f$ and $g$ be as in Lemma 3. Then
\[
\omega(A) \leq \frac{1}{2} \left( a + c + \sqrt{(a - c)^2 + 4b^2} \right),
\]
where,
\[
a = \frac{1}{2} \| f^2(|A_{11}|) + g^2(|A_{1}^*|) \|, \quad b = \frac{1}{2} (\| f(|A_{12}|) \| \| g(|A_{12}|) \| + \| f(|A_{21}|) \| \| g(|A_{21}|) \|),
\]
and $c = \frac{1}{2} \| f^2(|A_{22}|) + g^2(|A_{2}^*|) \|$. For $n > 2$, it is difficult to compute the right-hand side of the inequality (7). So, by using Lemma 6, we have the following theorem, which gives an estimate for the right-hand side of the inequality (6).

Theorem 11 Let $A = [A_{ij}]$ be an operator matrix in $B \left( \bigoplus_{i=1}^{n} H_i \right)$, and let $f$ and $g$ be as in Lemma 3. Then
Now the inequality (8) generalizes a related inequality for the numerical radii of matrices due to Johnson [8]. Based on Theorem 11, when $n = 2$ we obtain the following simple estimate for the numerical radius of an operator matrix.

**Corollary 12** Let $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ be an operator matrix in $B\left( H_1 \bigoplus H_2 \right)$, and let $f$ and $g$ be as in Lemma 3. Then

$$\omega(A) \leq \frac{1}{2} \max_{1 \leq i \leq n} \left( \|f^2(|A_{1i}|) + g^2(|A_{1i}^*|)\| + \frac{1}{2} \sum_{i \neq j} \|f(|A_{ij}|)\| \|g(|A_{ij}^*|)\| + \|f(|A_{ii}|)\| \|g(|A_{ii}^*|)\| \right). \tag{8}$$

**References**


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