On a Universal Property of the Equicontinuous Bornology on Modules of Linear Mappings

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Abstract

In this paper we prove that certain bornological modules of continuous linear mappings are isomorphic to bornological projective limits of bornological modules of continuous linear mappings.

Mathematics Subject Classification: 46A17, 46H25

Keywords: modules of continuous linear mappings, module bornologies, module topologies

1. Introduction and preliminaries

In this work we consider modules of continuous linear mappings between topological modules endowed with the (module) equicontinuous bornology and prove that, under certain conditions, such bornological modules of continuous linear mappings are isomorphic to bornological projective limits of bornological modules of continuous linear mappings. As a consequence, an analogous result for the (module) equihypococontinuous bornology is derived. The present paper was motivated by [9], where similar results in the special case of bornological vector spaces and topological vector spaces have been established.
Throughout this paper \((R, \tau_R)\) shall denote a commutative topological ring with a non-zero identity element such that \(0 \in \overline{R^x}\), where \(R^x\) is the multiplicative group of all invertible elements of \(R\), and \(B_R\) shall denote the bornology consisting of all \(\tau_R\)-bounded subsets of \(R\) ([10], §16). All \(R\)-modules under consideration shall be unitary \(R\)-modules. A bornology \(\mathcal{B}\) on an \(R\)-module \(E\) is an \(R\)-module bornology on \(E\), and \((E, \mathcal{B})\) is a bornological \(R\)-module, if the mappings \((x, y) \in (E \times E, \mathcal{B} \times \mathcal{B}) \mapsto x + y \in (E, \mathcal{B})\) and \((\lambda, x) \in (R \times E, \mathcal{B}_R \times \mathcal{B}) \mapsto \lambda x \in (E, \mathcal{B})\) are bounded. A topology \(\tau\) on an \(R\)-module \(E\) is an \(R\)-module topology on \(E\), and \((E, \tau)\) is a topological \(R\)-module, if the mappings \((x, y) \in (E \times E, \tau \times \tau) \mapsto x + y \in (E, \tau)\) and \((\lambda, x) \in (R \times E, \tau_R \times \tau) \mapsto \lambda x \in (E, \tau)\) are continuous. A \(r\)ésumé of the basic constructions of bornological modules [7] and topological modules [3, 10] which shall be needed in the sequel may be found in [5].

**Definition 1.** Given two topological \(R\)-modules \((E, \tau)\) and \((F, \theta)\), \(\mathcal{L}((E, \tau); (F, \theta))\) shall denote the \(R\)-module of all continuous \(R\)-linear mappings from \((E, \tau)\) into \((F, \theta)\) and \(\mathcal{E}_\theta^\#\) the equicontinuous bornology [6, 9] on \(\mathcal{L}((E, \tau); (F, \theta))\).

\(\mathcal{E}_\tau^\#\) is an \(R\)-module bornology. In fact, let \(\mathcal{X}_1, \mathcal{X}_2 \in \mathcal{E}_\tau^\#\) and \(L \in B_R\). Clearly, \(\mathcal{X}_1 + \mathcal{X}_2 \in \mathcal{E}_\tau^\#\). In order to prove that \(L \mathcal{X}_1 \in \mathcal{E}_\tau^\#\), let \(V\) be an arbitrary \(\theta\)-neighborhood of 0 in \(F\). By the continuity of the mapping \((\lambda, y) \in (R \times F, \tau_R \times \theta) \mapsto \lambda y \in (F, \theta)\) at \((0, 0)\), there are a \(\tau_R\)-neighborhood \(A\) of 0 in \(R\) and a \(\theta\)-neighborhood \(V_1\) of 0 in \(F\) such that \(AV_1 \subset V\) and, by the equicontinuity of \(\mathcal{X}_1\), there is a \(\tau\)-neighborhood \(U\) of 0 in \(E\) such that \(\mathcal{X}_1(U) \subset V_1\). Moreover, by the boundedness of \(L\), there is a \(\tau_R\)-neighborhood \(A_1\) of 0 in \(R\) such that \(A_1 L \subset A\). Thus

\[
(L \mathcal{X}_1)(A_1 U) = (A_1 L) \mathcal{X}_1(U) \subset AV_1 \subset V,
\]

\(A_1 U\) being a \(\tau\)-neighborhood of 0 in \(E\) because \(0 \in \overline{R^x}\). Hence \(L \mathcal{X}_1 \in \mathcal{E}_\tau^\#\).

Given two bornological \(R\)-modules \((E, \mathcal{B})\) and \((F, C)\), \(\mathcal{L}_b((E, \mathcal{B}); (F, C))\) shall denote the \(R\)-module of all bounded \(R\)-linear mappings from \((E, \mathcal{B})\) into \((F, C)\) and \(\mathcal{E}_C^\#\) the equibounded bornology [9] on \(\mathcal{L}_b((E, \mathcal{B}); (F, C))\). Clearly, \(\mathcal{E}_C^\#\) is an \(R\)-module bornology. For each topological \(R\)-module \((E, \tau)\), \(\mathcal{B}(\tau)\) shall represent the \(R\)-module bornology consisting of all \(\tau\)-bounded subsets of \(E\) ([10], §15).

**Example 2.** For \((E, \tau)\) and \((F, \theta)\) as in Definition 1, if \(\mathcal{X} \in \mathcal{E}_\tau^\#\), then \(\mathcal{X} \in \mathcal{E}_B^{\#(\theta)}\) by Theorem 25.5 of [10]. On the other hand, if \((E, \tau)\) is bornological and \(\mathcal{X} \in \mathcal{E}_B^{\#(\theta)}\), then \(\mathcal{X} \in \mathcal{E}_\tau^\#\) by the theorem established in [2].

### 2. The main result and applications

Let \(((E_\alpha, \tau_\alpha), u_{\beta\alpha})_{\alpha \in I}\) be an arbitrary inductive system of topological \(R\)-modules, \((E, \tau)\) the topological inductive limit of this system and, for each \(\alpha \in I\), \(E_\alpha\) a bornological \(R\)-module.
Proof. Each $\Phi(\lambda)$ projective limit of this system and, for each arbitrary projective system of topological $u$, because $(\lambda, \gamma) \in I \times J$ with $(\alpha, \lambda) \leq (\beta, \mu)$, let $\Phi^{(\beta, \mu)}_{(\alpha, \lambda)}$ be the $R$-linear mapping from $\mathcal{L}((E, \tau_\alpha); (F, \theta_\lambda))$ into $\mathcal{L}((E, \tau_\lambda); (F, \theta))$ given by $\Phi^{(\beta, \mu)}_{(\alpha, \lambda)}(\varphi) = v_{\lambda \mu} \circ \varphi \circ u_{\beta \alpha}$. Clearly, $\Phi^{(\beta, \mu)}_{(\alpha, \lambda)}$ is $\mathcal{E}_{\tau_\beta}^{\theta_\mu}$-$\mathcal{E}_{\tau_\lambda}^{\theta_\lambda}$-bounded.

\[
\begin{array}{c}
(E_\alpha, \tau_\alpha) \xrightarrow{u_{\beta \alpha}} (E_\beta, \tau_\beta) \\
\phi^{(\beta, \mu)}_{(\alpha, \lambda)}(\varphi) \downarrow \varphi \\
(F_\lambda, \theta_\lambda) \xleftarrow{v_{\lambda \mu}} (F_\mu, \theta_\mu)
\end{array}
\]

Proposition 3. $((\mathcal{L}((E_\alpha, \tau_\alpha); (F_\lambda, \theta_\lambda)), \mathcal{E}_{\tau_\alpha}^{\theta_\lambda}, \Phi^{(\beta, \mu)}_{(\alpha, \lambda)})_{(\alpha, \lambda) \in I \times J}$ is a projective system of bornological $R$-modules.

Proof. Each $\Phi^{(\alpha, \lambda)}_{(\alpha, \lambda)}$ is the identity mapping of $\mathcal{L}((E_\alpha, \tau_\alpha); (F_\lambda, \theta_\lambda))$ because $u_{\alpha \alpha} = 1_{E_\alpha}$ and $v_{\alpha \lambda} = 1_{F_\lambda}$, and $\Phi^{(\alpha, \lambda)}_{(\alpha, \lambda)} \circ \Phi^{(\gamma, \nu)}_{(\alpha, \lambda)} = \Phi^{(\gamma, \nu)}_{(\alpha, \lambda)}$ for $(\alpha, \lambda) \leq (\beta, \mu) \leq (\gamma, \nu)$ because $u_{\gamma \beta} \circ u_{\beta \alpha} = u_{\gamma \alpha}$ and $v_{\lambda \mu} \circ v_{\mu \nu} = v_{\lambda \nu}$.

For each $(\alpha, \lambda) \in I \times J$, let $\Psi_{(\alpha, \lambda)}$ be the $R$-linear mapping from $\mathcal{L}((E, \tau); (F, \theta))$ into $\mathcal{L}((E_\alpha, \tau_\alpha); (F_\lambda, \theta_\lambda))$ given by $\Psi_{(\alpha, \lambda)}(\psi) = v_\lambda \circ \psi \circ u_\alpha$.

\[
\begin{array}{c}
(E_\alpha, \tau_\alpha) \xrightarrow{u_\alpha} (E, \tau) \\
\Psi_{(\alpha, \lambda)}(\psi) \downarrow \psi \\
(F_\lambda, \theta_\lambda) \xleftarrow{v_\lambda} (F, \theta)
\end{array}
\]

It is obvious that $\Psi_{(\alpha, \lambda)}$ is $\mathcal{E}_{\tau}^{\theta}$-$\mathcal{E}_{\tau_\alpha}^{\theta_\lambda}$-bounded, and that the family $\left(\Psi_{(\alpha, \lambda)}(\psi)\right)_{(\alpha, \lambda) \in I \times J}$ belongs to the $R$-module $\varprojlim_{(\alpha, \lambda) \in I \times J} \mathcal{E}_{\tau_\alpha}^{\theta_\lambda}$ for all $\psi \in \mathcal{L}((E, \tau); (F, \theta))$ because

\[
\left(\Phi^{(\beta, \mu)}_{(\alpha, \lambda)} \circ \Psi_{(\beta, \mu)}\right)(\psi) = \Psi_{(\alpha, \lambda)}(\psi)
\]

for $(\alpha, \lambda) \leq (\beta, \mu)$.

Now, we can state our main result:
Theorem 4. Under the conditions above, the $R$-linear mapping

$$\Psi : \psi \in \left(\mathcal{L}(\mathcal{L}(E, \tau); (F, \theta), \mathcal{E}_\tau^\theta)\right)$$

$$\downarrow$$

$$\Psi(\psi) = \left(\Psi_{(\alpha, \lambda)}(\psi)\right)_{(\alpha, \lambda) \in I \times J} \in \left(\lim_{\leftarrow} \mathcal{L}(\mathcal{L}(E_\alpha, \tau_\alpha); (F_\lambda, \theta_\lambda), \mathcal{B})\right)$$

is a bornological $R$-module isomorphism, the codomain of $\Psi$ being the bornological projective limit of the system given in Proposition 3.

Before proving Theorem 4 we shall establish an auxiliary result:

Lemma 5. Let $\left((E_\alpha, \tau_\alpha)\right)_{\alpha \in I}$ be a non-empty family of topological $R$-modules, $E$ an $R$-module and, for each $\alpha \in I$, let $u_\alpha : E_\alpha \to E$ be an $R$-linear mapping. Assume that $E = \left(\bigcup_{\alpha \in I} \text{Im}(u_\alpha)\right)$ and let $\tau$ be the final $R$-module topology for the family $\left((E_\alpha, \tau_\alpha), u_\alpha\right)_{\alpha \in I}$ ([1], Proposition 1.4.2). Then, for each topological $R$-module $(F, \theta)$ and for each set $\mathcal{X}$ of $R$-linear mappings from $E$ into $F$, the following conditions are equivalent:

(a) $\mathcal{X} \in \mathcal{E}_\tau^\theta$;
(b) $\mathcal{X} \circ u_\alpha = \{\psi \circ u_\alpha; \psi \in \mathcal{X}\} \in \mathcal{E}_{\tau_\alpha}^\theta$ for all $\alpha \in I$.

Proof of Lemma 5. Since (a) obviously implies (b), let us verify that (b) implies (a). Indeed, let $(F, \theta)$ and $\mathcal{X}$ be as above and consider the $R$-module

$$\mathcal{Z}(\mathcal{X}; F) = \{f : \mathcal{X} \to F; \ f(\mathcal{X}) \text{ is } \theta\text{-bounded}\}$$

endowed with the topology $\tilde{\theta}$ of uniform convergence. By Lemma 1 of [2], $(\mathcal{Z}(\mathcal{X}; F), \tilde{\theta})$ is a topological $R$-module. Since $\mathcal{X}(x)$ is $\theta$-bounded for all $x \in E$, we can consider the $R$-linear mapping

$$u : x \in (E, \tau) \longmapsto u(x) \in (\mathcal{Z}(\mathcal{X}, F), \tilde{\theta}),$$

where $u(x)(\psi) = \psi(x)$ for $\psi \in \mathcal{X}$, and it is clear that the continuity of $u$ is equivalent to the equicontinuity of $\mathcal{X}$. Since $\tau$ is the final $R$-module topology for the family $\left((E_\alpha, \tau_\alpha), u_\alpha\right)_{\alpha \in I}$, $u$ is continuous if and only if $u \circ u_\alpha$ is continuous for all $\alpha \in I$. But (b) is equivalent to the assertion that $u \circ u_\alpha$ is continuous for all $\alpha \in I$, and therefore $\mathcal{X} \in \mathcal{E}_{\tau_\alpha}^\theta$, proving (a).

Now, let us turn to the

Proof of Theorem 4. Firstly, $\Psi$ is bounded. In fact, let $\mathcal{X} \in \mathcal{E}_{\tau_\alpha}^\theta$. By definition, $\Psi(\mathcal{X})$ is $\mathcal{B}$-bounded if and only if $\Psi_{(\alpha, \lambda)}(\mathcal{X}) \in \mathcal{E}_{\tau_\alpha}^\theta$ for all $(\alpha, \lambda) \in I \times J$. But this occurs since each $\Psi_{(\alpha, \lambda)}$ is $\mathcal{E}_{\tau}^\theta$-$\mathcal{E}_{\tau_\alpha}^\theta$-bounded.
Secondly, let us verify that $\Psi$ is an $R$-module isomorphism. Indeed, let $g = (g_{(\alpha, \lambda)})_{(\alpha, \lambda) \in I \times J} \in \lim \mathcal{L}((E_\alpha, \tau_\alpha); (F_\lambda, \theta_\lambda))$ be arbitrary. By the linear case of Theorem 2.4 of [5], there exists a unique $R$-linear mapping $\psi : E \to F$ such that $v_\alpha \circ \psi \circ u_\alpha = g_{(\alpha, \lambda)}$ for all $(\alpha, \lambda) \in I \times J$. Since $\tau$ is the final $R$-module topology for the family $((E_\alpha, \tau_\alpha), u_\alpha)_{\alpha \in I}$, $\psi \in \mathcal{L}((E, \tau); (F, \theta))$ if and only if $\psi \circ u_\alpha \in \mathcal{L}((E_\alpha, \tau_\alpha); (F, \theta))$ for all $\alpha \in I$. Let $\alpha \in I$ be arbitrary. Since $\theta$ is the initial topology for the family $((F_\lambda, \theta_\lambda), v_\lambda)_{\lambda \in J}$, $\psi \circ u_\alpha \in \mathcal{L}((E_\alpha, \tau_\alpha); (F, \theta))$ if and only if $v_\lambda \circ (\psi \circ u_\alpha) \in \mathcal{L}((E_\alpha, \tau_\alpha); (F, \theta))$ for all $\lambda \in J$. But this occurs if and only if $v_\lambda \circ \psi \circ u_\alpha = g_{(\alpha, \lambda)}$ for all $(\alpha, \lambda) \in I \times J$. Thus $\psi \in \mathcal{L}((E, \tau); (F, \theta))$, which concludes the proof that $\Psi$ is an $R$-module isomorphism.

Finally, let us prove that $\Psi^{-1}$ is bounded. Indeed, let $\mathcal{G}$ be an arbitrary $\mathcal{B}$-bounded set. For each $(\alpha, \lambda) \in I \times J$, let $G_{(\alpha, \lambda)}$ be the $R$-linear mapping $$(g_{(\beta, \mu)})_{(\beta, \mu) \in I \times J} \in \lim \mathcal{L}((E_\beta, \tau_\beta); (F_\mu, \theta_\mu)) \mapsto g_{(\alpha, \lambda)} \in \mathcal{L}((E_\alpha, \tau_\alpha); (F_\lambda, \theta_\lambda)).$$ Then $G_{(\alpha, \lambda)}(\mathcal{G}) \in \mathcal{E}_{\tau_\alpha}^\mathcal{B}$ for all $(\alpha, \lambda) \in I \times J$. By what we have just seen, for each $g = (g_{(\alpha, \lambda)})_{(\alpha, \lambda) \in I \times J} \in \mathcal{G}$ there exists a unique $\psi_g \in \mathcal{L}((E, \tau); (F, \theta))$ such that $\Psi(\psi_g) = g$, that is, such that $v_\alpha \circ \psi_g \circ u_\alpha = g_{(\alpha, \lambda)}$ for all $(\alpha, \lambda) \in I \times J$. We claim that $\mathcal{X} = \Psi^{-1}(\mathcal{G}) = \{\psi_g; g \in \mathcal{G}\} \in \mathcal{E}_{\tau_\alpha}^\mathcal{B}$. In fact, we have that $\mathcal{X} \in \mathcal{E}_{\tau_\alpha}^\mathcal{B}$ if and only if $v_\lambda \circ \mathcal{X} \in \mathcal{E}_{\tau_\alpha}^\mathcal{B}$ for all $\lambda \in J$. Let $\lambda \in J$ be fixed. Since Lemma 5 is applicable in the present case and $G_{(\alpha, \lambda)}(\mathcal{G}) = (v_\lambda \circ \mathcal{X}) \circ u_\alpha \in \mathcal{E}_{\tau_\alpha}^\mathcal{B}$ for all $\alpha \in I$, it follows that $v_\lambda \circ \mathcal{X} \in \mathcal{E}_{\tau_\alpha}^\mathcal{B}$. Therefore $\Psi^{-1}$ is bounded, which concludes the proof of the theorem.

Remark 6. In the particular case where each $(E_\alpha, \tau_\alpha)$ is a bornological topological $R$-module, Theorem 4 follows from the linear case of Theorem 3.6 of [5]. In fact, in this case $(E, \tau)$ is a bornological topological $R$-module by Corollary 1a) of [2]. Moreover, as we have already observed in Example 2, for each bornological topological $R$-module $(G, \xi)$, for each topological $R$-module $(F, \theta)$ and for each set $\mathcal{X}$ of $R$-linear mappings from $G$ into $F$, we have that $\mathcal{X} \in \mathcal{E}_{\xi}^\mathcal{B}$ if and only if $\mathcal{X} \in \mathcal{E}_{\xi}^{\mathcal{B}(\theta)}$.

Corollary 7. Let $((E_\alpha, \tau_\alpha))_{\alpha \in I}$ and $((F_\lambda, \theta_\lambda))_{\lambda \in J}$ be two non-empty families of topological $R$-modules, $(E, \tau)$ the topological direct sum of the family $((E_\alpha, \tau_\alpha))_{\alpha \in I}$ and $(F, \theta)$ the topological product of the family $((F_\lambda, \theta_\lambda))_{\lambda \in J}$. Then the bornological $R$-modules $$(\mathcal{L}((E, \tau); (F, \theta)), \mathcal{E}_{\tau}^\mathcal{B})$$ and $$\left(\prod_{(\alpha, \lambda) \in I \times J} \mathcal{L}((E_\alpha, \tau_\alpha); (F_\lambda, \theta_\lambda)), \prod_{(\alpha, \lambda) \in I \times J} \mathcal{E}_{\tau_\alpha}^\mathcal{B}\right)$$ are.
are isomorphic. In particular, let $X$ be a non-empty set, $(F, \theta)$ an arbitrary topological $R$-module and $(\mathcal{F}(X; F), T_s)$ the $R$-module of all mappings from $X$ into $F$ endowed with the $R$-module topology $T_s$ of simple convergence. Then the bornological $R$-modules

$$\left(\mathcal{L}((E, \tau); (\mathcal{F}(X; F), T_s)), E_{\tau}^s\right) \text{ and } \left(\prod_{\alpha \in I} \mathcal{L}((E_{\alpha}, \tau_{\alpha}); (F, \theta)), \prod_{\alpha \in I} E_{\tau_{\alpha}}^\theta\right)$$

are isomorphic.

**Proof.** Consider $I$ and $J$ endowed with their respective equality relations, and put $u_{\alpha \alpha} = 1_{E_{\alpha}}$ for all $\alpha \in I$ and $v_{\lambda \lambda} = 1_{F_{\lambda}}$ for all $\lambda \in J$. Then $((E_{\alpha}, \tau_{\alpha}), u_{\alpha \alpha})_{\alpha \in I}$ (resp. $((F_{\lambda}, \theta_{\lambda}), v_{\lambda \lambda})_{\lambda \in J}$) is an inductive system of topological $R$-modules (resp. a projective system of topological $R$-modules) whose topological inductive limit (resp. topological projective limit) coincides with $(E, \tau)$ (resp. $(F, \theta)$). Moreover, it is easily seen that the bornological $R$-modules

$$\left(\lim_{\alpha} \mathcal{L}((E_{\alpha}, \tau_{\alpha}); (F_{\lambda}, \theta_{\lambda})), \mathcal{B}\right) \text{ and } \left(\prod_{(\alpha, \lambda) \in I \times J} \mathcal{L}((E_{\alpha}, \tau_{\alpha}); (F_{\lambda}, \theta_{\lambda})), \prod_{(\alpha, \lambda) \in I \times J} E_{\tau_{\alpha}}^\theta\right)$$

coincide. Therefore the result follows immediately from Theorem 4. \hfill \blacksquare

In order to give another application of Theorem 4, let us recall [9] the following

**Definition 8.** Let $(E, \mathcal{B})$ be a bornological $R$-module and let $(F, \theta)$ and $(G, \xi)$ be two topological $R$-modules. An $R$-bilinear mapping $u : E \times F \to G$ is said to be $\mathcal{B}$-hypocontinuous if, for each $B \in \mathcal{B}$, the set $\{u_x; x \in B\}$ of $R$-linear mappings from $F$ into $G$ is equicontinuous, where $u_x (y) = u(x, y)$ for all $y \in F$. We shall denote by $\mathcal{L}_h((E, \mathcal{B}), (F, \theta); (G, \xi))$ the $R$-module of all such $\mathcal{B}$-hypocontinuous mappings. A subset $\mathcal{X}$ of $\mathcal{L}_h((E, \mathcal{B}), (F, \theta); (G, \xi))$ is said to be $\mathcal{B}$-equihypocontinuous if, for each $B \in \mathcal{B}$, the set $\{u_x; x \in B, u \in \mathcal{X}\}$ is equicontinuous.

It is easily seen that the set $E_{\mathcal{B}, \theta}^\xi$ of all $\mathcal{B}$-equihypocontinuous subsets of $\mathcal{L}_h((E, \mathcal{B}), (F, \theta); (G, \xi))$ is a bornology on $\mathcal{L}_h((E, \mathcal{B}), (F, \theta); (G, \xi))$. Moreover, $E_{\mathcal{B}, \theta}^\xi$ is an $R$-module bornology. In fact, let $\mathcal{X}_1, \mathcal{X}_2 \in E_{\mathcal{B}, \theta}^\xi$ and $L \in \mathcal{B}_R$, and let $B \in \mathcal{B}$ and $W$ a $\xi$-neighborhood of 0 in $G$. Let $W_1$ be a $\xi$-neighborhood of 0 in $G$ such that $W_1 + W_1 \subset W$. Then there exists a $\theta$-neighborhood $V$ of 0 in $F$ such that $u_x (V) \subset W_1$ and $v_x (V) \subset W_1$ for all $x \in B, u \in \mathcal{X}_1, v \in \mathcal{X}_2$, and hence $w_x (V) \subset W$ for all $x \in B, w \in \mathcal{X}_1 + \mathcal{X}_2$. Thus $x_1 + x_2 \in E_{\mathcal{B}, \theta}^\xi$. And, since $L B \in \mathcal{B}$, there exists a $\theta$-neighborhood $V$ of 0 in $F$ such that $u_{\lambda x} (V) = (\lambda u)_x (V) \subset W$ for all $\lambda \in L, x \in B, u \in \mathcal{X}_1$. Thus $L \mathcal{X}_1 \in E_{\mathcal{B}, \theta}^\xi$. 

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Example 9. Let \((E, \tau), (F, \theta)\) and \((G, \xi)\) be three topological \(R\)-modules, \((F, \theta)\) being barrelled \([8]\). Let \(\mathfrak{X}\) be a separately equicontinuous set of \(R\)-bilinear mappings from \(E \times F\) into \(G\). Then \(\mathfrak{X}\) is \(\mathcal{B}(\tau)\)-equihypocontinuous.

In fact, let \(B \in \mathcal{B}(\tau)\) be arbitrary and consider the subset \(\mathcal{Y} = \{u_x; \ x \in B, \ u \in \mathfrak{X}\}\) of \(\mathcal{L}((F, \theta); (G, \xi))\). For each \(y \in F\), \(\mathcal{Y}(y) \in \mathcal{B}(\xi)\), because the set \(\{x \in (E, \tau) \mapsto u(x, y) \in (G, \xi) ; \ u \in \mathfrak{X}\}\) is equicontinuous and hence \(\mathcal{Y}(y) \in \mathcal{B}(\xi)\) by Theorem 25.5 of \([10]\). Therefore \(\mathcal{Y}\) is equicontinuous by Theorem 3.1 of \([8]\).

Let \(u \in \mathcal{L}_h((E, \mathcal{B}), (F, \theta); (G, \xi))\) be arbitrary and put \(\Gamma_1(u)(x) = u_x\) for all \(x \in E\) \(\Gamma_1(u)(x) \in \mathcal{L}((F, \theta); (G, \xi))\) because \(\{x\} \in \mathcal{B}\). Then the \(\mathcal{B}\)-hypocontinuity of \(u\) guarantees that \(\Gamma_1(u) \in \mathcal{L}_h((E, \mathcal{B}); \mathcal{L}((F, \theta); (G, \xi)), \mathcal{E}_\xi^\mathcal{B})\).

**Proposition 10.** The \(R\)-linear mapping

\[
\Gamma_1 : u \in \left(\mathcal{L}_h((E, \mathcal{B}), (F, \theta); (G, \xi)), \mathcal{E}_\xi^\mathcal{B}\right) \mapsto \Gamma_1(u) \in \left(\mathcal{L}_h((E, \mathcal{B}); \mathcal{L}((F, \theta); (G, \xi)), \mathcal{E}_\xi^\mathcal{B}\right)
\]

is a bornological \(R\)-module isomorphism.

**Proof.** It is obvious that \(\Gamma_1\) is injective. Let \(\psi \in \mathcal{L}_h((E, \mathcal{B}); \mathcal{L}((F, \theta); (G, \xi)), \mathcal{E}_\xi^\mathcal{B})\) be arbitrary and define \(u : E \times F \to G\) by \(u(x, y) = \psi(x)(y)\). It is clear that \(u \in \mathcal{L}_h((E, \mathcal{B}), (F, \theta); (G, \xi))\) and \(\Gamma_1(u) = \psi\), and hence \(\Gamma_1\) is an \(R\)-module isomorphism. Moreover, for an arbitrary subset \(\mathfrak{X}\) of \(\mathcal{L}_h((E, \mathcal{B}), (F, \theta); (G, \xi))\), we have that \(\mathfrak{X} \in \mathcal{E}_\xi^\mathcal{B}\) if and only if \(\Gamma_1(\mathfrak{X}) \in \mathcal{E}_\xi^\mathcal{B}\) (by definition). This completes the proof. \[\blacksquare\]

Let \(((E_\alpha, \mathcal{B}_\alpha), u_{\beta, \alpha})_{\alpha \in I}\) be an arbitrary inductive system of bornological \(R\)-modules,

\(((F_\lambda, \theta_\lambda), v_{\mu, \lambda})_{\lambda \in J}\) an arbitrary inductive system of topological \(R\)-modules and

\(((G_\rho, \xi_\rho), w_{\rho, \sigma})_{\rho, \sigma \in K}\) an arbitrary projective system of topological \(R\)-modules. For \((\alpha, \lambda, \rho), (\beta, \mu, \sigma) \in I \times J \times K\), with \((\alpha, \lambda, \rho) \leq (\beta, \mu, \sigma)\), consider the \(R\)-linear mapping \(\Phi^{(\beta, \mu, \sigma)}_{(\alpha, \lambda, \rho)}\) from \(\mathcal{L}_h((E_\beta, \mathcal{B}_\beta), (F_\mu, \theta_\mu); (G_\sigma, \xi_\sigma))\) into \(\mathcal{L}_h((E_\alpha, \mathcal{B}_\alpha), (F_\lambda, \theta_\lambda); (G_\rho, \xi_\rho))\) given by

\[
\Phi^{(\beta, \mu, \sigma)}_{(\alpha, \lambda, \rho)}(u) = w_{\rho, \sigma} \circ u \circ (u_{\beta, \alpha} \times v_{\mu, \lambda}),
\]

where \((u_{\beta, \alpha} \times v_{\mu, \lambda})(x_\alpha, y_\lambda) = (u_{\beta, \alpha}(x_\alpha), v_{\mu, \lambda}(y_\lambda))\) for \((x_\alpha, y_\lambda) \in E_\alpha \times F_\lambda\).

Then \(\Phi^{(\beta, \mu, \sigma)}_{(\alpha, \lambda, \rho)}\) is \(\mathcal{E}_{\beta, \lambda, \rho}^\mathcal{E}_{\alpha, \mu, \sigma}\)-bounded. In fact, let \(\mathfrak{X} \in \mathcal{E}_{\beta, \lambda, \rho}^\mathcal{E}_{\alpha, \mu, \sigma}\), \(B_\alpha \in \mathcal{B}_\alpha\) and \(W_{\rho}\) a \(\xi_\rho\)-neighborhood of 0 in \(G_\rho\). Since \(w_{\rho, \sigma} : (G_\rho, \xi_\rho) \to (G_\rho, \xi_\rho)\) is continuous, there is a \(\xi_\sigma\)-neighborhood \(W_\sigma\) of 0 in \(G_\sigma\) such that \(w_{\rho, \sigma}(W_\sigma) \subset W_{\rho}\) and, since \(\mathfrak{X} \in \mathcal{E}_{\beta, \lambda, \rho}^\mathcal{E}_{\alpha, \mu, \sigma}\), there is a \(\theta_\mu\)-neighborhood \(V_\mu\) of 0 in \(F_\mu\) such that \(u_{\beta, \alpha}(x_\alpha)(y_\mu) = u(x_\alpha, y_\mu) \in W_\sigma\) for all \(x_\alpha \in B_\alpha, y_\mu \in V_\mu, u \in \mathfrak{X}\) (recall that \(u_{\beta, \alpha}(B_\alpha) \in \mathcal{B}_\beta\)). On the other hand, since \(v_{\mu, \lambda} : (F_\lambda, \theta_\lambda) \to (F_\mu, \theta_\mu)\) is
continuous, there is a $\theta_\lambda$-neighborhood $V_\lambda$ of 0 in $F_\lambda$ such that $v_{\mu\lambda}(V_\lambda) \subset V_\mu$. Consequently,

$$\begin{align*}
\left[\Phi^{(\beta, \mu, \sigma)}_{(\alpha, \lambda, \rho)}(u)\right](x_\alpha, y_\lambda) &= (w_{\rho\sigma} \circ u \circ (u_{\beta\alpha} \times v_{\mu\lambda}))(x_\alpha, y_\lambda) \\
&= w_{\rho\sigma}(u(u_{\beta\alpha}(x_\alpha), v_{\mu\lambda}(y_\lambda))) \\
&\subset w_{\rho\sigma}(W_\sigma) \subset W_\rho
\end{align*}$$

for all $x_\alpha \in B_\alpha$, $y_\lambda \in V_\lambda$, $u \in \mathcal{X}$. Therefore $\Phi^{(\beta, \mu, \sigma)}_{(\alpha, \lambda, \rho)}(\mathcal{X}) \in \mathcal{E}^{\xi_\rho}_{E_\alpha, \theta_\lambda}$.

**Proposition 11.** \(\left(\mathcal{L}_h((E_\alpha, B_\alpha), (F_\lambda, \theta_\lambda); (G_\rho, \xi_\rho)), \mathcal{E}^{\xi_\rho}_{E_\alpha, \theta_\lambda}\right), \Phi^{(\beta, \mu, \sigma)}_{(\alpha, \lambda, \rho)}\) is a projective system of bornological $R$-modules.

**Proof.** Straightforward. \(\blacksquare\)

For $(\alpha, \lambda, \rho), (\beta, \mu, \sigma) \in I \times J \times K$, with $(\alpha, \lambda, \rho) \leq (\beta, \mu, \sigma)$, consider the $R$-linear mapping $\tilde{\Phi}^{(\beta, \mu, \sigma)}_{(\alpha, \lambda, \rho)}$ from $\mathcal{L}_b((E_{\beta\lambda}, B_\beta); (\mathcal{L}((F_{\mu\lambda}, \theta_\mu); (G_{\rho\lambda}, \xi_\rho)), \mathcal{E}^{\xi_\rho}_{\theta_\mu, \theta_\lambda}))$ into $\mathcal{L}_b((E_\alpha, B_\alpha); (\mathcal{L}((F_\lambda, \theta_\lambda); (G_\rho, \xi_\rho)), \mathcal{E}^{\xi_\rho}_{\theta_\lambda, \theta_\lambda}))$ given by

$$\tilde{\Phi}^{(\beta, \mu, \sigma)}_{(\alpha, \lambda, \rho)}(\psi) = \Phi^{(\mu, \sigma)}_{(\alpha, \lambda, \rho)} \circ \psi \circ u_{\beta\alpha}.$$ 

It is easily verified that $\tilde{\Phi}^{(\beta, \mu, \sigma)}_{(\alpha, \lambda, \rho)}$ is $\mathcal{E}^{\xi_\rho}_{E_\beta, \theta_\beta} - \mathcal{E}^{\xi_\rho}_{E_\alpha, \theta_\alpha}$-bounded. An immediate consequence of Proposition 11 reads:

**Corollary 12.** \(\left(\mathcal{L}_b((E_\alpha, B_\alpha); (\mathcal{L}((F_\lambda, \theta_\lambda); (G_\rho, \xi_\rho)), \mathcal{E}^{\xi_\rho}_{\theta_\lambda, \theta_\lambda}), \mathcal{E}^{\xi_\rho}_{\theta_\lambda, \theta_\lambda}), \tilde{\Phi}^{(\beta, \mu, \sigma)}_{(\alpha, \lambda, \rho)}\right)\) is a projective system of bornological $R$-modules.

Let $(E, \mathcal{B})$ be the bornological inductive limit of the system $((E_\alpha, B_\alpha), u_{\beta\alpha})_{\alpha \in I}$, $(F, \theta)$ the topological inductive limit of the system $((F_\lambda, \theta_\lambda), v_{\mu\lambda})_{\lambda \in J}$ and $(G, \xi)$ the topological projective limit of the system $((G_\rho, \xi_\rho), w_{\rho\sigma})_{\rho \in K}$.

For each $(\alpha, \lambda, \rho) \in I \times J \times K$, let $\Psi_{(\alpha, \lambda, \rho)}$ be the $R$-linear mapping from $\mathcal{L}_h((E, \mathcal{B}), (F, \theta); (G, \xi))$ into $\mathcal{L}_h((E_\alpha, B_\alpha), (F_\lambda, \theta_\lambda); (G_\rho, \xi_\rho))$ given by

$$\Psi_{(\alpha, \lambda, \rho)}(u) = w_{\rho\sigma} \circ u \circ (u_\alpha \times v_\lambda),$$

where $u_\alpha : E_\alpha \to E$, $v_\lambda : F_\lambda \to F$ and $w_\rho : G \to G_\rho$ are the canonical $R$-linear mappings and $(u_\alpha \times v_\lambda)(x_\alpha, y_\lambda) = (u_\alpha(x_\alpha), v_\lambda(y_\lambda))$ for $(x_\alpha, y_\lambda) \in E_\alpha \times F_\lambda$. By arguing as before one sees that $\Psi_{(\alpha, \lambda, \rho)}$ is $\mathcal{E}^{\xi_\rho}_{E_\theta, \theta_\theta} - \mathcal{E}^{\xi_\rho}_{E_\theta, \theta_\theta}$-bounded.

Moreover, for all $u \in \mathcal{L}_h((E, \mathcal{B}), (F, \theta); (G, \xi))$, the family $\left(\Psi_{(\alpha, \lambda, \rho)}(u)\right)_{(\alpha, \lambda, \rho) \in I \times J \times K}$ belongs to the $R$-module $\lim \mathcal{L}_h((E_\alpha, B_\alpha), (F_\lambda, \theta_\lambda); (G_\rho, \xi_\rho))$ because

$$\left(\tilde{\Phi}^{(\beta, \mu, \sigma)}_{(\alpha, \lambda, \rho)} \circ \Psi^{(\beta, \mu, \sigma)}_{(\alpha, \lambda, \rho)}\right)(u) = \Psi^{(\alpha, \lambda, \rho)}(u)$$
for \((\alpha, \lambda, \rho) \leq (\beta, \mu, \sigma)\).

Now, we can state a result in whose proof Theorem 4 plays an important role:

**Proposition 13.** Under the conditions above, the \(R\)-linear mapping

\[
\Psi : u \in \left( \mathcal{L}_b((E, \mathcal{B}), (F, \theta); (G, \xi)), \mathcal{E}_{\mathcal{B}, \theta}^{\xi} \right) \\
\downarrow \\
\left( \Psi_{(\alpha, \lambda, \rho)}(u) \right) \in \left( \lim \mathcal{L}_b((E_{\alpha}, \mathcal{B}_{\alpha}), (F_{\lambda}, \theta_{\lambda}); (G_{\rho}, \xi_{\rho})), \mathcal{D} \right)
\]

is a bornological \(R\)-module isomorphism, the codomain of \(\Psi\) being the bornological projective limit of the system given in Proposition 11.

**Proof.** Let

\[
\Gamma_1 : \left( \mathcal{L}_h((E, \mathcal{B}), (F, \theta); (G, \xi)), \mathcal{E}_{\mathcal{B}, \theta}^{\xi} \right) \rightarrow \left( \mathcal{L}_b((E, \mathcal{B}); (L((F, \theta); (G, \xi)), \mathcal{E}_{\theta}^{\xi}), \mathcal{E}_{\mathcal{B}}^{\xi} \right)
\]

be the bornological \(R\)-module isomorphism given in Proposition 10. For each \((\lambda, \rho) \in J \times K\), let \(\Psi_{(\lambda, \rho)}\) be the bounded \(R\)-linear mapping from \(\mathcal{L}(F, \theta); (G, \xi)\), \(\mathcal{E}_{\theta}^{\xi}\) into \(\mathcal{L}(F_{\lambda}, \theta_{\lambda}); (G_{\rho}, \xi_{\rho}), \mathcal{E}_{\mathcal{B}}^{\xi}\) given by \(\Psi_{(\lambda, \rho)}(\varphi) = \omega_{\rho} \circ \varphi \circ v_{\lambda}\).

By the linear case of Theorem 3.6 of [5] and Theorem 4, we conclude that the \(R\)-linear mapping

\[
\Gamma_2 : \psi \in \left( \mathcal{L}_b((E, \mathcal{B}); (L((F, \theta); (G, \xi)), \mathcal{E}_{\theta}^{\xi}), \mathcal{E}_{\mathcal{B}}^{\xi} \right) \\
\downarrow \\
\Gamma_2(\psi) = \left( \Psi_{(\lambda, \rho)} \circ \psi \circ u_{\alpha} \right)_{(\alpha, \lambda, \rho) \in I \times J \times K} \in \left( \lim \mathcal{L}_b((E_{\alpha}, \mathcal{B}_{\alpha}); (L((F_{\lambda}, \theta_{\lambda}); (G_{\rho}, \xi_{\rho}), \mathcal{E}_{\mathcal{B}}^{\xi}_{\theta_{\lambda}})), \mathcal{D} \right)
\]

is a bornological \(R\)-module isomorphism, the codomain of \(\Gamma_2\) being the bornological projective limit of the system given in Corollary 12. For each \((\alpha, \lambda, \rho) \in I \times J \times K\), let

\[
\Gamma_{(\alpha, \lambda, \rho)} : \left( \mathcal{L}_b((E_{\alpha}, \mathcal{B}_{\alpha}); (L((F_{\lambda}, \theta_{\lambda}); (G_{\rho}, \xi_{\rho}), \mathcal{E}_{\mathcal{B}}^{\xi}_{\theta_{\lambda}})), \mathcal{E}_{\mathcal{B}}^{\xi}_{\theta_{\lambda}}) \right) \rightarrow \left( \mathcal{L}_h((E_{\alpha}, \mathcal{B}_{\alpha}), (F_{\lambda}, \theta_{\lambda}); (G_{\rho}, \xi_{\rho}), \mathcal{E}_{\mathcal{B}}^{\xi}_{\theta_{\lambda}}) \right)
\]

be the inverse of the \(R\)-linear mapping given in Proposition 10, which is a bornological \(R\)-module isomorphism. Then the \(R\)-linear mapping

\[
\Gamma_3 : \left( \Psi_{(\lambda, \rho)} \circ \psi \circ u_{\alpha} \right)_{(\alpha, \lambda, \rho) \in I \times J \times K} \in \left( \lim \mathcal{L}_b((E_{\alpha}, \mathcal{B}_{\alpha}); (L((F_{\lambda}, \theta_{\lambda}); (G_{\rho}, \xi_{\rho}), \mathcal{E}_{\mathcal{B}}^{\xi}_{\theta_{\lambda}})), \mathcal{D} \right)
\]
\[
\left( \Gamma_{(\alpha,\lambda,\rho)} \left( \Psi_{(\lambda,\rho)} \circ \psi \circ u_{\alpha} \right) \right)_{(\alpha,\lambda,\rho) \in I \times J \times K} \in \left( \lim \left\langle L_h \left( (E_{\alpha}, B_{\alpha}), (F_{\lambda}, \theta_{\lambda}); (G_{\rho}, \xi_{\rho}) \right), D \right\rangle \right)
\]

is a bornological \( R \)-module isomorphism. Finally, since it is clear that \( \Psi = \Gamma_3 \circ \Gamma_2 \circ \Gamma_1 \), then \( \Psi \) is a bornological \( R \)-module isomorphism, as was to be shown.

We end our paper by mentioning that other aspects of the study of hypocontinuous mappings between topological modules have been considered in [4].

References


Received: March, 2011