Uniquely Completion Sets in Some Latin Squares

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Abstract

In a Latin square $L$ of order $n$ constructed with symbol set $\{1, \ldots, 2n-2\}$, critical set $C$ is a subset of $L$ such that there exist a unique extension of $C$ to $L$ and no proper subset of $C$ has this property. Let $M$ and $N$ be two Latin squares of order $m$ and $n$ with symbol set $\{1, \ldots, 2m-2\}$ and $\{1, \ldots, 2n-2\}$ then $M \times N$ is a Latin square of order $mn$ with symbol set $\{1, \ldots, 2mn-(n+2)\}$ and the UC-set of product may be product of UC-sets.

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1 Introduction

The concept of critical set were invented by statistician John Nelder about 1977. There are many papers about critical set in Latin square with symbol set $\{1, \ldots, n\}$ but yet, there exists a problem on critical set with symbol set $\{1, \ldots, k\}$ where $n < k < 2n - 1$. [3].

D. A. Mojdeh and et.al found some of strong defining sets on coloring of graph $K_n \times K_n$ with $2n - 2$ colors. We build uniquely completion sets on Latin squares based on symbol set $\{1, \ldots, 2mn-(n+2)\}$ by product of this critical sets.
R. A. H. Gower conjectured that if $P$ and $Q$ are partial Latin squares with unique completion, then the completable product $P \times Q$ will also have unique completion. This conjecture has been shown to be true in number of cases:

- When either of $P$ or $Q$ has a strong completion: (Gower)
- When either of $P$ or $Q$ is not Bedford and Whitehouse totally weak (Bedford and Whitehouse).

However recently N.Cavenagh and et.al found whole class of counterexample to this conjecture.

In this paper we focus on partial Latin squares that strongly uniquely complete to Latin squares with symbol set $\{1, ..., 2n-2\}$ and product of them.

## 2 Preliminary Notes

A Latin square of order $n$ is an $n \times n$ array of integers chosen from the set $Y = \{1, \ldots, n\}$ such that each element of $Y$ occurs exactly once in each row and exactly once in each column. Then a back circulant Latin square has the integer $i + j \pmod n$ in cell $(i, j)$. A back circulant Latin square, of order $n$, corresponding to the cyclic group $C_n$. Hence $C_n$ will be used to represent such a Latin square. In this paper we assume $X = \{1, \ldots, 2n-2\}$.

A Partial Latin square $P$ of order $n$ is an $n \times n$ array with entries chosen from set $X$, such that each element of $X$ occurs at most once in each row and at most once in each column. A (partial) Latin square can be also written as a set of ordered triples $\{(i, j, k)\}$ such that symbol $k$ occurs in position $(i, j)$ of array $\{\}$.

A partial Latin square $C$ contained in a Latin square $L$ is said to be uniquely completable if $L$ is only Latin square of order $n$ with $k$ in the cell $(i, j)$ for every $(i, j; k) \in C$. A critical set $C$ contained in a Latin square $L$ is a partial Latin square such that is uniquely completable and no proper subset of $C$ satisfies this requirement.

Let $M, N$ be two Latin square of order $m$ and $n$ respectively with entries from the set $\{1, \ldots, m\}$ and $\{1, \ldots, n\}$ respectively, define $N^r$ to be the array obtained from $N$ by adding $rn$ to each entry of $N$, for $r = 1, \ldots, m$. (Then $N^r = rnJ + N$ where $J$ is the matrix whose entries are all 1's.) the Product of $M$ and $N$ is $L$, the Latin square of order $mn$ constructed by replacing the entry $r$ in $M$ by the array $N^r$. One writes $L = M \times N$. 
Uniquely completion sets in some Latin squares

\[ M \times N = ( (a-1)n + d, (b-1)n + e; (c-1)n + f ) : (a, b; c) \in M, (d, e; f) \in N \]

is the Product of two Latin squares of order \( m \) and \( n \).

If \( P \) is a partial Latin square of order \( n \) with unique completion to a Latin square \( L \) and there exist a set \( \{ P_1, P_2, ..., P_f \} \) of partial Latin squares of order \( n \) with \( f = n^2 - |C| \) such that:

1. \( P = P_1 \subset P_2 \subset ... \subset P_f \subset L \);
2. For any \( i, 2 \leq i \leq f \), where \( P_i = P_{i-1} \cup \{(r_{i-1}, s_{i-1}, t_{i-1})\} \), the set \( P_{i-1} \cup \{(r_{i-1}, s_{i-1}, t')\} \) is not a partial Latin square for any \( t' \in N \setminus \{ t \} \) then \( P \) is said to be strongly uniquely completable.

**THEOREM 1.**\([6]\) Every defining set of a \( k \)-regular \( k \)-vertex coloring graph is strong.

The integer number \( 2mn - (n + 2) \) is the greatest entry in array \( M \times N \) with symbol set \( X \) and \( Y \) respectively. Because the entries of product is form \( cn + f \) and maximum size of \( c \) in \( X \) is \( 2m - 2 \) and maximum size of \( f \) in \( Y \) is \( 2n - 2 \) then maximum size of \( cn + f \) is \( (2m-2)n - (2n-2) = 2mn - (n+2) \).

### 3 Main Results

**THEOREM 2.** Let \( M, N \) be two Latin square of order \( m, n \) and \( n \neq 3 \) with symbol set \( X = \{1, ..., 2m - 2\}, Y = \{1, ..., 2n - 2\} \). Then the product of:

\( (a, b; c). (d, e; f + m) = (x, y, z) \) and \( (a', b'; c + 1). (d', e'; f) = (x, y'; z) \)

are caused in \( x \) row there exist repeated symbol \( z \), and the product of

\( (a, d; c). (d, e; f + m) = (r, s; t) \) and \( (a', b; c + 1). (d', e; f) = (r', s; t) \)

are caused in \( s \) column there exist repeated symbol \( t \), where the first triples is belong to \( M \) and the second triples belongs to \( N \). It means that entries are caused repeated symbols that distance between two same-row or same-column cell is order of Latin square.

**PROOF.** Two element of product Latin square are \( cn + f \) and \( c'n + f' \). If in one row or column these elements are equal then are caused repeated symbols. It leads that \( c - c' = f - f' \). If \( \frac{f' - f}{n} \) be integer number then are caused repeated symbols. The maximum size of \( f' - f \) is \( (2n - 2) - 1 \) and minimum size is 0. If \( f' - f = n \) (so distance between symbols in \( N \) be \( n \)), then \( \frac{f' - f}{n} \) is integer number and cause repeated symbols. Also \( \frac{f' - f}{n} \) always is integer number for \( n = 3 \), hence we consider \( n \neq 3 \).
We study what needs to be done to obviate this repeated symbols. we should replace the rows or columns or rename the elements to replace the same-row or same-column elements that distance between them is order of Latin square.

For example, consider \( M \) be a Latin square of order 3 on symbol set \( \{1, ..., 4\} \) and \( N \) be a Latin square of order 4 on symbol set \( \{1, ..., 6\} \). The product of \( M \) and \( N \) is of order 12 on symbol set \( \{1, ..., 2mn - (n + 2) = 18\} \).

\[
M = \begin{bmatrix}
4 & 2 & 1 \\
2 & 3 & 4 \\
3 & 1 & 2 \\
\end{bmatrix}
\quad N = \begin{bmatrix}
1 & 2 & 3 & 4 \\
2 & 1 & 4 & 3 \\
5 & 6 & 1 & 2 \\
6 & 5 & 2 & 1 \\
\end{bmatrix}
\]

Product of these above Latin square has repeated symbols.

\[
M \times N = \begin{array}{cccccccccc}
13 & 14 & 15 & 16 & 5 & 6 & 7 & 8 & 1 & 2 & 3 & 4 \\
14 & 13 & 16 & 15 & 6 & 5 & 8 & 7 & 2 & 1 & 4 & 3 \\
17 & 18 & 13 & 14 & 9 & 10 & 5 & 6 & 5 & 6 & 1 & 2 \\
18 & 17 & 13 & 14 & 10 & 9 & 6 & 5 & 6 & 5 & 2 & 1 \\
5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\
6 & 5 & 8 & 7 & 10 & 9 & 12 & 11 & 14 & 13 & 16 & 15 \\
9 & 10 & 5 & 6 & 13 & 14 & 9 & 10 & 17 & 18 & 13 & 14 \\
10 & 9 & 6 & 5 & 14 & 13 & 10 & 9 & 18 & 17 & 14 & 13 \\
9 & 10 & 11 & 12 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
10 & 9 & 12 & 11 & 2 & 1 & 4 & 3 & 6 & 5 & 8 & 7 \\
13 & 14 & 9 & 10 & 5 & 6 & 1 & 2 & 9 & 10 & 5 & 6 \\
14 & 13 & 10 & 9 & 6 & 5 & 2 & 1 & 10 & 9 & 6 & 5 \\
\end{array}
\]

We need a replacement that obviate the same-row and same-column symbols with distance 4. This replacement is \( 1 \leftrightarrow 3 \) and \( 2 \leftrightarrow 4 \) in Latin square \( N \) and we have:

\[
M = \begin{bmatrix}
4 & 2 & 1 \\
2 & 3 & 4 \\
3 & 1 & 2 \\
\end{bmatrix}
\quad N = \begin{bmatrix}
3 & 4 & 1 & 2 \\
4 & 3 & 2 & 1 \\
5 & 6 & 3 & 4 \\
6 & 5 & 4 & 3 \\
\end{bmatrix}
\]
Therefore, we conjecture that this property is true for other Latin squares of order \( m \) and \( n \) as follows.

PROBLEM 1. There exists a permutation on Latin square of order \( n \) with symbol set \( \{1, \ldots, 2n-2\} \) such that there won’t be any two symbols by distance \( n \) in each row and each column of \( N \).

The following theorem generalize the Theorem 2.5 of [3].

THEOREM 3. Let \( P \) be a partial Latin square which is strongly uniquely completable to the Latin square \( M \) of order \( m \) on symbol set \( \{1, \ldots, 2m-2\} \) and let \( Q \) be a partial Latin square which is strongly uniquely completable to the Latin square \( N \) of order \( n \) on symbol set \( \{1, \ldots, 2n-2\} \). Then the partial Latin square \( R = P \times Q \) has uniquely completion to \( L \), the direct product of \( M \) and \( N \).

PROOF. Since \( P \) is strongly uniquely completable there is a set of partial Latin square, \( \{P_1, P_2, \ldots, P_f\} \), with \( f = m^2 - |P| \) satisfying the conditions of Definition. Then \( P_i = P_{i-1} \cup \{(r_{i-1}, s_{i-1}, t_{i-1})\} \) and it is known that for each \( t' \) in the set \( \{1, \ldots, 2m-2\} \setminus \{t_{i-1}\} \), \( t' \) is either in row \( r_{i-1} \) of \( P_{i-1} \) or in column \( s_{i-1} \) of \( P_{i-1} \). That is to say there is a cell \( (r, x, t') \in P_i \) for some \( x, x \neq s_i \), or there is a cell \( (x, s_i, t') \in P_i \) for some \( x, x \neq r_i \).

Recall that \( P_1 = P \) by the definition and define \( R_1 = P_1 \times Q = P \times Q = R \).

Now \( P_2 = P_1 \cup \{(r_1, s_1, t_1)\} \) which means that in the position of \( R_1 = R \) with \( f = m^2 - |P| \) satisfying the conditions of Definition. Then \( P_i = P_{i-1} \cup \{(r_{i-1}, s_{i-1}, t_{i-1})\} \) and it is known that for each \( t' \) in the set \( \{1, \ldots, 2m-2\} \setminus \{t_{i-1}\} \), \( t' \) is either in row \( r_{i-1} \) of \( P_{i-1} \) or in column \( s_{i-1} \) of \( P_{i-1} \). That is to say there is a cell \( (r, x, t') \in P_i \) for some \( x, x \neq s_i \), or there is a cell \( (x, s_i, t') \in P_i \) for some \( x, x \neq r_i \).

Recall that \( P_1 = P \) by the definition and define \( R_1 = P_1 \times Q = P \times Q = R \).

Now \( P_2 = P_1 \cup \{(r_1, s_1, t_1)\} \) which means that in the position of \( R_1 = R \) with \( f = m^2 - |P| \) satisfying the conditions of Definition. Then \( P_i = P_{i-1} \cup \{(r_{i-1}, s_{i-1}, t_{i-1})\} \) and it is known that for each \( t' \) in the set \( \{1, \ldots, 2m-2\} \setminus \{t_{i-1}\} \), \( t' \) is either in row \( r_{i-1} \) of \( P_{i-1} \) or in column \( s_{i-1} \) of \( P_{i-1} \). That is to say there is a cell \( (r, x, t') \in P_i \) for some \( x, x \neq s_i \), or there is a cell \( (x, s_i, t') \in P_i \) for some \( x, x \neq r_i \).
\(2 + t'n\) occurs exactly once in each these rows or it occurs exactly once of these columns. Hence the only elements of \(N_L = \{1, ..., 2mn-(n+2)\}\) which have not already occurred in the rows indexed from \((r_1-1)n\) to \(r_1n\) and columns indexed from \((s_1-1)n\) to \(s_1n\) are \(1 + (t_1-1)n, 2 + (t_1-1)n, ..., 2n - 2 + (t_1-1)n\). So these are the only entries which may be placed in the cells which are defined by the intersection of these rows and columns., but these cells already contain a copy of \(Q^t\) which has unique completion to \(N^t\) when restricted to this \(n \times n\) subarray of \(R = R_1\) is forced to complete to \(N^t\). This produces a new partial Latin square \(R_2 = P_2 \times Q\).

Similarly, a sequence of partial Latin squares \(R_1, R_2, ..., R_f\) is obtained with \(R_f = P_f \times Q\). Since \(P_f\) has one entry less than the Latin square \(M\), the partial Latin square \(R_f\) has only one subarray containing a copy of \(Q_i\) for some \(i\), there is only one possible completion from this point and it leads to the Latin square \(L = M \times N\).

The following example denotes the above Theorem:

**EXAMPLE.** Consider

\[
M = \begin{bmatrix}
4 & 2 & 1 \\
2 & 3 & 4 \\
3 & 1 & 2
\end{bmatrix} \quad N = \begin{bmatrix}
3 & 4 & 1 & 2 \\
4 & 3 & 2 & 1 \\
5 & 6 & 3 & 4 \\
6 & 5 & 4 & 3
\end{bmatrix}
\]

and

\[
P = \begin{bmatrix}
\bullet & \bullet & 1 \\
2 & 3 & \bullet \\
3 & 1 & \bullet
\end{bmatrix} \quad Q = \begin{bmatrix}
\bullet & \bullet & 1 & 2 \\
4 & \bullet & 2 & 1 \\
5 & 6 & \bullet & \bullet \\
6 & 5 & 4 & \bullet
\end{bmatrix}
\]

Since \(P\) is strongly uniquely completable then there exist set of partial Latin squares \(\{P_1, P_2, ..., P_f\}\), with \(f = m^2 - |P| = 4^2 - 6 = 10\) that satisfies conditions of definition. Then \(P_2 = P_1 \cup \{(1, 1, 4)\}\) and for each \(t' \in \{1, ..., 4\} \setminus \{4\} = \{1, 2, 3\}\), \(t'\) is either in row 1 of \(P_1\) or in column 1 of \(P_1\).

\[
P_1 = \begin{bmatrix}
\bullet & \bullet & 1 \\
2 & 3 & \bullet \\
3 & 1 & \bullet
\end{bmatrix} \quad P_2 = \begin{bmatrix}
\bullet & \bullet & 1 \\
4 & 3 & \bullet \\
2 & 3 & \bullet \\
3 & 1 & \bullet
\end{bmatrix}
\]

There is a cell \((2, x, t') \in P_2\) for some \(x\), \(x \neq 2\), or there is a cell \((x, 2, t') \in P_2\) for some \(x\), \(x \neq 2\).

Define the \(R_1 = P_1 \times Q = P \times Q = R\).

Now \(P_2 = P_1 \cup \{(1, 1, 4)\}\) means that in the position of \(R_1 = R\) with rows indexed from \((r_1 - 1)n + 1 = 1\) to \(r_1n = 4\) and columns indexed from \((s_1-1)n + 1 = 1\) to \(s_1n = 4\) there is a copy of \(Q^4\). ( \(Q^4\) means that \((t_1 - 1)n = 12 \) add to each entry of \(Q\).) Also, from the discussion above it can be concluded that for each \(t' \in \{1, ..., 4\} \setminus \{4\} = \{1, 2, 3\}\) there is a copy of \(N^{t'}\) which is posi-
tioned such that it is either in the rows indexed from 1 to 4 and columns indexed from 1 to 4. Each element of \{1+12, 2+12, ..., 6+12\} and \{1+8, 2+8, ..., 6+8\} and \{1+4, 2+4, ..., 6+4\} occurs exactly once in each these rows or it occurs exactly once of these columns. Hence the only elements of \(N_L = \{1, ..., 18\}\) which have not already occurred in the rows indexed from 1 to 4 and columns indexed from 1 to 4 are \(1 + 3 \times 4 = 13, 2 + 3 \times 4 = 14, ..., 6 + 3 \times 4 = 18\). So these are the only entries which may be placed in the cells which are defined by the intersection of these rows and columns. But these cells already contain a copy of \(Q^4\) which has unique completion to \(N^4\) when restricted to this \(n \times n\) subarray of \(R = R_1\) is forced to complete to \(N^4\). This produces a new partial Latin square \(R_2 = P_2 \times Q\). Similarly, a sequence of partial Latin squares, \(R_1, R_2, ..., R_6\) is obtained with \(R_6 = P_6 \times Q\). Since \(P_6\) has one entry less than the Latin square \(M\), the partial Latin square \(R_6\) has only one subarray containing a copy of \(Q^i\) for some \(i\), there is only one possible completion from this point and it leads to the Latin square \(L = M \times N\).

**PROBLEM 2.** Does the product of two critical sets be a critical set?

**References**


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