The Radical in Ideal Systems on Groups

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Abstract

The radical of an ideal has been studied not only in commutative rings, but also in ideals defined on semigroups or monoids. In this paper, the notion of radical in \( r \)-ideal systems defined on directed groups is introduced. In order to derive some of its algebraic properties, the structure of integral ideals is investigated and it is proved that the groups \( G_i \) are \( r_i \)-noetherian, \( i = 1, 2 \), if and only if the product \( G_1 \times G_2 \) is \( r_1 \otimes r_2 \)-noetherian. We observe that in \( r \)-ideal theory the radical does not lead in general to the definition of a new ideal system. The main result is that the analogue of Krull’s theorem holds when the \( r \)-system is of finite character, fact which implies that in this case the radical is an \( r \)-ideal.

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1 Introduction

Jaffard in [4] presents the theory of \( r \)-ideals on groups having as principal sources the work of Lorenzen ([9]) and Krull ([8]). Following Jaffard ([4]), we use multiplicative notation and all groups are considered to be directed abelian. For such a group \( G \), the set of its lower bounded subsets will be denoted by \( B(G) \) and the set of its integral elements by \( G^+ \). All the following introductory notions, except from those with a specific reference, can be found in [4]. The central notion is the \( r \)-ideal system or \( r \)-system that can be defined on a directed group \( G \) as a map \( r : B(G) \rightarrow P(G), X \mapsto X_r \), which satisfies, for every \( X, Y \in B(G) \) and every \( g \in G \), the following conditions:

(i) \( X \subseteq X_r \),
(ii) \( X \subseteq Y_r \Rightarrow X_r \subseteq Y_r \),
(iii) \( \{g\}_r = g \cdot G^+ = (g) \),
(iv) \( g \cdot X_r = (g \cdot X)_r \).

For any \( X \in B(G) \), the set \( X_r \) is called \( r \)-ideal of \( G \) generated by \( X \).

The set \( I_r(G) \) of all \( r \)-ideals of \( G \), which is a commutative monoid under the \( r \)-multiplication \( X_r \times_r Y_r = (X \cdot Y)_r \), contains a subset of particular interest, denoted by \( A_r(G) \), that consists of all integral \( r \)-ideals. An \( r \)-ideal is called integral if it is contained into \( G^+ \), relation that obviously holds if and only if its generator is a subset of \( G^+ \). An integral \( r \)-ideal \( X_r \), different from \( G^+ \), is called prime if the set \( G^+ \setminus X_r \) is closed under multiplication and it is called maximal if the relation \( X_r \subseteq Y_r \subseteq (1) \) implies, for any \( r \)-ideal \( Y_r \), \( Y_r = X_r \) or \( Y_r = (1) \).

On a directed group \( G \), there is always defined the \( s \)-system as \( X_s = \bigcup_{x \in X} (x) \), therefore the set \( R(G) \) of all \( r \)-ideal systems defined on \( G \) is non-empty.

Considering now two ideal systems \( r, q \in R(G) \), we can form the \( r \oplus q \)-ideal system, called sum of \( r \) and \( q \) ([7]), by \( X_{r \oplus q} = X_r \cap X_q \). A natural transition to cartesian products can be achieved using the product of ideal systems ([5]). We note that the cartesian product \( G_1 \times G_2 \) of two directed groups is a directed group with respect to pointwise multiplication and ordering. The product of \( r_1 \in R(G_1) \) and \( r_2 \in R(G_2) \) is the \( r_1 \otimes r_2 \)-ideal system, defined on \( G_1 \times G_2 \) by \( X_{r_1 \otimes r_2} = (p_1(X))_{r_1} \times (p_2(X))_{r_2} \), where \( p_i : G_1 \times G_2 \to G_i, i = 1, 2 \), are the usual projection maps.

Krull ([8]) defined the radical of an ordinary ideal in commutative rings, notion that has been studied not only in ring theory (we indicatively refer to [3]) but also in case of \( x \)-ideals defined on semigroups ([1]), as well as in case of weak ideal systems defined on monoids ([2]). The radical in those cases is defined for every ideal.

In this paper, we transfer the notion of radical to \( r \)-ideal systems under the restriction into integral \( r \)-ideals, fact that leads us to investigate at first the algebraic structure of integral \( r \)-ideals.

In section 2, we present some algebraic properties of \( A_r(G) \) and we describe the integral \( r_1 \otimes r_2 \)-ideals of a cartesian product. Especially, we investigate if the projections of a prime or maximal \( r_1 \otimes r_2 \)-ideal remain prime or maximal ideals, respectively. We also prove that the group \( G_1 \times G_2 \) is \( r_1 \otimes r_2 \)-noetherian if and only if the factors \( G_i \) are \( r_i \)-noetherian, \( i = 1, 2 \).

In section 3, we define the radical of an integral \( r \)-ideal and we prove some of its basic properties, including the connection with the sum and the product of ideal systems. Investigating whether the radical is an \( r \)-ideal or not, we give at first the definition of half-prime \( r \)-ideal. When the \( r \)-system is of finite character, we prove that the radical of an integral \( r \)-ideal \( X_r \) equals to the
intersection of all prime $r$-ideals which contain $X_r$. Finally, we observe that in $r$-ideal theory the radical does not lead in general to the definition of a new ideal system.

2 Integral $r$-ideals

Let $G$ be a directed group and $r \in \mathcal{R}(G)$. The $r$-multiplication is connected with the set theoretic multiplication of $r$-ideals, since $X_r \times_r Y_r = (X_r \cdot Y_r)_r$, for every $X_r, Y_r \in \mathcal{I}_r(G)$. The monoid $\mathcal{I}_r(G)$ becomes a complete lattice with respect to $X_r \leq Y_r \iff Y_r \subseteq X_r$, which implies that the non-empty intersection of $r$-ideals is also an $r$-ideal. We note that the subset $\mathcal{A}_r(G)$ is not considered to be partially ordered under the induced ordering but with respect to $X_r \leq Y_r \iff X_r \subseteq Y_r$. A detailed study of the above properties can be found in [4].

**Proposition 2.1.** Let $G$ be a directed group and $r \in \mathcal{R}(G)$. The following properties hold, for every $X_r, Y_r \in \mathcal{A}_r(G)$:

1. $X_r \cdot G^+ = X_r$ and $X \cdot G^+ \subseteq X_r$.
2. $X_r \times_r Y_r \subseteq X_r \cap Y_r$.

*Proof.* (1) It is clear that $X_r \subseteq X_r \cdot G^+$ and $X_r \cdot G^+ = \bigcup x \cdot G^+ = \bigcup \{x\}_r \subseteq X_r$, thus $X_r \cdot G^+ = X_r$. Therefore, $X \cdot G^+ \subseteq X_r$.

(2) Since $X_r \cdot Y_r \subseteq X_r \cdot G^+ \cap Y_r \cdot G^+ = X_r \cap Y_r$, it follows that $X_r \cap Y_r \neq \emptyset$ and $X_r \times_r Y_r = (X_r \cdot Y_r)_r \subseteq (X_r \cap Y_r)_r = X_r \cap Y_r$. □

**Proposition 2.2.** Let $G$ be a directed group and $r \in \mathcal{R}(G)$. The set of integral $r$-ideals of $G$ is a submonoid of $(\mathcal{I}_r(G), \times_r)$.

*Proof.* It is enough to observe that the $r$-ideal $\{1\}_r$, which is the neutral element of the monoid $(\mathcal{I}_r(G), \times_r)$, is integral and that for every $X_r, Y_r \in \mathcal{A}_r(G)$ the $r$-ideal $X_r \times_r Y_r$ is integral, since $X_r \times_r Y_r = (X \cdot Y)_r$ and $X \cdot Y \subseteq G^+$. □

**Proposition 2.3.** Let $G_1$ and $G_2$ be two directed groups, $r_1 \in \mathcal{R}(G_1)$ and $r_2 \in \mathcal{R}(G_2)$. The monoid $(\mathcal{A}_{r_1 \otimes r_2}(G_1 \times G_2), \times_{r_1 \otimes r_2})$ is isomorphic to the cartesian product of the monoids $(\mathcal{A}_{r_1}(G_1), \times_{r_1})$ and $(\mathcal{A}_{r_2}(G_2), \times_{r_2})$.

*Proof.* In [6], there was proved that the monoid of $r_1 \otimes r_2$-ideals is isomorphic to the cartesian product of the monoids of $r_1$-ideals and $r_2$-ideals. More precisely, there was proved that the map

$$\phi : \mathcal{I}_{r_1}(G_1) \times \mathcal{I}_{r_2}(G_2) \to \mathcal{I}_{r_1 \otimes r_2}(G_1 \times G_2),$$

$$\phi((X_1)_{r_1}, (X_2)_{r_2}) = (X_1 \times X_2)_{r_1 \otimes r_2},$$
is a monoid-isomorphism. Obviously, the monoid $A_{r_1}(G_1) \times A_{r_2}(G_2)$ is a submonoid of $\mathcal{I}_{r_1}(G_1) \times \mathcal{I}_{r_2}(G_2)$. We can easily verify that the map
\[
\tilde{\phi} : A_{r_1}(G_1) \times A_{r_2}(G_2) \rightarrow A_{r_1 \otimes r_2}(G_1 \times G_2),
\]
\[
\tilde{\phi}((X_1)_{r_1}, (X_2)_{r_2}) = (X_1 \times X_2)_{r_1 \otimes r_2},
\]
is the restriction of $\phi$ into $A_{r_1}(G_1) \times A_{r_2}(G_2)$. Hence, it is enough to prove that the map $\tilde{\phi}$ is onto. In fact, if $X_{r_1 \otimes r_2} \subseteq (X_2)_{r_2} \subseteq (X_3)_{r_2} \subseteq \cdots$ of integral $r$-ideals, there exists $n \in \mathbb{N}$ such that $(X_i)_{r_i} = (X_n)_{r_i}$ for every $i \geq n$. When the group $G$ is $r$-noetherian, the $r$-ideal system is also called noetherian.

**Theorem 2.4.** Let $G_1$ and $G_2$ be two directed groups, $r_1 \in \mathcal{R}(G_1)$ and $r_2 \in \mathcal{R}(G_2)$. The group $G_1 \times G_2$ is $r_1 \otimes r_2$-noetherian if and only if the group $G_1$ is $r_1$-noetherian and the group $G_2$ is $r_2$-noetherian.

**Proof.** We suppose, at first, that the group $G_1 \times G_2$ is $r_1 \otimes r_2$-noetherian. Let $(Z_i)_{r_1} \subseteq (Z_2)_{r_1} \subseteq (Z_3)_{r_1} \subseteq \cdots$ be an ascending sequence of integral $r_1$-ideals of $G_1$. Then, the sets $Y_i = Z_i \times \{1\}$, $i \in \mathbb{N}$, are non-empty subsets of $(G_1 \times G_2)^+$ and the sequence $((Y_i)_{r_1 \otimes r_2})_{i \in \mathbb{N}}$ is an ascending sequence of integral $r_1 \otimes r_2$-ideals of $G_1 \times G_2$. Thus, there exists $n \in \mathbb{N}$ such that $(Y_i)_{r_1 \otimes r_2} = (Y_n)_{r_1 \otimes r_2}$, for every $i \geq n$. Therefore $(p_1(Y_i))_{r_1} = (p_1(Y_n))_{r_1}$, which means that $(Z_i)_{r_1} = (Z_n)_{r_1}$, for every $i \geq n$. Hence, the group $G_1$ is $r_1$-noetherian. Following the analogous procedure for $G_2$, we can prove that it is an $r_2$-noetherian group.

Conversely, we suppose that the group $G_1$ is $r_1$-noetherian and that the group $G_2$ is $r_2$-noetherian. Let $(X_i)_{r_1 \otimes r_2} \subseteq (X_2)_{r_1 \otimes r_2} \subseteq (X_3)_{r_1 \otimes r_2} \subseteq \cdots$ be an ascending sequence of integral $r_1 \otimes r_2$-ideals of $G_1 \times G_2$. It is clear that the sequence $((p_1(X_i))_{r_1})_{i \in \mathbb{N}}$ is an ascending sequence of integral $r_1$-ideals of $G_1$ and that the sequence $((p_2(X_i))_{r_2})_{i \in \mathbb{N}}$ is an ascending sequence of integral $r_2$-ideals of $G_2$. Thus, since $G_1$ is an $r_1$-noetherian group and $G_2$ is an $r_2$-noetherian group, there exist $n, m \in \mathbb{N}$ such that $(p_1(X_j))_{r_1} = (p_1(X_n))_{r_1}$ and $(p_2(X_k))_{r_2} = (p_2(X_m))_{r_2}$, for every $j \geq n$ and every $k \geq m$. We assume that $n \leq m$. Then
\[
(X_n)_{r_1 \otimes r_2} = (p_1(X_n))_{r_1} \times (p_2(X_n))_{r_2}
\]
\[
(X_{n+1})_{r_1 \otimes r_2} = (p_1(X_{n+1}))_{r_1} \times (p_2(X_{n+1}))_{r_2} = (p_1(X_n))_{r_1} \times (p_2(X_{n+1}))_{r_2}
\]
\[
\vdots
\]
\[
(X_m)_{r_1 \otimes r_2} = (p_1(X_m))_{r_1} \times (p_2(X_m))_{r_2}
\]
\[
(X_{m+1})_{r_1 \otimes r_2} = (p_1(X_{m+1}))_{r_1} \times (p_2(X_{m+1}))_{r_2} = (p_1(X_n))_{r_1} \times (p_2(X_{m+1}))_{r_2}
\]
\[
\vdots
\]
that means that \((X_i)_{r_1 \otimes r_2} = (X_m)_{r_1 \otimes r_2}\), for every \(i \geq m\), thus the sequence 
\(((X_i)_{r_1 \otimes r_2})_{i \in \mathbb{N}}\) is finally constant. In case that \(m < n\) we derive, using analogous 
arguments, the same result. Hence, the group \(G_1 \times G_2\) is \(r_1 \otimes r_2\)-noetherian. \(\square\)

It is known ([4]), as an equivalent definition, that an integral \(r\)-ideal \(X_r\) is prime if and only if the relations \(x \cdot y \in X_r\) and \(x \notin X_r\) imply \(y \in X_r\), for any \(x, y \in G^+\). In proposition 2.5 we characterise a prime \(r\)-ideal using its subsets. We denote by \(\mathcal{P}_r(G)\) and \(\mathcal{M}_r(G)\) the sets of prime and maximal \(r\)-ideals of \(G\), respectively.

**Proposition 2.5.** Let \(G\) be a directed group, \(r \in \mathcal{R}(G)\) and \(X, Y, Z\) non-empty 
subsets of \(G^+\). The following are equivalent :

1. \(Z_r \in \mathcal{P}_r(G)\).
2. If \(X \cdot Y \subseteq Z_r\), then \(X \subseteq Z_r\) or \(Y \subseteq Z_r\).
3. If \(X_r \times_r Y_r \subseteq Z_r\), then \(X_r \subseteq Z_r\) or \(Y_r \subseteq Z_r\).

**Proof.** (1) \(\Rightarrow\) (2) Let \(X \cdot Y \subseteq Z_r\) and \(X \not\subseteq \frac{Z_r}{r}\), so there exists \(x \in X\) with \(x \notin Z_r\). Moreover, for every \(y \in Y\) there holds \(x \cdot y \in Z_r\). Thus \(y \in Z_r\), hence \(Y \subseteq Z_r\).

(2) \(\Rightarrow\) (3) Let \(X_r \times_r Y_r \subseteq Z_r\) and \(X_r \not\subseteq \frac{Z_r}{r}\). Then \(X_r \cdot Y_r \subseteq (X_r \cdot Y_r)_r = X_r \times_r Y_r \subseteq Z_r\). Thus \(Y_r \subseteq Z_r\).

(3) \(\Rightarrow\) (1) Let \(x, y \in G^+\) with \(x \cdot y \in Z_r\). Then \(\{x \cdot y\}_r \subseteq Z_r\), that is \(\{x\}_r \times_r \{y\}_r \subseteq Z_r\). Hence, \(\{x\}_r \subseteq Z_r\) or \(\{y\}_r \subseteq Z_r\), so \(x \in Z_r\) or \(y \in Z_r\). Thus, \(Z_r\) is a prime \(r\)-ideal. \(\square\)

**Proposition 2.6.** Let \(G_1\) and \(G_2\) be two directed groups, \(r_1 \in \mathcal{R}(G_1)\), \(r_2 \in \mathcal{R}(G_2)\) and \(X_{r_1 \otimes r_2}\) an integral \(r_1 \otimes r_2\)-ideal of \(G_1 \times G_2\) different from \((G_1 \times G_2)^+\).

(i) If \(X_{r_1 \otimes r_2}\) is a prime \(r_1 \otimes r_2\)-ideal of \(G_1 \times G_2\), then \((p_1(X))_{r_1} \in \mathcal{P}_{r_1}(G_1)\) or \((p_2(X))_{r_2} \in \mathcal{P}_{r_2}(G_2)\).

(ii) If \(X_{r_1 \otimes r_2}\) is a maximal \(r_1 \otimes r_2\)-ideal of \(G_1 \times G_2\), then \((p_1(X))_{r_1} \in \mathcal{M}_{r_1}(G_1)\) or \((p_2(X))_{r_2} \in \mathcal{M}_{r_2}(G_2)\).

**Proof.** Since \(X_{r_1 \otimes r_2} \in \mathcal{A}_{r_1 \otimes r_2}(G_1 \times G_2)\), it is clear that \((p_1(X))_{r_1} \in \mathcal{A}_{r_1}(G_1)\) and \((p_2(X))_{r_2} \in \mathcal{A}_{r_2}(G_2)\). In addition, the hypothesis that \(X_{r_1 \otimes r_2}\) is different from \((G_1 \times G_2)^+\), assures that \((1, 1) \notin X_{r_1 \otimes r_2}\), thus \(1 \notin (p_1(X))_{r_1}\) or \(1 \notin (p_2(X))_{r_2}\). We assume that \(1 \notin (p_1(X))_{r_1}\), that means \((p_1(X))_{r_1} \notin G_1^+\). Then :

(i) If \(X_{r_1 \otimes r_2} \in \mathcal{P}_{r_1 \otimes r_2}(G_1 \times G_2)\), we will prove that \((p_1(X))_{r_1}\) is a prime \(r_1\)-ideal of \(G_1\). In fact, let \(x, y \in G_1^+\) such that \(x \cdot y \in (p_1(X))_{r_1}\) and \(x \notin (p_1(X))_{r_1}\). Then \((x, 1) \notin X_{r_1 \otimes r_2}\), while \((x, 1) \cdot (y, z) = (x \cdot y, z) \in X_{r_1 \otimes r_2}\) for any \(z \in p_2(X)\). Thus \((y, z) \in X_{r_1 \otimes r_2}\), so \(y \in (p_1(X))_{r_1}\). Hence, \((p_1(X))_{r_1} \in \mathcal{P}_{r_1}(G_1)\).
(ii) If \( X_{r_1 \otimes r_2} \in \mathcal{M}_{r_1 \otimes r_2}(G_1 \times G_2) \), we will prove that \((p_1(X))_{r_1}\) is a maximal \(r_1\)-ideal of \(G_1\). In fact, let \(Y_{r_1}\) be an integral \(r_1\)-ideal of \(G_1\) such that \((p_1(X))_{r_1} \subseteq Y_{r_1} \subseteq (1)\) and \(Y_{r_1} \neq (p_1(X))_{r_1}\). Then \((Y \times p_2(X))_{r_1 \otimes r_2}\) is an integral \(r_1 \otimes r_2\)-ideal of \(G_1 \times G_2\), such that \(X_{r_1 \otimes r_2} \subseteq (Y \times p_2(X))_{r_1 \otimes r_2} \subseteq (1)\) and \(X_{r_1 \otimes r_2} \neq (Y \times p_2(X))_{r_1 \otimes r_2}\). Thus, \((Y \times p_2(X))_{r_1 \otimes r_2} = (G_1 \times G_2)_1^+\), that means \(Y_{r_1} = G_1^1\). Hence, \((p_1(X))_{r_1} \in \mathcal{M}_{r_1}(G_1)\).

Assuming now that \(1 \notin (p_2(X))_{r_2}\), we can prove using analogous arguments that if \(X_{r_1 \otimes r_2}\) is a prime (resp. maximal) \(r_1 \otimes r_2\)-ideal of \(G_1 \times G_2\) then \((p_2(X))_{r_2}\) is a prime (resp. maximal) \(r_2\)-ideal of \(G_2\).

3 Radical of integral \(r\)-ideals

**Definition 3.1.** Let \(G\) be a directed group, \(r \in \mathcal{R}(G)\) and \(X_r\) an integral \(r\)-ideal. The set

\[
\tau(X_r) = \{x \in G^+| (\exists n \in \mathbb{N}) x^n \in X_r\}
\]

is called radical of \(X_r\).

**Proposition 3.2.** Let \(G\) be a directed group and \(r \in \mathcal{R}(G)\). For every \(X_r, Y_r \in \mathcal{A}_r(G)\) and every \(g \in G^+\), the following properties hold:

1. \(X_r \subseteq \tau(Y_r)\).
2. \(\tau(X_r) = G^+\) if and only if \(X_r = G^+\).
3. If \(X_r \subseteq \tau(Y_r)\), then \(\tau(X_r) \subseteq \tau(Y_r)\).
4. \(g \cdot \tau(X_r) \subseteq \tau((g \cdot X)_r)\).
5. \(\tau(X_r \times_r Y_r) = \tau(X_r \cap Y_r) = \tau(X_r) \cap \tau(Y_r)\).
6. \(\tau(X_r) = \bigcup_{x \in \tau(X_r)} \{x\}_r\).

**Proof.** The first two properties are directly derived from the definition of radical.

3. Let \(X_r \subseteq \tau(Y_r)\) and \(x \in \tau(X_r)\). Then, there exists \(n \in \mathbb{N}\) such that \(x^n \in X_r\), so \(x^n \in \tau(Y_r)\). Thus, there exists \(m \in \mathbb{N}\) such that \(x^{n-m} \in Y_r\), hence \(x \in \tau(Y_r)\).

4. Let \(x \in g \cdot \tau(X_r)\), that is \(x = g \cdot w\), where \(w \in \tau(X_r)\). Then, there exists \(n \in \mathbb{N}\) such that \(w^n \in X_r\). In case that \(n = 1\), so \(w \in X_r\), we have \(x = g \cdot (g \cdot X)_r\), thus \(x \in \tau((g \cdot X)_r)\). We suppose now that \(n > 1\). Then, \(x^n = g \cdot (g^{n-1} \cdot w^n)\) and \(g^{n-1} \cdot w^n \in \{w^n\}_r\), thus \(x^n \in g \cdot \{w^n\}_r \subseteq g \cdot X_r = (g \cdot X)_r\), that means that \(x \in \tau((g \cdot X)_r)\).
(5) Using the previous properties (1) and (3) in combination with the relation \(X_r \times Y_r \subseteq X_r \cap Y_r\) from proposition 2.1, it follows that \(\tau(X_r \times Y_r) \subseteq \tau(X_r \cap Y_r)\). Similarly, since \(X_r \cap Y_r \subseteq X_r\) and \(X_r \cap Y_r \subseteq Y_r\), there holds \(\tau(X_r \cap Y_r) \subseteq \tau(X_r) \cap \tau(Y_r)\). Moreover, if \(x \in \tau(X_r) \cap \tau(Y_r)\), then there exist \(m, n \in \mathbb{N}\) such that \(x^m \in X_r\) and \(x^n \in Y_r\). Then \(x^{m+n} \in X_r \cdot Y_r \subseteq (X_r \cdot Y_r)_r = X_r \times Y_r\), which means that \(x \in \tau(X_r \times Y_r)\), hence \(\tau(X_r) \cap \tau(Y_r) \subseteq \tau(X_r \times Y_r)\).

(6) Obviously \(\tau(X_r) \subseteq \bigcup_{x \in \tau(X_r)} \{x\}_r\). Conversely, let \(w \in \bigcup_{x \in \tau(X_r)} \{x\}_r\). Then, there exists \(x_w \in \tau(X_r)\) with \(w \in \{x_w\}_r\), which means that there exists \(n \in \mathbb{N}\) such that \(x_w^n \in X_r\) and \(w \geq x_w^n\). Hence \(w^n \in \{x_w^n\}_r \subseteq X_r\), thus \(w \in \tau(X_r)\).

\(\square\)

**Proposition 3.3.** Let \(G\) be a directed group and \(r, q \in \mathcal{R}(G)\). The radical of an integral \(r \oplus q\)-ideal \(X_{r \oplus q}\) of \(G\) is equal to \(\tau(X_r) \cap \tau(X_q)\).

**Proof.** Let \(X\) be a non-empty subset of \(G^+\). It is easily verified that \(\tau(X_{r \oplus q}) \subseteq \tau(X_r) \cap \tau(X_q)\). Conversely, if \(x \in \tau(X_r) \cap \tau(X_q)\), then there exist \(m, n \in \mathbb{N}\) such that \(x^m \in X_r\) and \(x^n \in X_q\). We suppose that \(m \leq n\). Then \(x^{n-m} \in G^+\) and \(x^n = x^{n-m} \cdot x^m \in x^{n-m} \cdot X_r \subseteq G^+ \cdot X_r = X_r\). Hence \(x^n \in X_{r \oplus q}\), so \(x \in \tau(X_{r \oplus q})\). Following the analogous procedure in case that \(n < m\) we conclude that \(\tau(X_r) \cap \tau(X_q) \subseteq \tau(X_{r \oplus q})\).

\(\square\)

**Proposition 3.4.** Let \(G_1\) and \(G_2\) be two directed groups, \(r_1 \in \mathcal{R}(G_1)\) and \(r_2 \in \mathcal{R}(G_2)\). If \(X_{r_1 \oplus r_2}\) is an integral \(r_1 \oplus r_2\)-ideal of \(G_1 \times G_2\), then \(\tau(X_{r_1 \circ r_2}) = \tau((p_1(X))_{r_1}) \times \tau((p_2(X))_{r_2})\).

**Proof.** Let \(X\) be a non-empty subset of \((G_1 \times G_2)^+\). If \(x = (x_1, x_2) \in \tau(X_{r_1 \circ r_2})\), then there exists \(k \in \mathbb{N}\) such that \(x^k \in X_{r_1 \circ r_2}\), that is \(x^k_1 \in (p_1(X))_{r_1}\) and \(x^k_2 \in (p_2(X))_{r_2}\). Thus, \(x_1 \in \tau((p_1(X))_{r_1})\) and \(x_2 \in \tau((p_2(X))_{r_2})\), that is \(x \in \tau((p_1(X))_{r_1}) \times \tau((p_2(X))_{r_2})\).

Conversely, if \(y = (y_1, y_2) \in \tau((p_1(X))_{r_1}) \times \tau((p_2(X))_{r_2})\), then there exist \(m, n \in \mathbb{N}\) such that \(y_1^n \in (p_1(X))_{r_1}\) and \(y_2^n \in (p_2(X))_{r_2}\). We suppose that \(m \leq n\). Then \(y_2^{n-m} \in G_2^+\) and \(y_2^n = y_2^{n-m} \cdot y_2^m \cdot (p_2(X))_{r_2} \subseteq (p_2(X))_{r_2} \cdot G_2^+ = (p_2(X))_{r_2}\), that is \(y_2^n \in (p_2(X))_{r_2}\). Thus \(y^n = (y_1^n, y_2^n) \in (p_1(X))_{r_1} \times (p_2(X))_{r_2} = X_{r_1 \circ r_2}\).

\(\square\)

**Proposition 3.5.** Let \(G_1\) and \(G_2\) be two directed groups, \(r_1 \in \mathcal{R}(G_1)\), \(r_2 \in \mathcal{R}(G_2)\) and \(f : G_1 \to G_2\) an \((r_1, r_2)\)-morphism. Then \(f(\tau(X_{r_1})) \subseteq \tau((f(X))_{r_2})\), for every integral \(r_1\)-ideal \(X_{r_1}\) of \(G_1\).
Proof. Let $X \subseteq G^+_1$ and $g \in f(\tau(X_{r_1}))$. There exist $a \in \tau(X_{r_1})$ such that $f(a) = g$. Moreover, there exist $n \in \mathbb{N}$ such that $a^n \in X_{r_1}$. Then $g^n \in f(X_{r_1}) \subseteq (f(X))_{r_2}$, that means $g \in \tau((f(X))_{r_2})$. \qed

An arising question is whether the radical of an arbitrary integral $r$-ideal is an $r$-ideal or not. A first observation (proposition 3.2(6)) is that every radical is an $s$-ideal. We also prove that the radical of any prime $r$-ideal $X_r$ equals to $X_r$, property that leads normally to the notion of half-prime $r$-ideal.

**Proposition 3.6.** Let $G$ be a directed group and $r \in \mathcal{R}(G)$. If $X_r$ is a prime $r$-ideal, then $\tau(X_r) = X_r$.

**Proof.** It is enough to prove that $\tau(X_r) \subseteq X_r$, for a prime $r$-ideal $X_r$. We suppose that there exists $x \in \tau(X_r)$ with $x \notin X_r$. Consider $n$ to be the smallest natural number such that $x^n \in X_r$. Then $n > 1$ and $x, x^{n-1} \in G^+$ with $x \cdot x^{n-1} \in X_r$ and $x \notin X_r$. Thus $x^{n-1} \in X_r$, which is absurd. \qed

**Definition 3.7.** Let $G$ be a directed group and $r \in \mathcal{R}(G)$. An integral $r$-ideal $X_r$ is called half-prime if it is equal to its radical. The $r$-system is called half-prime if every integral $r$-ideal is half-prime.

**Proposition 3.8.** Let $G$ be a directed group and $r \in \mathcal{R}(G)$. An integral $r$-ideal $X_r$ is half-prime if and only if the relation $a^2 \in X_r$ implies $a \in X_r$, for any $a \in G^+$.

**Proof.** Obviously, for every half-prime $r$-ideal $X_r$ and any $a \in G^+$ such that $a^2 \in X_r$, it follows $a \in X_r$. Conversely, let $X_r \in \mathcal{A}_r(G)$ satisfying that for any $a \in G^+$ the relation $a^2 \in X_r$ implies $a \in X_r$, and $g \in \tau(X_r)$. Then, there exists $n \in \mathbb{N}$ such that $g^n \in X_r$ and for a sufficiently large $m \in \mathbb{N}$, $g^{2^m} = g^{2^{m-n}} \cdot g^n \in X_r$. Using the giving condition repeatedly, we conclude that $g^2 \in X_r$ that is $g \in X_r$. So far, we have proved that $\tau(X_r) \subseteq X_r$, thus $X_r$ is half-prime. \qed

An $r$-ideal system is called of finite character ([4]) if $X_r = \bigcup_{K \subseteq X_r, K \neq \emptyset} K_r$ for every $r$-ideal. On a directed group $G$, there is always defined ([4]) the $v$-system as $X_v = \bigcap_{X \subseteq \{x\}}$, which leads to the definition of a system of finite character, called $t$-system, by $X_t = \bigcup_{Y \subseteq X, Y \neq \emptyset} Y_t$. Therefore, the set $\mathcal{R}_{fin}(G)$ of all $r$-ideal systems of finite character defined on $G$ is non-empty.

Considering a system $r \in \mathcal{R}_{fin}(G)$, we prove in theorem 3.10 that the radical of an integral $r$-ideal $X_r$ equals to the intersection of all prime ideals that contain $X_r$, from which we derive that every radical is an $r$-ideal. This result is analogous to Krull’s theorem ([8]) where the case of ordinary ideals in commutative rings is investigated.
Theorem 3.9. ([4]) Let $G$ be a directed group and $r \in \mathcal{R}_{f_{\text{in}}}(G)$. If $X_r$ is an integral $r$-ideal of $G$ and $S$ a closed under multiplication subset of $G^+$ such that $X_r \cap S = \emptyset$, then there exists a prime $r$-ideal $P_r$ which contains $X_r$ and $P_r \cap S = \emptyset$.

Theorem 3.10. Let $G$ be a directed group and $r \in \mathcal{R}_{f_{\text{in}}}(G)$. For every $X_r \in \mathcal{A}_r(G)$, $X_r \neq G^+$, there holds

$$\tau(X_r) = \bigcap_{Z_r \in P_r(G) \atop X_r \subseteq Z_r} Z_r.$$ 

Proof. Let $X_r \in \mathcal{A}_r(G)$, $X_r \neq G^+$. The set $\{1\}$ is a closed under multiplication subset of $G^+$ and $1 \notin X_r$. Then, from theorem 3.9, there exists a prime $r$-ideal that contains $X_r$. Thus, $\bigcap_{Z_r \in P_r(G) \atop X_r \subseteq Z_r} Z_r \neq \emptyset$. Let $Z_r \in P_r(G)$ with $X_r \subseteq Z_r$.

Using propositions 3.2 and 3.6, it follows that $\tau(X_r) \subseteq Z_r$. Hence,

$$\tau(X_r) \subseteq \bigcap_{Z_r \in P_r(G) \atop X_r \subseteq Z_r} Z_r.$$ 

Since $X_r \neq G^+$, there holds $\tau(X_r) \neq G^+$ that means $G^+ \setminus \tau(X_r) \neq \emptyset$. For every $g \in G^+ \setminus \tau(X_r)$ we consider the set $S_g = \{g^n \mid n \in \mathbb{N}\}$ which is a closed under multiplication subset of $G^+$ and $S_g \cap X_r = \emptyset$. Then, from theorem 3.9, it follows that there exists $(Z_g)_r \in P_r(G)$ such that $X_r \subseteq (Z_g)_r$ and $(Z_g)_r \cap S_g = \emptyset$, that is $(Z_g)_r \subseteq G^+ \setminus S_g$. Thus,

$$\bigcap_{Z_r \in P_r(G) \atop X_r \subseteq Z_r} Z_r \subseteq \bigcap_{g \in G^+ \setminus \tau(X_r)} (Z_g)_r \subseteq \bigcap_{g \in G^+ \setminus \tau(X_r)} G^+ \setminus S_g \subseteq \bigcap_{g \in G^+ \setminus \tau(X_r)} \bigcup_{g \in G^+ \setminus \tau(X_r)} S_g \subseteq \tau(X_r).$$ 

\[\square\]

Corollary 3.11. Let $G$ be a directed group and $r \in \mathcal{R}_{f_{\text{in}}}(G)$. If $X_r \in \mathcal{A}_r(G)$, $X_r \neq G^+$, then the set $\tau(X_r)$ is an integral $r$-ideal of $G$ and $\tau(\tau(X_r)) = \tau(X_r)$.

Proof. Let $X_r \in \mathcal{A}_r(G)$, $X_r \neq G^+$. The set $\tau(\tau(X_r))$ is well defined and non-empty, since $\tau(X_r)$ is an integral $r$-ideal as it equals to the intersection of all prime $r$-ideals that contain $X_r$. Let $g \in \tau(\tau(X_r))$. Then, there exists $n \in \mathbb{N}$ such that $g^n \in \tau(X_r)$, that means that there exists $m \in \mathbb{N}$ such that $(g^n)^m \in X_r$. Thus, $g \in \tau(X_r)$. \[\square\]
Remark 3.12. Considering a finitary weak ideal system $r$ on a monoid, it is known ([2]) that a new system can be defined having as ideal generated by $X$ the radical of $X_r$. In case of groups, the radical does not lead in general to the definition of a new system, not even when the system we began with is of finite character. Let $r$ be an arbitrary ideal system on a directed group $G$. It is not difficult to see that a necessary condition for the map $\tau_r : B(G) \to P(G)$, $\tau_r(X) = X_{\tau_r} = \tau(X_r)$, to be an ideal system on $G$ is that any integral principal $r$-ideal is equal to its radical. In fact, for every $x \in G^+$ the relation $\{x\}_{\tau(r)} = x \cdot G^+$ should be true, that is $\tau(\{x\}_r) = \{x\}_r$. This condition is not automatically satisfied by an $r$-system of finite character. For example, let $G$ be the directed group $\mathbb{Z} \times \mathbb{Z}$ with respect to pointwise addition and ordering. We consider the $t$-system on $G$, for which it is easy to verify that $X_t = \{\inf X\}_t$, for every $X \in B(G)$. Let

$$A_t = \{(4,4)\}_t = \{(a,b) \in G \mid a \geq 4 \text{ and } b \geq 4\}.$$ 

Then $(2,2) \in \tau(A_t)$, since $2 \cdot (2,2) = (4,4) \in A_t$, while $(2,2) \notin A_t$.

References


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