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On Symmetric Bi-Derivations of Incline Algebras

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Abstract. In this paper, we introduce the notion of a symmetric bi-derivation of an incline algebra and investigate some of its properties. We prove some result on symmetric bi-derivation of an incline and integral incline.

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1. Introduction

The notion of incline algebras was introduced by [14] and their applications were studied by [9, 10, 11, 13]. Inclines are a special type of a semiring, also are a generalization of both Boolean and fuzzy algebras, and they give a way to combine algebras and ordered structures to express the degree of intensity binary relations.

N. O. Alshehri [8] introduced the notion of derivation of an incline algebra and proved some result on derivation of an incline and integral incline. We [12] introduced the notion of an f-derivation of an incline algebra and investigated some of its properties.

The notion of the symmetric bi-derivation was defined in [1, 2] by Maksa and a lot of researchers studied the symmetric bi-derivation in rings, near-rings and lattices [3, 4, 5, 6, 7].

In this paper, we apply the notion of symmetric bi-derivation in rings, near rings and lattices to incline algebras. We introduce the concept of symmetric bi-derivation of incline algebras and investigate some of its properties. We prove that if D is a symmetric bi-derivation for an incline algebra R, and $a \in R$ such that $a * D(x,y) = 0$ or $D(x,y) * a = 0$ for all $x \in R$, then either $a = 0$ or $D = 0$. Also, we prove that for a commutative integral incline algebra R if there exists joinitive symmetric bi-derivations D_1 and D_2 such that $D_1(d_2(x), x) = 0$ for all $x \in R$ where d_1, d_2 denote the trace of D_1 and D_2 respectively, then either $d_1 = 0$ or $d_2 = 0$.

2. Preliminaries

Definition 2.1. [14] An incline algebra is a non-empty set R with binary operations denoted by $+$ and $*$ satisfying the following axioms for all $x, y, z \in R$:

(RI) $x + y = y + x$,

(RII) $x + (y + z) = (x + y) + z$,

(RIII) $x * (y * z) = (x * y) * z$,

(RIV) $x * (y + z) = (x * y) + (x * z)$,

(RV) $(y + z) * x = (y * x) + (z * x)$,

(RVI) $x + x = x$,

(RVII) $x + (x * y) = x$,

(RVIII) $y + (x * y) = y$.

Furthermore, an incline algebra R is said to be commutative if $x * y = y * x$ for all $x, y \in R$.

For convenience, we pronounce "$+$" (resp."$*$") as addition (resp. multiplication). Every distributive lattice is an incline. An incline is a distributive lattice (as a semiring) if and only if $x*x = x$ for all $x \in R$ ([3, Proposition (1.1.1)]). A subincline of an incline R is a nonempty subset M of R which is closed under addition and multiplication. An ideal in an incline R is a subincline $M \subseteq R$ such that if $x \in R$ and $y \leq x$ then $y \in M$. An element 0 in an incline algebra R is a zero element if $x + 0 = x = 0 + x$ and $x * 0 = 0 * x = 0$ for any $x \in X$. An element 1 ($\neq$ zero element) in an incline algebra R is called multiplicative identity if for any $x \in R$, $x * 1 = 1 * x = x$. A non-zero element a in an incline algebra R with a zero element is said to be a left (resp. right) zero divisor if there exists a non-zero element $b \in R$ such that $a * b = 0$($resp.$ $b * a = 0$). A zero divisor is an element of R which is both a left zero divisor and a right zero divisor. An incline algebra R with a multiplicative identity 1 and a zero element 0 is called an integral incline if it has no zero divisors.

Note that $x \leq y$ if and only if $x + y = y$ for all $x, y \in R$. It is easy to see that $\leq$ is a partial order on R and that for any $x, y \in R$, the element $x + y$ is the least upper bound of x, y. We say that $\leq$ is induced by operation $+$. It follows that

(1) $x * y \leq x$ and $x * y \leq y$ for all $x, y \in R$.

(2) $y \leq z$ implies $x * y \leq x$ and $y * x \leq z * x$ for any $x, y, z \in R$.

(3) If $x \leq y$, $a \leq b$, then $x + a \leq y + b$, $x * a \leq y * b$.

Definition 2.2. Let R be an incline algebra. A mapping $D(.,.) : R\times R \rightarrow R$ is called symmetric if $D(x, y) = D(y, x)$ holds for all $x, y \in R$.

Definition 2.3. Let R be an incline algebra. A mapping $d : R \rightarrow R$ defined by $d(x) = D(x, x)$ is called trace of $D(.,.)$, where $D(.,.) : R \times R \rightarrow R$ is a symmetric mapping.

3. The Symmetric Bi-Derivations on Incline Algebras

The following definition introduces the notion of symmetric bi-derivation for an incline algebra.

Definition 3.1. Let R be an incline algebra and $D : R \times R \rightarrow R$ be a symmetric mapping. We call D a symmetric bi-derivation on R, if it satisfies the following condition

$$D(x * y, z) = (D(x, z) * y) + (x * D(y, z))$$

for all $x, y, z \in R$.

Obviously, a symmetric bi-derivation D on R satisfies the relation $D(x, y* z) = (D(x, y) * z)) + (y * D(x, z))$ for all $x, y, z \in R$.

Example 3.1. *Let R be a commutative incline algebra and define a mapping on R by $D(x, y) = x*y$ for all $x, y \in R$. Then we can see that D is a symmetric bi-derivation on R.*

Example 3.2. *Let R be a commutative incline algebra and $a \in R$. Define a mapping on R by $D(x, y) = (x * y) * a$ for all $x, y \in R$. Then we can see that D is a symmetric bi-derivation on R.*

Proposition 3.2. *Let R be a commutative incline algebra and D be a symmetric bi-derivation on R. Then the following hold for all $x, y, z \in R$:*

1) $D(x*y,z) \leq D(x,z)+D(y,z)$.
2) If $x \leq y$, *then* $D(x*y,z) \leq y$.
3) If R *is a distributive lattice then* $D(x,y) \leq x$ *and* $D(x,y) \leq y$.

Proof:

1) Let $x,y,z \in R$. By using (1) we can write $D(x,z)*y \leq D(x,z)$ and $x*D(y,z) \leq D(y,z)$. Then by using (3) we get $(D(x,z)*y)+(x*D(y,z)) \leq D(x,z)+D(y,z)$. Hence we find that $D(x*y,z) \leq D(x,z)+D(y,z)$.

2) Let $x \leq y$. Then by using (3) and (1) we get $x*D(y,z) \leq y*D(y,z) \leq y$. Also by using (1) we can write $D(x,z)*y \leq y$.
Thereby, we have $D(x*y,z) = (D(x,z)*y)+(x*D(y,z)) \leq y+y$. Hence we get $D(x*y,z) \leq y$.

3) Let R be a distributive lattice, then $D(x,y) = D(x*x,y) = (D(x,y)*x)+(x*D(x,y))$ and so, $D(x,y)+x = ((D(x,y)*x)+(x*D(x,y)))+x = ((D(x,y)*x)+(D(x,y)*x))+x = (D(x,y)*x)+x$.

By using $R(8)$ we get $D(x,y)+x = x$. Therefore, we have $D(x,y) \leq x$. Similarly we can have $D(x,y) \leq y$.

Proposition 3.3. *Let* R *be a commutative incline algebra with a zero element and* d *be the trace of symmetric bi-derivation* D *of* R. *Then* $d(0) = 0$.

Proof: Let $x \in R$, then we can write
$d(0) = D(0,0) = D(x*0,0) = (D(x,0)*0)+(x*D(0,0)) = 0+(x*D(0,0)) = x* \ D(0,0)$.

If we take $x = 0$, then we get $d(0) = 0$.

Proposition 3.4. *Let* R *be a commutative incline algebra with a multiplicative identity and* d *be the trace of symmetric bi-derivation* D *of* R. *Then the following hold for all* $x,y \in R$:
1) $x*D(1,y) \leq D(x,y)$.
2) If $d(1) = 1$, *then* $x \leq D(x,1)$.

Proof:

1) Let $x \in R$, then we can write $D(x,y) = D(x*1,y) = (D(x,y)*1)+(x*D(1,y)) = D(x,y)+(x*D(1,x))$. Therefore, $x*D(1,y) \leq \ D(x,y)$.

2) It can be derived from 1).

Proposition 3.5. *Let Let* R *be a commutative integral incline and* D *be a symmetric bi-derivation of* R *and* a *be an element of* R. *Then for all* $x,y \in R$ *we have:*
1) $a*D(x,y) = 0$ *implies that* $a = 0$ *or* $D = 0$.
2) $D(x,y)*a = 0$ *implies that* $a = 0$ *or* $D = 0$

Proof:

1) Let $a*D(x,y) = 0$ for all $x,y \in R$. If we replace x by $x*z$ for $z \in R$ we get
$0 = a*D(x*z,y) = a*((D(x,y)*z)+(x*D(z,y)) = a*(D(x,y)*z)+a*$

$(x * D(z,y)) = a * (x * D(z,y))$. By putting $x = 1$ we have $a * (D(z,y)) = 0$. We know that R is an integral incline; i.e, it has no zero divisors, so $a = 0$ or $D(z,y) = 0$ for all $y, z \in R$. Therefore, we have $a = 0$ or $D = 0$.

2) Proof is similar with the previous one.

Definition 3.6. Let R be a commutative incline algebra and $D : R\times\ R \to\ R$ be a symmetric mapping. We call D a joinitive mapping if it satisfies

$$D(x+y,z) = D(x,z) + D(y,z)$$

for all $x, y, z \in R$.

Proposition 3.7. *Let R be a commutative integral algebra and d be the trace of jointive symmetric bi-derivation D of R. Then the following hold for all $x, y \in R$:*
1) $d(x+y) = d(x) + d(y) + D(x,y)$ and $d(x) + d(y) \leq d(x+y)$,
2) $D(x*y,x) \leq d(x)$,
3) D is an isotone symmetric bi-derivation of R.

Proof:
1) Let $x, y \in R$, then we have
$d(x+y) = D(x+y, x+y) = D(x, x+y) + D(y, x+y) = D(x,x) + D(x,y) + D(y,x) + D(y,y) = D(x,x) + D(y,y) + D(x,y)$. Hence we get $d(x+y) = d(x) + d(y) + D(x,y)$ and $d(x) + d(y) \leq d(x+y)$
.

2) Let $x, y \in R$ and by using $R(7)$ we can write
$d(x,x) = D(x,x) = D(x+(x*y),x) = D(x,x)+D(x*y,x)$. Hence we have $D(x*y,x) \leq d(x)$.

3) Let $(x,y) \leq (z,t)$ for $x, y, z, t \in R$ then we have $x + z = z$ and $y + t = t$, and $(x,y) + (z,t) = (z,t)$. Therefore, we can write that
$D(z,t) = D((x,y) + (z,t)) = D(x,y) + D(z,t)$. Hence we have $D(x,y) \leq D(z,t)$.

Proposition 3.8. *Let R be a commutative integral incline algebra. Suppose there exists joinitive symmetric bi-derivations D_1 and D_2 such that $D_1(d_2(x),x) = 0$ for all $x \in R$ where d_1, d_2 denote the trace of D_1 and D_2 respectively. In this case either $d_1 = 0$ or $d_2 = 0$.*

Proof: Let $D_1(d_2(x),x) = 0$ where d_1, d_2 denote the trace of symmetric bi-derivations D_1 and D_2, respectively. Then by using $(R7)$ we can write
$D_1(d_2(x) + (d_2(x) * x), x) = D_1(d_2(x), x) + D_1(d_2(x) * x, x) = D_1(d_2(x), x) * x + d_2(x) * D_1(x,x) = d_2(x) * d_1(x)$.
Since R is a commutative integral incline algebra we have $d_1 = 0$ or $d_2 = 0$.
Additionally, if we have a symmetric bi-derivation D such that $D(d(x), x) = 0$ for all $x \in R$ where d denote the trace of D. In this case we have $d = 0$.

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