The Numerical Index of the $L_{(p)}$ Space of Dimension Two

N. Faried, F. A. Ibrahim and A. A. Bakery

Department of Mathematics, Faculty of Science
Ain Shams University, Cairo, Egypt
n_faried@hotmail.com
awad_bakery@yahoo.com

Abstract. We give a lower bound for the numerical index of the two-dimensional real $L_{(p)}$ space, where $(p) = (p_1, p_2)$.

Keywords: numerical index, numerical radius, Nakano sequence space, $L_{(p)}$ -spaces.

1. Introduction

The numerical index of a Banach space is a constant relating the norm and the numerical radius of the (bounded linear) operators on the space. Let us start by recalling the relevant definitions. Given a Banach space $X$, we will write $X^*$ for its topological dual and $L(X)$ for the Banach algebra of all (bounded linear) operators on $X$. For an operator $T \in L(X)$, its numerical radius is defined as

$$\nu(T) := \sup \{|x^*(Tx)| : x^* \in X^*, x \in X, ||x^*|| = ||x|| = x^*(x) = 1 \}.$$ 

Finally, the numerical index of the Banach space $X$ is the constant defined by

$$n(X) := \inf \{\nu(T) : T \in L(X), ||T|| = 1 \}.$$ Obviously, $n(X)$ is the greatest constant
k ≥ 0 such that k||T||≤ν(T) for every T ∈ L(X). The concept of numerical index was introduced by G. Lumer in 1968 and it appeared for the first time in the 1970 paper [3] where it was deeply studied. Classical references here are the aforementioned paper [3] and the monographs by F. Bonsall and J. Duncan [1, 2] from the seventies. The reader will find the state-of-the-art on the subject in the recent survey paper [4] and references therein. We refer to all these references for background. Let us comment on some results regarding the numerical index which will be relevant in the sequel. First, it is clear that 0 ≤ n(X) ≤ 1 for every Banach space X. In the real case, these inequalities are the best possible. In the complex case one has 1/ e ≤ n(X) ≤ 1 and all of these values are possible. Let us also mention that n(X) ≤ n(X*), and that the reverse inequality does not always hold. Anyhow, when X is a reflexive space (in particular when X is finite-dimensional), one gets n(X) = n(X*). Second, there are some classical Banach spaces for which the numerical index has been calculated. For instance, the numerical index of L₁(μ) is 1, and this property is shared by any of its isometric preduals. In particular, C(K) has numerical index 1 for every compact space and the same is true for all finite-dimensional subspaces of C[0, 1]. For any real Hilbert space H of dimension greater than one it is known that n(H) = 0 (actually, there is T ∈ L(H) with ||T|| = 1 and ν(T) = 0). In the complex case, n(H) =1/2 for every Hilbert space of dimension greater than one. Finally, the numerical indices of those real two dimensional spaces whose unit balls are regular polygons have been recently achieved and can be expressed in terms of the number of vertices.

By Δ₀, we shall denote the space of all real or complex sequences and the set of natural numbers will denote by \( \mathbb{N} = \{1, 2, \ldots\} \). The Nakano sequence space \( \ell_{(p)} \) is defined by \( \ell_{(p)} = \{ x = (x_n) \in \Delta_0 : \sigma(\tau x) < \infty \text{ for some } \tau > 0 \} \), where \( \sigma(x) = \sum_{k=1}^{\infty} |x_k|^{p_k} \) and \( (p_k) \) is a sequence of positive real numbers with \( p_k \geq 1, \forall k \in \mathbb{N} \).

The space \( \ell_{(p)} \) is a Banach space with the norm
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$\|x\| = \inf\left\{ t > 0 : \sigma\left(\frac{x}{t}\right) \leq 1\right\}$. If $p = (p_k)$ is bounded, we can simply write

$\ell_{(p)} = \{ x \in \omega : \sum_{k=1}^{\infty} |x_k|^{p_k} < \infty \}$. Also, some geometric properties of $\ell(p)$ were studied in [6].

**Lemma 1:**

For any, $x \in \ell_{(p)}$ the following assertions are satisfied:

(i) If $\|x\| < 1$, then $\sigma(x) \leq \|x\|$.

(ii) If $\|x\| > 1$, then $\sigma(x) \geq \|x\|$.

(iii) $\|x\| = 1$ if and only if $\sigma(x) = 1$.

(iv) If $0 < r < 1$ and $\|x\| > r$, then $\sigma(x) > r^M$ where $M = \sup_p r$.

(v) If $r \geq 1$ and $\|x\| < r$, then $\sigma(x) < r^M$.

**Proof:** It can be proved with standard techniques in a similar way as in [6].

**2. Main results**

**Lemma 2:**

Let $1 < p_1, p_2 < \infty$, and $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ an operator in $L(\ell_{(p)}^{(2)})$. Then,

$$
\nu(T) = \max\left\{ \max_{r \in (0,1]} \frac{|a + dt^{p_2}| + |bt + ct^{p_2 - 1}|}{(1 + t^{p_2})^r}, \max_{r \in (0,1]} \frac{|d + at^{p_2}| + |ct + bt^{p_2 - 1}|}{(1 + t^{p_2})^r} \right\},
$$

where

$r = \min(1, \frac{1}{p_2} + \frac{1}{q_1} + \frac{1}{p_1} + \frac{1}{q_2})$ and $\frac{1}{p_i} + \frac{1}{q_i} = 1$, for $i = 1, 2$. In particular,

$$
\nu\left( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right) = \max_{r \in (0,1]} \frac{|t^{p_2 - 1} - t|}{(1 + t^{p_2})^r}.
$$
Proof

Take \( y \in S_{(p)}^{(2)} \) i.e., \( \|y\| = 1 \),
\[
y = \begin{bmatrix}
\frac{1}{(1 + t^{p_2})^{1/p_1}} \\
\varepsilon^t \\
\frac{1}{(1 + t^{p_2})^{1/p_1}}
\end{bmatrix},
\]
where \( \varepsilon \in \{1, -1\} \), and \( y^* \in \ell^{(2)}_{(q)} \) (i.e.,) \( \|y^*\| = 1 \).
\[
y^* = \begin{bmatrix}
\frac{1}{(1 + t^{p_2})^{1/q_1}} \\
\varepsilon t^{(p_2 - 1)} \\
\frac{1}{(1 + t^{p_2})^{1/q_1}}
\end{bmatrix}.
\]

For \( T \in \ell^{(2)}_{(p)} \), we have
\[
|y^*(Ty)| = \frac{a + t^{p_2}d}{(1 + t^{p_2})} + \frac{\varepsilon bt}{(1 + t^{p_2})^{1/p_1}} + \frac{\varepsilon t^{p_2 - 1}c}{(1 + t^{p_2})^{1/p_1 + 1/q_2}}.
\]

Since \( \nu(T) \geq |y^*(Ty)| \) we get
\[
\nu(T) \geq \max_{\varepsilon \in \{1, -1\}} \left| \frac{a + t^{p_2}d}{(1 + t^{p_2})} + \frac{\varepsilon bt}{(1 + t^{p_2})^{1/p_1}} + \frac{\varepsilon t^{p_2 - 1}c}{(1 + t^{p_2})^{1/p_1 + 1/q_2}} \right|.
\]

Since \( r = \min \left( 1, \frac{1}{p_2}, \frac{1}{q_1}, \frac{1}{p_1}, \frac{1}{q_2} \right) \) we get
\[
\nu(T) \geq \left| \frac{a + t^{p_2}d}{(1 + t^{p_2})} \right| + \left| \frac{bt}{(1 + t^{p_2})^{1/p_1}} \right| + \left| \frac{ct^{p_2 - 1}}{(1 + t^{p_2})^{1/p_1 + 1/q_2}} \right|.
\]

Replace \( a \to d \) and \( b \to c \) we have
\[
\nu(T) \geq \max_{\varepsilon \in \{0, 1\}} \left| \frac{a + dt^{p_2}}{(1 + t^{p_2})^{\varepsilon}} \right| + \left| \frac{ct^{p_2 - 1}}{(1 + t^{p_2})^{\varepsilon}} \right|.
\]

From (1) and (2) we get
\[
\nu(T) = \max \left\{ \max_{\varepsilon \in \{0, 1\}} \left| \frac{a + dt^{p_2}}{(1 + t^{p_2})^{\varepsilon}} \right|, \max_{\varepsilon \in \{0, 1\}} \left| \frac{ct^{p_2 - 1}}{(1 + t^{p_2})^{\varepsilon}} \right| \right\}.
\]
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**Theorem 3:**

Let $1 < p_1 \leq p_2 < \infty$ and $\frac{1}{p_i} + \frac{1}{q_i} = 1$, for $i = 1, 2$. Then in the real case,

$$n(\ell^{(2)}_{(p)}) \geq \max \left\{ \frac{1}{2^{p_2}}, \frac{1}{2^{q_2}} \right\} \max_{t \in [0,1]} \frac{|t^{p_2-1} - t|}{(1 + t^{p_2})^r} \quad \text{and for } \quad 1 < p_2 \leq p_1 < \infty$$

$$n(\ell^{(2)}_{(p)}) \geq \max \left\{ \frac{1}{2^{p_1}}, \frac{1}{2^{q_1}} \right\} \max_{t \in [0,1]} \frac{|t^{p_2-1} - t|}{(1 + t^{p_2})^r} .$$

**Proof:**

Let us start with the case in which $1 < p_1 \leq p_2 \leq 2$. We fix an operator $T: \ell^{(2)}_{(p)} \rightarrow \ell^{(2)}_{(p)}$. In order to estimate $\|T\|$, we observe that

$$\|T\| = \sup_{(x,y) \in S_{(p)}^{(2)}} \|Tx\| = \sup_{(x,y) \in S_{(p)}^{(2)}} \left\| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \right\|$$

$$= \sup_{(x,y) \in S_{(p)}^{(2)}} \|ax + by, cx + dy\| \leq \sup_{(x,y) \in S_{(p)}^{(2)}} (|ax + by|^{p_1} + |cx + dy|^{p_2})^{1/M} ,$$

where $M = \max (p_1, p_2)$, we get

$$\|T\| \leq \sup_{(x,y) \in S_{(p)}^{(2)}} (|ax + by|^{p_1/M} + |cx + dy|^{p_2/M}) \leq \sup_{(x,y) \in S_{(p)}^{(2)}} (|ax + by| + |cx + dy|)

\leq \sup_{(x,y) \in S_{(p)}^{(2)}} (|a| + |c| |x| + (|b| + |d|) |y|) ,\text{ so}

$$\|T\| \leq \max \{ |a| + |c|, |b| + |d| \} . \sup_{(x,y) \in S_{(p)}^{(2)}} (|x| + |y|).$$

Since $\sup_{(x,y) \in S_{(p)}^{(2)}} (|x| + |y|) = \sup(|x| + |y|; |x|^{p_1} + |y|^{p_2} = 1)$

$$= \sup_{(x,y) \in S_{(p)}^{(2)}} (|x|^{1/p_1} + |y|^{1/p_2}; |x| + |y| = 1) \leq \sup_{t \in [0,1]} \left[ t^{1/p_1} + (1 - t)^{1/p_2} \right] \leq \frac{1}{2^{q_1}} .$$

Then we get $\|T\| \leq 2^{q_1} \max \{ |a| + |c|, |b| + |d| \}$.  

(3)
To state the lower estimation for $\nu(T)$ we distinguish two cases. We assume first that $|a| + |c| \geq |b| + |d|$ and, by using Lemma 2, we obtain the following for each $t \in [0, 1]$:

$$
\nu(T) \geq \frac{|a| - |d| + |c| + t^{p_2} - |b| - t}{(1 + t^{p_2})^r}.
$$

$$
\nu(T) \geq (|a| + |c|) \cdot \frac{t^{p_2} - t}{(1 + t^{p_2})^r} + |a| \cdot \frac{1 - t^{p_2}}{(1 + t^{p_2})^r} + |a| \cdot \frac{t + |c| - |b| - |d|}{(1 - t^{p_2})^r}.
$$

$$
\nu(T) \geq \frac{t^{p_2} - t}{(1 + t^{p_2})^r}.
$$

If otherwise $|b| + |d| \geq |a| + |c|$ then, for each $t \in [0, 1]$, we get

$$
\nu(T) \geq \frac{|a| + |c|, |b| + |d|} \cdot \max_{t \in [0, 1]} \frac{|t^{p_2} - t|}{(1 + t^{p_2})^r}.
$$

$$
\nu(T) \geq (|b| + |d|) \cdot \frac{t^{p_2} - t}{(1 + t^{p_2})^r} + |d| \cdot \frac{1 - t^{p_2}}{(1 + t^{p_2})^r} +
$$

$$
\nu(T) \geq (|b| + |d|) \cdot \frac{t^{p_2} - t}{(1 + t^{p_2})^r} + |d| \cdot \frac{1 - t^{p_2}}{(1 + t^{p_2})^r} +
$$

$$
+ ((|a| - |b| - |c|) \cdot \frac{t}{(1 + t^{p_2})^r}.
$$
\[ \nu(T) \geq (|b| + |d|) \frac{t^{p_2 - 1}}{1 + t^{p_2}}. \]

\[ \nu(T) \geq \max \{|a| + |c|, |b| + |d|\} \left( \frac{|t^{p_2 - 1} - t|}{(1 + t^{p_2})^r} \right). \]

In both cases we can take maximum with \( t \in [0, 1] \) to obtain

\[ \nu(T) \geq \max \{|a| + |c|, |b| + |d|\} \cdot \max_{t \in [0, 1]} \left( \frac{|t^{p_2 - 1} - t|}{(1 + t^{p_2})^r} \right). \] (4)

From (3), (4), we get

\[ n(\ell^{(2)}_{(p)}) \geq \frac{1}{\frac{2^{p_2}}{t^{p_2}}} \max_{t \in [0, 1]} \left( \frac{|t^{p_2 - 1} - t|}{(1 + t^{p_2})^r} \right). \] (5)

which finishes the proof when \( 1 < p_1 \leq p_2 \leq 2 \).

If \( 2 \leq p_1 \leq p_2 < \infty \), we observe that \( 1 < q_1, q_2 \leq 2 \) and \( n(\ell^{(2)}_{(p)}) = n((\ell^{(2)}_{(p)})^*) = n(\ell^{(2)}_{(q)}) \).

By the result already proved, we have

\[ n(\ell^{(2)}_{(q)}) \geq \frac{1}{\frac{2^{p_2}}{t^{q_2}}} \max_{t \in [0, 1]} \left( \frac{|t^{q_2 - 1} - t|}{(1 + t^{q_2})^r} \right). \] (6)

Finally, the substitution \( s = t^{q_2 - 1} \) gives

\[
\max_{t \in [0, 1]} \left( \frac{|t^{q_2 - 1} - t|}{(1 + t^{q_2})^r} \right) = \max_{s \in [0, 1]} \left( \frac{|s^{p_2 - 1} - t|}{(1 + t^{p_2})^r} \right).
\]

In particular

\[
\nu\begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix} = \max_{t \in [0, 1]} \left( \frac{|t^{p_2 - 1} - t|}{(1 + t^{p_2})^r} \right).
\]

For \( 1 < p_2 \leq p_1 < \infty \), by the same techniques in a similar way we get

\[ n(\ell^{(2)}_{(p)}) \geq \max \left( \frac{1}{2^{p_1}}, \frac{1}{2^{p_2}} \right) \max_{t \in [0, 1]} \left( \frac{|t^{p_2 - 1} - t|}{(1 + t^{p_2})^r} \right). \]

Remark 4:

The above proof shows that, for every operator \( T \in L(\ell^{(2)}_{(p)}) \), \( 1 < p_1 \leq p_2 \leq 2 \), such that \( \|T\| \leq \max \{|a| + |c|, |b| + |d|\} \), one has
\[
\max_{t \in [0,1]} \left| t^{p_i-1} - t \right| \leq \frac{\nu(T)}{\| T \|}.
\]

**COROLLARY 5.** If \( p_1 = p_2 = p \), for \( 1 < p < \infty \) and \( \frac{1}{p} + \frac{1}{q} = 1 \). We get in the real case,

\[
\max \left\{ \frac{1}{2^{1/p}}, \frac{1}{2^{1/q}} \right\} \max_{t \in [0,1]} \left| t^{p-1} - t \right| \leq n(\ell^2_p). \]
See [5]

**References**


Received: March, 2011