On Direct Product of \((\lambda, \mu)\)-Fuzzy Subrings

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Abstract

In this paper the direct product of \((\lambda, \mu)\)-fuzzy subrings (ideals) has been discussed. We have proved that if the direct product of two \((\lambda, \mu)\)-fuzzy subsets is a \((\lambda, \mu)\)-fuzzy subring (ideal) then at least one of the \((\lambda, \mu)\)-fuzzy subsets must be a \((\lambda, \mu)\)-fuzzy subring (ideal).

Keywords: Direct product of \((\lambda, \mu)\)-fuzzy subsets, \((\lambda, \mu)\)-fuzzy subrings and \((\lambda, \mu)\)-fuzzy ideals

1 Introduction

The notions of fuzzy ideals were introduced by S-Abou-Zaid in 1991[1, 2]. Using the notation of a fuzzy subset introduced by Zadeh[13], W.Liu[4]
defined fuzzy set and fuzzy ideals of a ring. The notion of fuzzy subgroup was introduced by A. Rosenfeld [8] in his pioneering paper. Subsequently the definition of fuzzy subgroup was generalized by Negoita and Ralescu [5] and by Anthony and Sherwood [3] product of fuzzy subgroups were first defined by C.V. Negoita and D.A. Ralescu [5]. H. Sherwood [3] also studied product of fuzzy subgroups in a generalized form. Ray [7] also obtained some results on product of fuzzy subgroups. Fuzzy ideals of a ring were first introduced by Liu [11]. In this paper Liu [12] introduced the concept of operations of fuzzy ideals of a ring. Here we have obtained a result relating to direct product of fuzzy subrings (ideals). This is an extension of the result of Tazid Ali [10] to rings.

2 Preliminaries

Definition 2.1 [10] By a fuzzy subset of a set X, we mean a function from X into [0,1]. The set of all fuzzy subsets of X is called fuzzy power set of X and is denoted by $I^X = [0,1]^X$.

Definition 2.2 [10] A fuzzy subset $A$ of a ring $R$ is said to be a fuzzy subring of $R$ if $\forall a, b \in R$,

\[
A(a - b) \geq A(a) \land A(b) \\
A(ab) \geq A(a) \land A(b)
\]

The set of all fuzzy subrings of $R$ is denoted by $I(R)$.

Definition 2.3 [10] A fuzzy subset $A$ of a ring $R$ is called a fuzzy ideal of $R$ if $\forall a, b \in R$,

\[
A(a - b) \geq A(a) \land A(b) \\
A(ab) \geq A(a) \lor A(b)
\]

Definition 2.4 [10] Let $A, B$ be fuzzy subsets of the sets $X$ and $Y$ respectively. The product of $A$ and $B$, denoted by $A \times B$, is a fuzzy subset of $X \times Y$ defined as follows

\[(A \times B)(x, y) = A(x) \land B(y), \forall (x, y) \in X \times Y\].
3 On direct product of \((\lambda, \mu)\)-fuzzy subrings

Let \(R\) be a ring with the zero element 0 and \(S\) be a ring with the zero element \(0'\). The product of \(R\) and \(S\) of \((\lambda, \mu)\), denoted by \((R \times S) \vee \lambda\), is the set \((R \times S) \vee \lambda = \{(r, s) \land \mu; r \in R, s \in S\}\). Then \((R \times S) \vee \lambda\) is a ring where addition, multiplication and inverse are defined as

\[
((r_1, s_1) + (r_2, s_2)) \vee \lambda = (r_1 + r_2, s_1 + s_2) \land \mu
\]

\[
(r_1, s_1)(r_2, s_2) \vee \lambda = (r_1 r_2, s_1 s_2) \land \mu \quad \text{and}
\]

\[-(r, s) \vee \lambda = (-r, -s) \land \mu.
\]

Then zero element of \((R \times s) \vee \lambda\) is \((0, 0') \land \mu\).

**Definition 3.1** By a \((\lambda, \mu)\)-fuzzy subset of a set \(X\), we mean a function from \(X\) into \([0,1]\). The set of all \((\lambda, \mu)\)-fuzzy subsets of \(X\) is called \((\lambda, \mu)\)-fuzzy power set of \(X\) and is denoted by \(I^X_{(\lambda, \mu)} = [0,1]^X\).

**Definition 3.2** A \((\lambda, \mu)\)-fuzzy subset \(A\) of a ring \(R\) is said to be a \((\lambda, \mu)\)-fuzzy subring of \(R\) if \(\forall a, b \in R\),

\[
A(a - b) \vee \lambda \geq A(a) \land A(b) \land \mu
\]

\[
A(ab) \vee \lambda \geq (A(a) \land A(b)) \land \mu
\]

The set of all \((\lambda, \mu)\)-fuzzy subrings of \(R\) is denoted by \(I^{(R)}_{(\lambda, \mu)}\).

**Lemma 3.3** If \(A\) is a \((\lambda, \mu)\)-fuzzy subring of a ring \(R\) then

\[
A(0) \vee \lambda \geq A(r) \land \mu, \forall r \in R
\]

**Proof.** The proof is straightforward and omitted. ■

**Lemma 3.4** Let \(A \in I^{(R)}_{(\lambda, \mu)}\). Then \(A\) is a \((\lambda, \mu)\)-fuzzy subrings of \(R\) if and only if the level subset \(A_t\) is a subring of \(R\) \(\forall t \in \text{Im}(\mu)\).

**Proof.** The proof is straightforward and omitted. ■

**Definition 3.5** A \((\lambda, \mu)\)-fuzzy subset \(A\) of a ring \(R\) is called a \((\lambda, \mu)\)-fuzzy ideal of \(R\) if \(\forall a, b \in R\),

\[
A(a - b) \vee \lambda \geq A(a) \land A(b) \land \mu
\]

\[
A(ab) \vee \lambda \geq (A(a) \lor A(b)) \land \mu
\]
Theorem 3.6 A \((\lambda, \mu)\)-fuzzy subset \(A\) of a ring \(R\) is a \((\lambda, \mu)\)-fuzzy ideal of \(R\) if and only if the level subset \(A_t\) is an ideal of \(R\) \(\forall t \in \text{Im}(A)\).

\textbf{Proof.} The proof is straightforward and omitted. \hfill \blacksquare

\textbf{Definition 3.7} Let \(A, B\) be \((\lambda, \mu)\)-fuzzy subsets of the sets \(X\) and \(Y\) respectively. The product of \(A\) and \(B\), denoted by \((A \times B) \vee \lambda\), is a \((\lambda, \mu)\)-fuzzy subset of \((X \times Y) \vee \lambda\) defined as follows

\[((A \times B)(x, y)) \vee \lambda = (A(x) \wedge B(y)) \wedge \mu, \forall (x, y) \in X \times Y.\]

\textbf{Theorem 3.8} Let \(A\) be a \((\lambda, \mu)\)-fuzzy subring of the ring \(R\) and \(B\) be a \((\lambda, \mu)\)-fuzzy subring of the ring \(S\). Then \(A \times B\) is a \((\lambda, \mu)\)-fuzzy subring of the ring \(R \times S\).

\textbf{Proof.} Let \(((r_1, s_1), (r_2, s_2)) \vee \lambda \in (R \times S) \vee \lambda\). Then \(((r_1, s_1)(r_2, s_2)) \vee \lambda = ((r_1r_2, s_1s_2)) \wedge \mu\). Now,

\[((A \times B)((r_1, s_1) - (r_2, s_2))) \vee \lambda = [(A \times B)(r_1 - r_2, s_1 - s_2)] \vee \lambda = [A(r_1 - r_2) \wedge B(s_1 - s_2)] \wedge \mu = [A(r_1 - r_2) \wedge \mu] \wedge [B(s_1 - s_2) \wedge \mu] \geq [(A(r_1) \wedge A(r_2) \wedge \mu] \wedge [B(s_1) \wedge B(s_2) \wedge \mu] \wedge \mu = [A(r_1) \wedge B(s_1) \wedge A(r_2) \wedge B(s_2)] \wedge \mu = [(A \times B)(r_1, s_1) \wedge (A \times B)(r_2, s_2)] \wedge \mu

Also

\[((A \times B)((r_1, s_1)(r_2, s_2))) \vee \lambda = [(A \times B)((r_1r_2, s_1s_2))] \vee \lambda = [A(r_1r_2) \wedge B(s_1s_2)] \wedge \mu = [A(r_1r_2) \wedge \mu] \wedge [B(s_1s_2) \wedge \mu] \geq [A(r_1) \wedge A(r_2) \wedge \mu] \wedge [B(s_1) \wedge B(s_2) \wedge \mu] = [A(r_1) \wedge B(s_1) \wedge A(r_2) \wedge B(s_2)] \wedge \mu = [(A \times B)(r_1, s_1) \wedge (A \times B)(r_2, s_2)] \wedge \mu

Hence \(A \times B\) is a \((\lambda, \mu)\)-fuzzy subring of \(R \times S\). \hfill \blacksquare

\textbf{Theorem 3.9} Let \(A\) be a \((\lambda, \mu)\)-fuzzy ideal of the ring \(R\) and \(B\) be the \((\lambda, \mu)\)-fuzzy ideal of the ring \(S\). Then \((A \times B)\) is a \((\lambda, \mu)\)-fuzzy ideal of the ring \((R \times S)\).

However if \(A\) and \(B\) are \((\lambda, \mu)\)-fuzzy subsets of \(R\) and \(S\) respectively, such that \(A \times B\) is a \((\lambda, \mu)\)-fuzzy subring (ideal) of \(R \times S\) it is not necessarily true that both \(A\) and \(B\) are \((\lambda, \mu)\)-fuzzy subrings (ideals) of \(R\) and \(S\) respectively as is evident from the following example.
Example 3.10 Consider $X_1 = \{.2, .3\}$ and $X_2 = \{.2, .3, .4\}$ be a any set and $\lambda = .1$ and $\mu = .9$. Let $A$ and $B$ be $(\lambda, \mu)$-fuzzy subsets of $X_1$ and $X_2$ respectively given by $A = \{(2, .7), (.3, .6)\}$ and $B = \{(2, .8), (.3, .3), (.4, .7)\}$. The $(\lambda, \mu)$-fuzzy subset $A \times B$ of $R \times S$ is given by $A \times B = \{(2, .2), .7), ((2, .3), .7), ((2, .4), .7), ((3, .2), .6), ((3, .3), .6), ((3, .4), .6)\}$. Then $A \times B$ is a $(\lambda, \mu)$-fuzzy ideal of $R \times S$ but $B$ is not a $(\lambda, \mu)$-fuzzy ideal of $S$ as $B_1 = \{.3\}$ is not an ideal of $S$. We will show that if $A \times B$ is a $(\lambda, \mu)$-fuzzy subring (ideal) of $R \times S$ then at least one of $A$ or $B$ is a $(\lambda, \mu)$-fuzzy subring (ideal) of $R$ or $S$.

Theorem 3.11 Let $A$ and $B$ be the $(\lambda, \mu)$-fuzzy subsets of $R$ and $S$, respectively. If $(A \times B)$ is a $(\lambda, \mu)$-fuzzy subring of $R \times S$, then at least one of the following statements must hold.

\[
A(0) \lor \lambda \geq B(s) \land \mu, \forall s \in S \quad (i)
\]

\[
B(0') \lor \lambda \geq A(r) \land \mu, \forall r \in R \quad (ii)
\]

Proof. Since $R$ and $S$ are rings, $R \times S$ is also a ring. Let $A \times B$ be a $(\lambda, \mu)$-fuzzy subring of $R \times S$. By contraposition, suppose that none of the statements $(i)$ and $(ii)$ hold. Then we can find $r \in R$ and $s \in S$ such that $A(r) \lor \lambda > B(0') \land \mu$ and $B(s) \lor \lambda > A(0) \land \mu$. Now, $[(A \times B)(r, s)] \lor \lambda = [A(r) \land B(s)] \land \mu = [A(r) \lor \mu] \land [B(s) \lor \mu] > [B(0') \land \mu] \land [A(0) \land \mu] = [B(0') \land A(0)] \land \mu = [(A \times B)(0, 0')] \land \mu$. Thus $(r, s) \in (R \times S)$, $(0, 0')$ is the zero of the ring $R \times S$, and $A \times B$ is a $(\lambda, \mu)$-fuzzy subring of $R \times S$ satisfying $[(A \times B)(r, s)] \lor \lambda \geq [(A \times B)(0, 0')] \land \mu$. which contradicts the lemma 3.3 hence either $A(0) \lor \lambda \geq B(s) \land \mu, \forall s \in S$, $B(0') \lor \lambda \geq A(r) \land \mu, \forall r \in R$.

Theorem 3.12 Let $A$ and $B$ be $(\lambda, \mu)$-fuzzy subsets of $R$ and $S$, respectively. If $A \times B$ is a $(\lambda, \mu)$-fuzzy subring of $R \times S$, then either $A$ is a $(\lambda, \mu)$-fuzzy subring of $R$ or $B$ is a $(\lambda, \mu)$-fuzzy subring of $S$.

Proof. Suppose $A \times B$ is a $(\lambda, \mu)$-fuzzy subring of $R \times S$. Then by Theorem 3.11 one of the following statements hold:
\( A(0) \lor \lambda \geq B(s) \land \mu, \forall s \in S \quad (i) \)
\( B(0') \lor \lambda \geq A(r) \land \mu, \forall r \in R \quad (ii) \)

Suppose (ii) holds. Since \( R \) and \( S \) are rings, \( R \times S \) is also a ring with zero element \((0, 0')\). Let \( x, y \in R \), then \((x, 0'), (y, 0') \in R \times S\). Now using the property \( B(0') \lor \lambda \geq A(r) \land \mu, \forall r \in R \), we have, \( \forall x, y \in R, A(x - y) \lor \lambda = [A(x - y) \land B(0')] \land \mu = [A(x - y) \land B(0') - 0'] \land \mu \)
\( = [(A \land B)(x - y, 0' - 0')] \land \mu \)
\( = [(A \land B)(x, 0') \land (A \land B)(0', 0')] \land \mu = [(A \land B)(x, 0') \land (A \land B)(0', 0')] \land \mu = [(A \land B)(x, 0') \land (A \land B)(0', 0')] \land \mu \)
\( = [A(x) \land A(0')] \land \mu. \)

Also \( A(xy) \lor \lambda = [A(xy) \land B(0')] \land \mu = [(A \land B)(xy, 0'0')] \land \mu = [(A \land B)(xy, 0'0')] \land \mu \)
\( \geq [(A \land B)(x, 0') \land (A \land B)(y, 0')] \land \mu = [(A \land B)(x, 0') \land (A \land B)(y, 0')] \land \mu \)
\( \geq [(A \land B)(x, 0') \land (A \land B)(y, 0')] \land \mu. \)

Hence \( A \) is a \((\lambda, \mu)\)-fuzzy subring of \( R \).
Similarly we can show that, using the property \( A(0) \lor \lambda \geq B(s) \land \mu, \forall s \in S, \)
\( B \) is a \((\lambda, \mu)\)-fuzzy subring of \( S \).

**Theorem 3.13** Let \( A \) and \( B \) be fuzzy subsets of \( R \) and \( S \), respectively. If \( A \times B \) is a \((\lambda, \mu)\)-fuzzy ideal of \( R \times S \), then either \( A \) is a \((\lambda, \mu)\)-fuzzy ideal of \( R \) or \( B \) is a \((\lambda, \mu)\)-fuzzy ideal of \( S \).

**Proof.** We have already shown that under the stated condition either \( A \) is a \((\lambda, \mu)\)-fuzzy subring of \( R \) or \( B \) is a \((\lambda, \mu)\)-fuzzy subring of \( S \). Suppose \( A \) is a \((\lambda, \mu)\)-fuzzy subring of \( R \). We will show that \( A \) is a \((\lambda, \mu)\)-fuzzy ideal of \( R \).

Now using the property \( B(0') \lor \lambda \geq A(r) \land \mu, r \in R \), we have, \( x, y \in R, A(xy) \lor \lambda = [A(xy) \land B(0')] \land \mu = [(A \land B)(xy, 0'0')] \land \mu = [(A \land B)(xy, 0'0')] \land \mu \)
\( \geq [(A \land B)(x, 0') \land (A \land B)(y, 0')] \land \mu \)
\( \geq [(A \land B)(x, 0') \land (A \land B)(y, 0')] \land \mu. \)

Hence \( A \) is a \((\lambda, \mu)\) fuzzy ideal of \( R \).

**Corollary 3.14** Let \( A_1, A_2, ..., A_n \) be \((\lambda, \mu)\)-fuzzy subsets of the rings \( R_1, R_2, ..., R_n \) respectively. If \( A_1 \times A_2 \times ... \times A_n \) is a \((\lambda, \mu)\)-fuzzy subring(ideal) of \( R_1 \times R_2 \times ... \times R_n \), then atleast for one \( i, A_i(0_i) \lor \lambda \geq A_k(x) \land \mu, \forall x \in R_k, k = 1, 2, ... n \) where \( 0_i \) denotes the zero element of \( R_i \).

**Corollary 3.15** Let \( A_1, A_2, ..., A_n \) be \((\lambda, \mu)\)-fuzzy subsets of the rings \( R_1, R_2, ..., R_n \) respectively.
On direct product of \((\lambda, \mu)\)-fuzzy subrings respectively. If \(A_1 \times A_2 \times \ldots \times A_n\) is a \((\lambda, \mu)\)-fuzzy subring(ideal) of \(R_1 \times R_2 \times \ldots \times R_n\), then atleast for one \(i\), \(A_i\) is a \((\lambda, \mu)\)-fuzzy subring of \(R_i\).

References

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