Best Proximity Points for Semi-Cyclic Contractive Pairs in Banach Spaces

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Abstract

We introduce a new notion of semi-cyclic contractiveness for a pair \((S, T)\) of mappings in a Banach space. We then prove a theorem on the existence and convergence of best proximity points for a semi-cyclic contraction pair \((S, T)\). In case that \(S = T\), the semi-cyclic contraction pair reduces to the cyclic contraction map already considered by Eldred and Veeramani in [3].

1 Introduction

Let \(A\) and \(B\) be two nonempty closed subsets of a complete metric space \(X = (X, d)\). A self mapping \(T\) on \(A \cup B\) is said to be cyclic if \(T(A) \subseteq B\) and \(T(B) \subseteq A\). Assume further that there exists \(0 < \alpha < 1\) such that

\[d(Tx, Ty) \leq \alpha d(x, y) \quad x \in A, y \in B.\]

It follows that \(A \cap B \neq \emptyset\) and that the cyclic map \(T\) has a unique fixed point in \(A \cap B\) (see [2]). According to [3] a self mapping \(T\) on \(A \cup B\) is said to be a cyclic contraction if \(T\) is cyclic and satisfies

\[\exists 0 < \alpha < 1; d(Tx, Ty) \leq \alpha d(x, y) + (1 - \alpha)dist(A, B), \quad x \in A, y \in B. \quad (1)\]

Note that the condition (1) does not entail that \(A \cap B \neq \emptyset\), therefore it makes no sense to ask for a fixed point of \(T\), however, one may ask for a best proximity point, that is, a point \(x\) in \(A \cup B\) such that \(d(x, Tx) = dist(A, B)\).

In [3] Eldred and Veeramani proved the following theorem:
Theorem 1.1. ([3]) Let $X$ be a uniformly convex Banach space and let $A, B$ be two nonempty closed convex subsets of $X$ and $T$ be a cyclic contraction. Then $T$ has a unique best proximity point, that is there is $x \in A \cup B$ such that $\|x - Tx\| = \text{dist}(A, B)$. Furthermore, if $x_0 \in A$ and $x_{n+1} = Tx_n$, $n \geq 0$, then $\{x_{2n}\}$ converges to the best proximity point.

In the following definition, we introduce the new concept of a semi-cyclic contraction pair.

Definition 1.2. Let $A, B$ be two nonempty closed subsets of a complete metric space $(X, d)$, and let $S, T$ be two self maps on $A \cup B$. We call $(S, T)$ a semi-cyclic contraction pair if the following conditions hold:

1. $S(A) \subseteq B$, $T(B) \subseteq A$;
2. $\exists 0 < \alpha < 1 : d(Sx, Ty) \leq \alpha d(x, y) + (1 - \alpha)\text{dist}(A, B)$, $x \in A, y \in B$.

Note that in case $S = T$, a semi-cyclic contraction pair reduces to a cyclic contraction, a notion already studied by Eldred and Veeramani in [3], so that, in a sense, the result we obtain (see Theorem 3.9) generalizes Theorem 1.1.

2 Preliminaries

We begin by recalling some basic definitions which are necessary for our future work. For $A, B$ subsets of a metric space $(X, d)$, we define

$$d(x, A) = \inf\{d(x, y) : y \in A\},$$

and

$$\text{dist}(A, B) = \inf\{d(x, y) : x \in A, y \in B\}.$$

It is well known that if $A$ is a nonempty closed convex set in a reflexive Banach space, then $A$ contains an element with minimum norm. Moreover, if $X$ is strictly convex, then this element is unique. It follows that if $A, B$ are two nonempty closed convex sets in a uniformly convex Banach space $X$, then there are unique $a \in A$ and $b \in B$ such that $\|a - b\| = \text{dist}(A, B)$.

3 The main result

This section is devoted to the main result of the paper. In the following proposition we introduce the iterative sequence which will approximate the best proximity points.
Proposition 3.1. Let \((S, T)\) be a semi-cyclic contraction pair. Consider \(x_0 \in A\) and define:

\[
\begin{cases}
x_{n+1} = Ty_n, \\
y_n = Sx_n,
\end{cases}
\quad n = 0, 1, 2, \ldots
\]

Then \(\{x_n\}\) and \(\{y_n\}\) are sequences in \(A, B\), respectively. Moreover

\[
d(x_n, Sx_n) \to \operatorname{dist}(A, B), \quad d(y_n, Ty_n) \to \operatorname{dist}(A, B).
\]

Proof. First we note that

\[
d(x_n, Sx_n) = d(Ty_n, Sx_n) \leq \alpha d(y_{n-1}, x_n) + (1 - \alpha) \operatorname{dist}(A, B)
= \alpha d(Sx_{n-1}, Ty_{n-1}) + (1 - \alpha) \operatorname{dist}(A, B)
\leq \alpha (\alpha d(x_{n-1}, y_{n-1}) + (1 - \alpha) \operatorname{dist}(A, B)) + (1 - \alpha) \operatorname{dist}(A, B)
= \alpha^2 d(x_{n-1}, Sx_{n-1}) + (1 - \alpha^2) \operatorname{dist}(A, B)
\leq \cdots \leq \alpha^n d(x_0, x_1) + (1 - \alpha^n) \operatorname{dist}(A, B).
\]

Therefore \(d(x_n, Sx_n) \to \operatorname{dist}(A, B)\). Similarly, it can be shown that \(d(y_n, Ty_n)\) converges to \(\operatorname{dist}(A, B)\). \hfill \Box

Proposition 3.2. Let \((S, T)\) be semi-cyclic contraction pair. Consider the iterative sequences defined by (2). If both \(\{x_n\}\) and \(\{y_n\}\) have a convergent subsequence in \(A\) and \(B\), respectively, then there exist \(x \in A\) and \(y \in B\) such that

\[
d(x, Sx) = \operatorname{dist}(A, B) = d(y, Ty).
\]

Proof. Let \(\{y_{n_k}\}\) be a subsequence of \(\{y_n\}\) such that \(y_{n_k} \to y\). Since

\[
dist(A, B) \leq d(Ty_{n_k}, y) \leq d(y, y_{n_k}) + d(y_{n_k}, Ty_{n_k}),
\]

it follows from Proposition 3.1 that \(d(y, Ty_{n_k}) \to \operatorname{dist}(A, B)\). On the other hand

\[
dist(A, B) \leq d(Ty, y_{n_k}) = d(Ty, Sx_{n_k}) \leq \alpha d(y, x_{n_k}) + (1 - \alpha) \operatorname{dist}(A, B)
= \alpha d(y, Ty_{n_k-1}) + (1 - \alpha) \operatorname{dist}(A, B).
\]

Now we let \(k \to \infty\) to conclude that \(d(Ty, y) = \operatorname{dist}(A, B)\). Similarly, it can be proved that \(d(x, Sx) = \operatorname{dist}(A, B)\). \hfill \Box

Proposition 3.3. Let \((S, T)\) be a semi-cyclic contraction pair. Then the iterative sequences defined by (2) are bounded.
Proof. Since \(d(x_n, Sx_n) \to \text{dist}(A, B)\), it is enough to show that the sequence \(\{Sx_n\}\) is bounded in \(B\). Suppose not, then there exists \(N_0\) such that
\[
d(x_1, Sx_{N_0}) > M \text{ and } d(x_1, Sx_{N_0-1}) \leq M
\]
where \(M > \max\left(\frac{2d(x_0, Sx_0)}{1/\alpha^2 - 1} + \frac{1}{1-\alpha^2} \text{dist}(A, B), d(x_1, Sx_0)\right)\). Since \((S, T)\) is a semi-cyclic contraction pair, we have
\[
M < d(x_1, Sx_{N_0}) = d(Ty_0, Sx_{N_0}) \leq \alpha d(y_0, x_{N_0}) + (1 - \alpha) \text{dist}(A, B)
\]
\[
\leq \alpha^2 d(x_0, y_{N_0-1}) + (1 - \alpha^2) \text{dist}(A, B).
\]
Therefore
\[
(1/\alpha^2)(M - (1 - \alpha^2) \text{dist}(A, B)) < d(x_0, Sx_{N_0-1}),
\]
and consequently
\[
(1/\alpha^2)(M - \text{dist}(A, B)) + \text{dist}(A, B) < d(x_0, Sx_{N_0-1})
\]
\[
\leq d(x_0, Ty_0) + d(Ty_0, Sx_{N_0-1}) \leq d(x_0, Ty_0) + M
\]
\[
\leq d(x_0, Sx_0) + d(Sx_0, Ty_0) + M
\]
\[
\leq d(x_0, Sx_0) + \alpha d(x_0, y_0) + (1 - \alpha) \text{dist}(A, B) + M
\]
\[
\leq 2d(x_0, Sx_0) + \text{dist}(A, B) + M.
\]
This says that
\[
M(1/\alpha^2 - 1) < 2d(x_0, Sx_0) + 1/\alpha^2 \text{dist}(A, B)
\]
or equivalently
\[
M < \frac{2d(x_0, Sx_0)}{1/\alpha^2 - 1} + \frac{1}{1-\alpha^2} \text{dist}(A, B),
\]
which is a contradiction. The boundedness of \(\{y_n\}\) in \(B\) is proved similarly. \(\square\)

**Definition 3.4.** A subset \(K\) of a metric space \((X, d)\) is said to be boundedly compact if each bounded sequence in \(K\) has a subsequence converging to a point in \(K\).

**Theorem 3.5.** Let \((S, T)\) be a semi-cyclic contraction pair in a complete metric space \(X\). If either \(A\) or \(B\) is boundedly compact, then there exists \(z \in A \cup B\) such that either \(d(z, Sz) = \text{dist}(A, B)\) or \(d(z, Tz) = \text{dist}(A, B)\).

**Proof.** The result is an immediately consequence of Propositions 3.2 and 3.3. \(\square\)
Corollary 3.6. Let \((S, T)\) be a semi-cyclic contraction pair in a Banach space \(X\). If either the span of \(A\) or the span of \(B\) is a finite dimensional subspace of \(X\) then there exists \(z \in A \cup B\) such that either \(d(z, Tz) = \text{dist}(A, B)\) or \(d(z, Sz) = \text{dist}(A, B)\).

Proof. Since every finite dimensional subspace is boundedly compact, the result follows from Theorem 3.5. \(\square\)

Lemma 3.7. ([3]) Let \(A\) be a nonempty closed convex subset, and \(B\) be a nonempty closed subset of a uniformly convex Banach space \(X\). Let \(\{x_n\}, \{z_n\}\) be two sequences in \(A\), and \(\{y_n\}\) be a sequence in \(B\) such that

(i) \(\|z_n - y_n\| \to \text{dist}(A, B)\),

(ii) \(\forall \varepsilon > 0, \exists N_0\) such that for all \(m > n \geq N_0\) : \(\|x_m - y_n\| \leq \text{dist}(A, B)\).

Then for every \(\varepsilon > 0\) there exists \(N_1\) such that for all \(m > n \geq N_1\) we have \(\|x_m - z_n\| \leq \varepsilon\).

Lemma 3.8. ([3]) Let \(A\) be a nonempty closed convex subset, and \(B\) be a nonempty closed subset of a uniformly convex Banach space \(X\). Let \(\{x_n\}, \{z_n\}\) be two sequences in \(A\), and \(\{y_n\}\) be a sequence in \(B\) such that

(i) \(\|x_n - y_n\| \to \text{dist}(A, B)\),

(ii) \(\|z_n - y_n\| \to \text{dist}(A, B)\).

Then \(\|x_n - z_n\| \to 0\).

Now time is ripe for the main result of this paper.

Theorem 3.9. Let \(A, B\) be two nonempty closed convex subsets of a uniformly convex Banach space \(X\). Let \((S, T)\) be a semi-cyclic contraction pair.

(i) if \(\text{dist}(A, B) = 0\), then \(S, T\) have a unique common fixed point in \(A \cap B\).

(ii) if \(\text{dist}(A, B) > 0\), then each mapping has a unique best proximity point.

Moreover either of fixed point or best proximity points can be approximated by some iterative sequences.

Proof. Assume that \(\text{dist}(A, B) = 0\). Then we have

\(\forall x \in A, \forall y \in B : \|Sx - Ty\| \leq \alpha \|x - y\|.\)
Define a sequence \( \{z_n\}_{n \geq 1} \) in \( A \cup B \) in the following manner:

\[
z_n = \begin{cases} 
Ty_k & n = 2k \\
Sx_k & n = 2k - 1.
\end{cases}
\]

We show that \( \{z_n\} \) is a Cauchy sequence in \( A \cup B \). If \( n = 2k \) we have

\[
\|z_{n+1} - z_n\| = \|Sx_{k+1} - Ty_k\| \leq \alpha \|x_{k+1} - y_k\| = \alpha \|Ty_k - Sx_k\| \leq \alpha^2 \|y_k - x_k\| \leq \cdots \leq \alpha^{2k} \|y_1 - x_1\| \rightarrow 0, \quad k \rightarrow \infty.
\]

Similarly, for \( n = 2k - 1 \), we can get the same conclusion, so that for \( m > n \) we have

\[
\|z_m - z_n\| \leq \sum_{k=n}^{m-1} \alpha^{2k} \|y_1 - x_1\| \rightarrow 0, \quad n, m \rightarrow \infty.
\]

Then there exists \( z \in A \cup B \) such that \( z_n \rightarrow z \). Assume that \( z \in A \). Since \( \{z_{2k-1}\} \subseteq B \), it follows that \( z \in B \), and finally \( z \in A \cap B \). In case that \( z \in B \), the same argument again shows that \( z \in A \cap B \). On the other hand,

\[
\|z - Tz\| = \lim_k \|y_k - Tz\| = \lim_k \|Sx_k - Tz\| \leq \lim_k \alpha \|x_k - z\| = 0.
\]

This implies that \( Tz = z \). Similarly, we see that \( Sz = z \). Hence \( T, S \) have a common fixed point. We claim that the fixed point \( z \) is unique. In fact if \( Tw = w = Sw \) for some \( w \in A \cap B \), then \( \|z - w\| = \|Tz - Sw\| \leq \alpha \|z - w\| \).

This implies that \( z = w \).

Assume now that \( \text{dist}(A, B) > 0 \). Since \( (S, T) \) is a semi-cyclic contraction pair, we have

\[
\|y_n - Ty_n\| = \|Sx_n - Ty_n\| \leq \alpha \|x_n - y_n\| + (1 - \alpha)\text{dist}(A, B).
\]

This together with Proposition 3.1 implies that \( \|y_n - x_{n+1}\| \rightarrow \text{dist}(A, B) \).

Similarly, we see that \( \|y_{n+1} - x_{n+1}\| \rightarrow \text{dist}(A, B) \). Now it follows from Lemma 3.8 that \( \|y_n - y_{n+1}\| \rightarrow 0 \). Similarly \( \|x_n - x_{n+1}\| \rightarrow 0 \). Now we claim that for every \( \varepsilon > 0 \) there exists \( N_0 \) such that for all \( m > n > N_0 \) we have \( \|y_m - Ty_n\| = \|y_m - x_{n+1}\| \leq \text{dist}(A, B) + \varepsilon \). Suppose not; then there exists \( \varepsilon > 0 \) such that for all \( k \geq 1 \) there exist \( m_k > n_k \geq k \) for which \( \|y_{m_k} - Ty_{n_k}\| \geq \text{dist}(A, B) + \varepsilon \). This \( m_k \) can be chosen in such a way that it is the least integer greater than \( n_k \) to satisfy the above inequality. Now

\[
\text{dist}(A, B) + \varepsilon \leq \|y_{m_k} - Ty_{n_k}\| \leq \|y_{m_k} - y_{m_k-1}\| + \|y_{m_k-1} - Ty_{n_k}\| \leq \|y_{m_k} - y_{m_k-1}\| + \text{dist}(A, B) + \varepsilon.
\]
Hence \( \|y_{m_k} - Ty_{n_k}\| \to dist(A, B) + \varepsilon \). Then

\[
\|y_{m_k} - Ty_{n_k}\| \leq \|y_{m_k} - y_{m_k+1}\| + \|y_{m_k+1} - Ty_{n_k+1}\| + \|Ty_{n_k+1} - Ty_{n_k}\|
\]

\[
\leq \|y_{m_k} - y_{m_k+1}\| + \alpha^2\|y_{m_k} - Ty_{n_k}\| + (1 - \alpha^2)dist(A, B) + \|Ty_{n_k+1} - Ty_{n_k}\|
\]

If in the above inequality \( k \to \infty \), we obtain

\[
dist(A, B) + \varepsilon \leq \alpha^2(dist(A, B) + \varepsilon) + (1 - \alpha^2)dist(A, B) = dist(A, B) + \alpha^2\varepsilon
\]

which is a contradiction. Therefore \( \{y_n\} \) is a Cauchy sequence by Lemma 3.7. Therefore we can find a \( y \in B \) such that \( \{y_n\} \) converges to \( y \). It now follows from Proposition 3.2 that \( \|y - Ty\| = dist(A, B) \). Similarly we can prove that the sequence \( \{x_n\} \) is convergent to some \( x \in A \) and \( \|x - Sx\| = dist(A, B) \).

As for the uniqueness let \( w \in A \) is such that \( \|w - Sw\| = dist(A, B) \). Since

\[
dist(A, B) \leq \|TSx - Sx\| \leq \alpha\|Sx - x\| + (1 - \alpha)dist(A, B) = dist(A, B),
\]

it follows that \( \|TSx - Sx\| = \|x - Sx\| \). This in turn entails \( TSx = x \). Similarly, we see that \( TSw = w \). Now if \( w \neq x \), then \( \|x - Sw\| > dist(A, B) \) from which we obtain

\[
\|Sx - w\| = \|Sx - TSw\| \leq \alpha\|x - Sw\| + (1 - \alpha)dist(A, B)
\]

\[
< \alpha\|x - Sw\| + (1 - \alpha)\|x - Sw\|
\]

\[
= \|x - Sw\| = \|TSx - Sx\|
\]

\[
\leq \alpha\|Sx - w\| + (1 - \alpha)dist(A, B)
\]

\[
\leq \|Sx - w\|,
\]

which is a contradiction. \( \square \)

In the following example we shall see that if the space \( X \) is not uniformly convex, then the uniqueness of best proximity point may fail.

**Example.** Let \( X = \mathbb{R}^2 \) and for all \( (x, y) \in \mathbb{R}^2 \) define \( \|(x, y)\| = \max\{|x|, |y|\} \). Let \( A = \{(x, y) \in \mathbb{R}^2 : 1/2 \leq x \leq 1, y = 0\} \), \( B = \{(x, y) \in \mathbb{R}^2 : x = 0, 1 \leq y \leq 2\} \). Clearly \( A \) and \( B \) are closed and \( dist(A, B) = 1 \). Define \( S, T : A \cup B \to A \cup B \) by

\[
S(x, y) = \begin{cases} (0, 1), & (x, y) \in A \\ (x, y), & (x, y) \in B \end{cases}, \quad T(x, y) = \begin{cases} (y/2, 0), & (x, y) \in B \\ (x, y), & (x, y) \in A \end{cases}
\]

Obviously \( S(A) \subseteq B \), \( T(B) \subseteq A \). Note also that neither \( S \) nor \( T \) is cyclic. On the other hand if \( b = (0, y) \in B \) and \( a = (x, 0) \in A \) then

\[
\|Tb - Sa\| = \|T(0, y) - S(x, 0)\| = \|(y/2, 1)\| = 1.
\]

Similarly \( \|a - b\| = \max\{x, y\} = y \). Therefore

\[
\|T(b) - S(a)\| = 1 \leq (1/2)|y| + 1/2 = (1/2)\|b - a\| + (1/2)dist(A, B).
\]
References


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