Abstract. The intent of this paper is to introduce the notion of converse commuting maps and prove a common fixed point theorem for single and set valued maps without considering the completeness of the space and continuity of maps in fuzzy metric space using an implicit relation.

Mathematics Subject Classification: 47H10, 54H25.

Keyword and Phrases. Converse commuting maps, fixed points and fuzzy metric space

1. Introduction

Lii [1] introduced the concept of converse commuting maps which discuss the relation from the reverse and proved fixed point theorems for single valued maps.
in metric spaces. Recently, Qi-Kaun Liu and Xin-qi-Hu [2] and V.Pop a [5] introduced the new concept of converse commuting multivalued mappings and proved some fixed point theorems for converse commuting multivalued mappings. The main purpose of our paper is to introduce the concept of converse commuting maps in fuzzy metric space and to prove common fixed point theorem for single and multi valued maps under strict contractive condition in fuzzy metric spaces.

Our improvements in this paper are four-fold as;

(i) Relaxed the continuity of maps completely
(ii) Completeness of the space removed
(iii) Minimal type contractive condition used
(iv) The condition \( \lim_{t \to \infty} M(x, y, t) = 1 \) not used

We first give some preliminaries and definitions.

### 1. Preliminaries

**Definition 2.1.** A binary operation \( \ast : [0,1] \times [0,1] \to [0,1] \) is continuous \( t \)-norm if *

is satisfying the following conditions:

(i) \( \ast \) is commutative and associative

(ii) \( \ast \) is continuous

(iii) \( a \ast 1 = a \) for all \( a \in [0,1] \)

(iv) \( a \ast b \leq c \ast d \) whenever \( a \leq c \) and \( b \leq d \), \( a, b, c, d \in [0,1] \).

**Definition 2.2.** A 3-triplet \((X, M, \ast)\) is said to be a fuzzy metric space if \(X\) is an arbitrary set, \( \ast \) is a continuous \( t \)-norm and \(M\) is a fuzzy set on \(X^2 \times (0, \infty)\) satisfying the following;

\( \text{(FM-1) } M(x, y, t) > 0 \)

\( \text{(FM-2) } M(x, y, t) = 1 \text{ if and only if } x = y. \)

\( \text{(FM-3) } M(x, y, t) = M(y, x, t) \)

\( \text{(FM-4) } M(x, y, t) \ast M(y, z, s) \leq M(x, z, t+s) \)

\( \text{(FM-5) } M(x, y, \bullet) : (0, \infty) \to (0,1] \text{ is continuous.} \)

Throughout the paper \(X\) will represent the fuzzy metric space \((X, M, \ast)\) and
$CB(X)$, the set of all non-empty closed and bounded sub-sets of $X$. For $A, B \in CB(X)$ and for every $t > 0$, denote
$H(A,B,t) = \sup\{M(a,b,t); a \in A, b \in B\}$ and
$\delta_M(A,B,t) = \inf\{M(a,b,t); a \in A, b \in B\}$.
If $A$ consists of a single point $a$, we write $\delta_M(A,B,t) = \delta_M(a,B,t)$. If $B$ also consists of a single point $b$, we write $\delta_M(A,B,t) = M(a,b,t)$

**Definition 2.3.** A point $x \in X$ is called a commuting point of $A: X \to X$, $B: X \to CB(X)$ if $ABx = BAx$

**Definition 2.4.** Maps $A: X \to X$ and $B: X \to CB(X)$ are said to be converse commuting if $ABx = BAx$ implies $Ax = Bx$.
Let $C(A, S)$ denotes the set of converse commuting points of $A$ and $S$.

### 3. Main Result

**Theorem 1.** Let $(X,M,*)$ be a fuzzy metric space with $t* t = t$ for all $t \in [0,1]$ and let $A,B : X \to X$ and $S , T : X \to CB(X)$ be single and set valued mappings respectively such that the pairs $(A,S)$ and $(B,T)$ are converse commuting maps satisfying

\[
\int_0^t (\phi(t)dt > \int_0^t \phi(t)dt)
\]

$\forall x,y \in X, k \in (0,1)$ where $\phi:[0,1] \to [0,1]$ is a function which is summable, Lebesque integrable, non-negative and such that $\int_0^t \phi(t)dt > 0$ for each $\epsilon > 0$, and

\[
m(x,y,t) = \min \{M(Ax,By,t), H(Ax,Sx,t), H(By,Ty,t), H(Ax,Ty,at)*H(By,Sx,(2-\alpha)t)\}
\]

$\forall x,y \in X, t \in (0,1)$ and $\alpha \in (0,2)$. If $A$ and $S$, $B$ and $T$ have a commuting point, then $A, B, S$ and $T$ have a unique common fixed point in $X$.

**Proof.** Let $u \in C(A,S)$ and $v \in C(B,T)$, therefore, $ASu = SAu \Rightarrow Au \in Su$, hence $M(Au,Su,t) = 1$, also $Au \in Su \Rightarrow AAu \in SAu$, and hence $M(AAu, SAu,t) = 1$. Similarly, $BTv = TBv \Rightarrow Bv \in Tv$ therefore, $M(Bv,Tv,t) = 1$ and so $M(BBv,TBv,t) = 1$.
First we prove that $Au = Bv$. Using (1.2) for $x = Au, y = Bv$
(1.3) \( m(Au, Bv, t) \geq \min \left\{ \begin{array}{l}
M(AAu, BBv, t), H(AAu, SAu, t), H(BBv, TBv, t), \\
H(AAu, TBv, \alpha t) * H(BBv, SAu, (2 - \alpha)t) \end{array} \right\} \)

Since \( * \) is continuous, letting \( \alpha \to 1 \) in (1.3), we get
\[ m(Au, Bv, t) \geq \min \left\{ M(A^2u, B^2v, t), H(AAu, SAu, t), H(BBv, TBv, t), \\
H(AAu, TBv, t) * H(BBv, SAu, t) \right\} \]

As \( Au \in Su \) so \( AAu \in ASu = SAu \), \( Bv \in Tv \) so \( BBv \in BTv = TBv \) and \( M(A^2u, B^2v, t) \geq \delta_M(SAu, TBv, t) \) hence

(1.4) \( m(Au, Bv, t) \geq \min \left\{ \delta_M(SAu, TBv, t), 1,1, \\
\delta_M(SAu, TBv, t) * \delta_M(SAu, TBv, t) \right\} = \delta_M(SAu, TBv, t) \)

From (1.1) and (1.4), we have
\[ \int_0^\delta \phi(t) dt < \int_0^\delta \phi(t) dt \geq \int_0^\delta \phi(t) dt, \text{a contradiction. Hence } Au = Bv. \]

Now, we claim that \( Au = u \). It not, then considering (1.2) for \( Au = x, u = y \)
\[ m(Au, u, t) \geq \min \left\{ M(A^2u, Bu, t), H(A^2u, SAu, t), H(Bu, Tu, t), \\
H(A^2u, Tu, t) * H(Bu, SAu, t) \right\} \]

\[ \geq \min \left\{ \delta_M(SAu, Tu, t), 1,1, \\
\delta_M(SAu, Tu, t) * \delta_M(SAu, Tu, t) \right\} = \delta_M(SAu, Tu, t) \]

(1.5) \( M(Au, u, t) \geq \delta_M(SAu, Tu, t) \)

From (1.1) and (1.5) we have
\[ \int_0^\delta \phi(t) dt > \int_0^\delta \phi(t) dt \geq \int_0^\delta \phi(t) dt, \text{which is again a contradiction and} \]

hence \( Au = u \). Similarly, we can get \( Bv = v \). Thus \( A, B, S \) and \( T \) have a common fixed point in \( X \) and uniqueness of fixed point follows easily from (1.1) and hence the theorem

Now, we furnish our theorem with an example.

Example 1. Let \((X, M, *)\) be a fuzzy metric space in which \( X = R^+ \),
\( a * b = \min\{a, b\} \) for all \( a, b \in [0,1] \) such that such that \( M(x, y, t) = \frac{t}{t + |x - y|} \) \( \forall t > 0 \)

Define the maps \( A, B : X \to X \) and \( S, T : X \to CB(X) \) as follows;
Here the pairs \((A,S)\) and \((B,T)\) are conversely commuting. Define \(\phi : [0,1] \rightarrow [0,1]\) as \(\phi(0)=0, \phi(1)=1\) and \(\phi(s) = \sqrt{s}\) for \(0 < s < 1\), then the contractive condition (1.1) is satisfied for all \(t > 0\). Thus all the conditions of our theorem are satisfied and '1' is the common fixed point of \(A, B, S, T\).

**Corollary 1.** Let \((X,M,\ast)\) be a fuzzy metric space with \(t \ast t = t\) for all \(t \in [0,1]\) and let \(A,B : X \rightarrow X\) and \(S,T : X \rightarrow CB(X)\) be single and set valued mappings respectively such that the pairs \((A,S)\) and \((B,T)\) are converse commuting maps satisfying (1.1) and

\[
(1.6) \quad m(x,y,t) \geq \min \left\{ M(Ax,By,t), H(Ax,Sx,\alpha t), H(By,Ty,(2-\alpha)t) \right\},
\]

\[
H(Ax,Ty,\beta t) \ast H(By,Sx,((2-\beta)t)
\]

\[\forall x, y \in X, t \in (0,1) \text{ and } \alpha, \beta \in (0,2).\] If \(A\) and \(S\), \(B\) and \(T\) have a commuting point. Then \(A, B, S, T\) have unique common fixed point in \(X\).

If we take \(A=B\) and \(S=T\) in above result, then we get the following corollary

**Corollary 2.** Let \((X,M,\ast)\) be a fuzzy metric space with \(t \ast t = t\) for all \(t \in [0,1]\) and let \(A : X \rightarrow X\) and \(S : X \rightarrow CB(X)\) be single and set valued mappings respectively such that the pairs \((A,S)\) is converse commuting maps satisfying

\[
(1.7) \quad \int_{0}^{\infty} \phi(t)dt > \int_{0}^{\infty} \phi(t)dt
\]

\[\forall x, y \in X, k \in (0,1) \text{ where } \phi : [0,1] \rightarrow [0,1] \text{ is a function which is summable, Lebesque integrable, non-negative and such that } \int_{0}^{\epsilon} \phi(t)dt > 0 \text{ for each } \epsilon > 0.\]

where

\[
(1.8) \quad m(x,y,t) \geq \min \left\{ M(Ax,Ay,t), H(Ax,Sx,\alpha t), H(Ay,Sy,(2-\alpha)t), \right\},
\]

\[
H(Ax,Ty,\beta t) \ast H(Ay,Sx,((2-\beta)t)\right\)
\]

\[\forall x, y \in X, t \in (0,1) \text{ and } \alpha, \beta \in (0,2).\] If \(A\) and \(S\) have a commuting point. Then \(A\) and \(S\) have unique common fixed point in \(X\).
References


Common fixed point theorems

Received: October, 2010