Common Fixed Point Theorems for Occasionally Weakly Compatible Maps in Fuzzy Metric Spaces

M. Alamgir Khan
Department of Mathematics, Eritrea Institute of Technology
Asmara, Eritrea (N. E. Africa)
alam3333@gmail.com

Sumitra
Department of Mathematics, Eritrea Institute of Technology
Asmara, Eritrea (N. E. Africa)
mathsqueen_d@yahoo.com

Renu Chugh
Department of Mathematics, M.D.U, Rohtak-Haryana, India
chughrenu@yahoo.com

Abstract. The intent of this paper is to introduce the notion of occasionally weakly compatible (owc) maps and prove common fixed point theorems for single and set valued maps by using a contractive condition of integral type in fuzzy metric spaces. Our results are independent of the continuity requirement of the maps and completeness of the space. Several known results are generalized in this note.

Mathematics Subject Classification: 47H10, 54H25

Keywords: Weak Compatible maps, Occasionally weakly compatible maps, fixed points and fuzzy metric space
1. Introduction

The concept of fuzzy set coined by Zadeh [15] proved a turning point in the development of Mathematics and laid the foundation of fuzzy mathematics. Consequently, the last three decades remained productive for various authors like Deng [9], Erceg [10], Kaleva and Siekkala [12], Kramosil and Michalek [11], Pant [13], S. Sessa [14] etc.

In recent years several fixed point theorems for single and set valued maps are proved and have numerous applications and by now, there exists a considerable and rich literature in this domain.

Various authors have discussed and studied extensively various results on coincidence, existence and uniqueness of fixed and common fixed points by using the concept of weak commutativity, compatibility, non-compatibility and weak compatibility for single and set valued maps satisfying certain contractive conditions in different spaces and they have been applied to diverse problems.

Note that common fixed point theorems for single and set valued maps are interesting and play a major role in many areas.

More recently, Al-Thagafi and N. Shahzad [2] weakened the concept of compatibility by giving a new notion of occasionally weakly compatible (owc) maps which is most general among all the commutativity concepts. No wonder that the notion of occasionally weakly compatible (owc) maps has become an area of interest for specialists in fixed point theory; see [1], [3], [4], [5], [6], [7] and [8]

The main purpose of our paper is to introduce the concept of occasionally weakly compatible (owc) maps in fuzzy metric space and to prove common fixed point theorems for single and set valued maps under strict contractive condition of integral type.

Our improvements in this paper are five-fold as;

(i) Relaxed the continuity of maps completely

(ii) Completeness of the space removed

(iii) Minimal type contractive condition used

(iv) The condition \( \lim_{t \to \infty} M(x, y, t) = 1 \) not used

(v) Weakened the concept of compatibility by a more general concept of occasionally weak compatible (owc) maps.
Common fixed point theorems

2. Preliminaries

Definition 2.1. A binary operation \( \ast : [0,1] \times [0,1] \rightarrow [0,1] \) is continuous \( t \)-norm if \( \ast \) satisfies the following conditions:

(i) \( \ast \) is commutative and associative
(ii) \( \ast \) is continuous
(iii) \( a \ast 1 = a \) for all \( a \in [0,1] \)
(iv) \( a \ast b \leq c \ast d \) whenever \( a \leq c \) and \( b \leq d \), \( a, b, c, d \in [0,1] \).

Definition 2.2. A triplet \( (X, M, \ast) \) is said to be a fuzzy metric space if \( X \) is an arbitrary set, \( \ast \) is a continuous \( t \)-norm and \( M \) is a fuzzy set on \( X \times [0, \infty) \) satisfying the following:

(FM-1) \( M(x, y, t) > 0 \)
(FM-2) \( M(x, y, t) = 1 \) if and only if \( x = y \).
(FM-3) \( M(x, y, t) = M(y, x, t) \)
(FM-4) \( M(x, y, t) \ast M(y, z, s) \leq M(x, z, t + s) \)
(FM-5) \( M(x, y, \ast) : [0, \infty) \rightarrow [0,1] \) is left continuous for all \( x, y, z \in X \) and \( s, t > 0 \).

Note that \( M(x, y, t) \) can be thought of as the degree of nearness between \( x \) and \( y \) with respect to \( t \).

Example 1. (Induced fuzzy metric space) Let \( (X, d) \) be a metric space and \( a \ast b = ab \) for all \( a, b \in [0,1] \) and let \( M_d \) be fuzzy set on \( X \times [0, \infty) \) defined as follows;

\[
M_d(x, y, t) = \frac{t}{t + d(x, y)}.
\]

Then \( (X, M_d, \ast) \) is a fuzzy metric space. We call this fuzzy metric induced by a metric \( d \).

Throughout the paper \( X \) will represent the fuzzy metric space \( (X, M, \ast) \) and \( CB(X) \), the set of all non-empty closed and bounded sub-sets of \( X \). For \( A, B \in CB(X) \) and for every \( t > 0 \), denote

\[
H(A, B, t) = \sup \{M(a, b, t); a \in A, b \in B\}
\]
and \( \delta_d (A,B,t) = \inf \{ M(a,b,t); a \in A, b \in B \} \).

If \( A \) consists of a single point \( a \), we write \( \delta_d (A,B,t) = \delta_d (a,b,t) \). If \( B \) also consists of a single point \( b \), we write \( \delta_d (A,B,t) = M(a,b,t) \).

It follows immediately from definition that

\[
\delta_d (A,B,t) = \delta_d (B,A,t) \geq 0
\]

\[
\delta_d (A,B,t) = 1 \iff A = B = \{a\} \text{ for all } A,B \in CB(X).
\]

**Definition 2.3.** A point \( x \in X \) is called a coincidence point (resp. fixed point) of \( A : X \to X \), \( B : X \to CB(X) \) if \( Ax \in Bx \) (resp. \( x = Ax \in Bx \)).

**Definition 2.4.** Maps \( A : X \to X \) and \( B : X \to CB(X) \) are said to be compatible if \( ABx \in CB(X) \) for all \( x \in X \) and \( \lim_{n \to \infty} H(ABx_n, BAx_n, t) = 1 \) whenever \( \{x_n\} \) is a sequence in \( X \) such that \( Bx_n \to M \in CB(x) \) and \( Ax_n \to x \in M \).

**Definition 2.5.** Maps \( A : X \to X \) and \( B : X \to CB(X) \) are said to be weakly compatible if they commute at coincidence points, i.e., if \( ABx = BAx \), whenever \( Ax \in Bx \).

**Definition 2.6.** Maps \( A : X \to X \) and \( B : X \to CB(X) \) are said to be occasionally weakly compatible (owc) if there exists some point \( x \in X \) such that \( Ax \in Bx \) and \( ABx \subseteq BAx \).

Clearly weakly compatible maps are occasionally weakly compatible (owc). However, the converse is not true in general as shown in the following example.

**Example 2.** Let \( X = [0, \infty) \) with \( a \ast b = \min \{a,b\} \) for all \( a,b \in [0,1] \) and

\[
M(x,y,t) = \frac{t}{t + d(x,y)} \text{ for all } t > 0.
\]

Define the maps \( A : X \to X \) and \( B : X \to CB(X) \) by setting

\[
Ax = \begin{cases} 
0, 0 \leq x < 1 \\
\{0\}, 0 \leq x < 1 \\
\{0\}, x + 1, 1 \leq x < \infty \end{cases}, \quad Bx = \begin{cases} 
\{0\}, 0 \leq x < 1 \\
[1,x + 2], 1 \leq x < \infty
\end{cases}
\]

Here '1' is a coincidence point of \( A \) and \( B \) but \( A \) and \( B \) are not weakly compatible as
Common fixed point theorems

But A and B are occasionally weakly compatible (owc) as A and B are weakly compatible at \( x = 0 \) as \( A(0) \subseteq B(0) \) and \( AB(0) \subseteq BA(0) \). i.e., \( A \{ 0 \} = 0 \subseteq B(0) = \{ 0 \} \)

3. Main Result

Now, we prove our main result.

**Theorem 1.** Let \((X, M, *)\) be a fuzzy metric space with \( t * t = t \) for all \( t \in [0,1] \). Let \( A, B : X \to X \) and \( S, T : X \to CB(X) \) be single and set valued mappings respectively such that the maps \((A,S)\) and \((B,T)\) are (owc) and satisfy the inequality

\[
\delta_m(Sx Ty, t) \geq m(x, y) \int_0^t \phi(t) \, dt > \int_0^t \phi(t) \, dt \quad \text{for all } x, y \in X, \quad k \in (0,1) \quad \text{where } \phi : [0,1] \to [0,1]\]

is a function which is sum able, Lebesque integrable, non-negative and such that

\[
\int_0^\varepsilon \phi(t) \, dt > 0 \quad \text{for each } \varepsilon > 0 .
\]

where

\[
(1.2) \quad m(x, y, t) = \min \left\{ M \left( A x, B y, t \right), H \left( A x, S x, t \right), \frac{1}{2} M \left( H \left( B y, T y, t \right), H \left( A x, T y, t \right) \right) \right\}
\]

for every \( x, y \in X, t > 0, \alpha \in (0,2) \). Then A, B, S and T have unique common fixed point in X.

**Proof.** Since the pairs \((A,S)\) and \((B,T)\) are occasionally weakly compatible (owc)

maps, therefore, there exist two elements \( u, v \) in X such that \( Au \subseteq Su \), \( ASu \subseteq SAu \)

and \( Bv \subseteq Tv \) , \( BTv \subseteq TBv \).

First we prove that \( Au = Bv \).

As \( Au \subseteq Su \) so \( AAu \subseteq ASu \subseteq SAu \), \( Bv \subseteq Tv \) so \( BBv \subseteq BTv \subseteq TBv \) and hence

\[
M \left( A u, B v, t \right) \geq \delta_m \left( SAu, TBv, t \right) \quad \text{and if } Au \neq Bv, \quad \text{then } \delta_m \left( SAu, TBv, t \right) < 1. \quad \text{Using}
\]
(1.2) for \( x = Au, y = Bv \)

\[
m(Au, Bv, t) = \min \left\{ M(AAu, BBv, t), H(AAu, SAu, t), H(BBv, TBv, t), \right. \\
\left. H(AAu, TBv, \alpha t) * H(BBv, SAu, (2 - \alpha)t) \right\}
\]

\[
\geq \min \left\{ M(AAu, BBv, t), M(AAu, SAu, t), M(BBv, TBv, t), \right. \\
\left. M(AAu, TBv, \alpha t) * M(BBv, SAu, (2 - \alpha)t) \right\} \tag{1.3}
\]

Since \( * \) is continuous, letting \( \alpha \to 1 \) in (1.3), we get

\[
m(Au, Bv, t) \geq \min \left\{ M(A^2u, B^2v, t), M(A^2u, SAu, t), \\
M(B^2v, TBv, t), M(A^2u, TBv, t) * M(B^2v, SAu, t) \right\}
\]

\[
\geq \min \left\{ \delta_M(SAu, TBv, t), 1, \delta_M(SAu, TBv, t) * \delta_M(TBv, SAu, t) \right\}
\]

\[
= \delta_M(SAu, TBv, t) \tag{1.4}
\]

From (1.1) and (1.4), we have

\[
\delta_M(SAu, TBv, t) \int_0^m(Au, Bv, t) \geq \int_0^m(Au, Bv, t) \geq \int_0^m \phi(t) dt \geq 0, \text{ a contradiction.}
\]

Hence \( Au = Bv \).

Also, \( M(A^2u, Bu, t) \geq \delta_M(SAu, Tu, t), M(A^2u, Tu, t) \geq \delta_M(SAu, Tu, t) \).

Now, we claim that \( Au = u \). It not, then \( \delta_M(SAu, Tu, t) < 1 \).

Considering (1.2) for \( Au = x, u = y, \alpha = 1 \)

\[
m(Au, u, t) = \min \left\{ M(AAu, Bu, t), H(AAu, SAu, t), H(Bu, Tu, t), \right. \\
\left. H(AAu, Tu, t) * H(Bu, SAu, t) \right\}
\]

\[
= \min \left\{ M(A^2u, Bu, t), H(A^2u, SAu, t), H(Bu, Tu, t), \right. \\
\left. H(A^2u, Tu, t) * H(Bu, SAu, t) \right\}
\]
Common fixed point theorems

\[ \min \left\{ M\left(A^2u,Bu,t\right), M\left(A^2u,SAu,t\right), M\left(Bu,Tu,t\right), M\left(A^2u,Tu,t\right) * M\left(Bu,SAu,t\right) \right\} \]

\[ \geq \min \left\{ \delta^M(SAu,Tu,t), 1, 1, \delta^M(SAu,Tu,t) * \delta^M(Tu,SAu,t) \right\} = \delta^M(SAu,Tu,t) \]

i.e., \[ m(Au,u,t) \geq \delta^M(SAu,Tu,t) \] (1.5)

From (1.1) and (1.5) we have

\[ \int_0^\infty \phi(t)dt > \int_0^\infty \phi(t)dt \geq \int_0^\infty \phi(t)dt, \] which is again a contradiction and hence \( Au = u \).

Similarly, we can get \( Bv = v \).

Thus A, B, S and T have a common fixed point in X.

For uniqueness let \( u \neq u' \) be another fixed point of A, B, S and T, then (1.2) gives

\[ m(u,u',t) = \min \left\{ M\left(Au,Bu',t\right), H\left(Au,Su,t\right), H\left(Bu',Tu',t\right), H\left(Au,Tu',\alpha t\right) * H\left(Bu',Su,(2-\alpha)t\right) \right\} \]

Letting \( \alpha \to 1 \),

\[ m(u,u',t) = \min \left\{ M\left(Au,Bu',t\right), \delta^M(Au,Su,t), \delta^M(Bu',Tu',t), \delta^M(Au,Tu',t) * \delta^M(Bu',Su,t) \right\} \]

\[ m(u,u',t) = \min \left\{ M\left(Su,Tu',t\right), 1, 1, \delta^M(Su,Tu',t) \right\} \]

\[ = \delta^M(Su,Tu',t) \] (1.6)

Again from (1.1) and (1.6), we obtain
\[ \delta_{\mu} \{ Su, Tu, \delta_t \} \geq m_{(u, w, x)} \cdot \delta_{\mu} \{ Su, Tu, \delta_t \} \]

Which yields \( Su = Tu \), i.e., \( u = u' \).

Thus, A, B, S and T have unique common fixed point.

Now, we furnish our theorem with an example.

**Example 3.** Let \( (X, M, *) \) be a fuzzy metric space in which \( X = R^+ \), \( a * b = \min \{a, b\} \) for all \( a, b \in [0,1] \) such that such that \( M (x, y, t) = \frac{t}{x+y} \) for all \( t > 0 \).

Define the maps A, B, S and T on X as follows;

\[
A(x) = \begin{cases} 2x - 1, & x \leq 5 \\ 2x, & x > 5 \end{cases}, \quad B(x) = \begin{cases} 3 - 2x, & x \leq 1 \\ x + 1, & x > 1 \end{cases},
\]

\[
S(x) = \begin{cases} 1, & x < 2 \\ [2x, 2x + 5], & x \geq 2 \end{cases}, \quad T(x) = \begin{cases} 1, & x = 1 \\ [x, x + 2], & \text{otherwise} \end{cases}
\]

Here the pairs (A, S) and (B, T) are owc.

Define \( \phi : [0,1] \to [0,1] \) as \( \phi(0) = 0 \), \( \phi(1) = 1 \) and \( \phi(s) = \sqrt{s} \) for \( 0 < s < 1 \), then the contractive condition (1.1) is satisfied for all \( t > 0 \).

Thus all the conditions of our theorem are satisfied and '1' is the common fixed point of A, B, S and T.

**Corollary 1.** Let \( (X, M, *) \) be a fuzzy metric space with \( t * t = t \) for all \( t \in [0,1] \). Let \( A, B : X \to X \) and \( S, T : X \to CB(X) \) be single and set valued mappings respectively such that the maps (A, S) and (B, T) are (owc) and satisfy the inequality

\[
\delta_{\mu} \{ Sx, Ty, \delta_t \} \geq m_{(x, y, \delta)} \cdot \delta_{\mu} \{ Sx, Ty, \delta_t \}
\]

for all \( x, y \in X \), \( k \in (0,1) \) where \( \phi : [0,1] \to [0,1] \) is a Lebesgue integrable mapping which is summable, non-negative and such that
Common fixed point theorems

\[ \int_0^\varepsilon \phi(t) \, dt > 0 \quad \text{for each} \quad \varepsilon > 0. \]

where

\[ (1.7) \quad m(x, y, t) = \min \left\{ M(Ax, By, t), H(Ax, Sx, \alpha t) * H(By, Ty, (2 - \alpha) t) \right\} \]

for every \( x, y \in X, t > 0, \alpha, \beta \in (0, 2) \). Then \( A, B, S \) and \( T \) have unique common fixed point in \( X \).

If we take \( A = B \) and \( S = T \) in our theorem, then we get the following results.

**Corollary 2.** Let \( (X, M, \ast) \) be a fuzzy metric space with \( t \ast t = t \) for all \( t \in [0, 1] \). Let \( A : X \to X \) and \( S : X \to CB(X) \) be single and set valued mappings respectively such that the pair \( (A, S) \) is owc and satisfy the inequality

\[ (1.1) \quad \int_0^\varepsilon \phi(t) \, dt > \int_0^k \phi(t) \, dt \quad \text{for all} \quad x, y \in X, \quad k \in (0,1) \quad \text{where} \quad \phi : [0,1] \to [0, 1] \]

is a Lebesgue integral mapping which is summable, non-negative and such that

\[ \int_0^\varepsilon \phi(t) \, dt > 0 \quad \text{for each} \quad \varepsilon > 0 \]

where,

\[ (1.8) \quad m(x, y, t) = \min \left\{ M(Ax, Ay, t), H(Ax, Sx, \alpha t) * H(Ay, Sy, (2 - \alpha) t) \right\} \]

for every \( x, y \in X, t > 0, \alpha, \beta \in (0, 2) \). Then \( A \) and \( S \) have unique common fixed point in \( X \).

**Corollary 3.** Let \( (X, M, \ast) \) be a fuzzy metric space with \( t \ast t = t \) for all \( t \in [0, 1] \). Let \( A : X \to X \) and \( S : X \to CB(X) \) be single and set valued mappings respectively such that the pair \( (A, S) \), \( (A, T) \) are owc and satisfy the inequality
is a Lebesgue integral mapping which is summable, non-negative and such that
\[ \int_0^\varepsilon \phi(t) dt > 0 \]
for each \( \varepsilon > 0 \).

Now, we give the projection of above results in metric spaces.

Let \((X, d)\) denotes a metric space and \(CB(X)\) the family of all non-empty closed and bounded subsets of \(X\). Let \(H\) be the Hausdorff metric on \(CB(X)\) induced by the metric \(d\); i.e.,
\[
H(A, B) = \max \{ \sup_{x \in A} d(x, B), \sup_{y \in B} d(A, y) \}
\]
for \(A, B\) in \(CB(X)\), where
\[
d(x, A) = \inf \{ d(x, y) : y \in A \}\]

**Theorem 2.** Let \((X, d)\) be a metric space. Let \(A, B : X \to X\) and \(S, T : X \to CB(X)\) be single and set valued mappings respectively such that the maps \((A, S)\) and \((B, T)\) are owc and satisfy the inequality
\[
\delta_d (Sx, Ty, kx) \int_0^m (x, y, t) \phi(t) dt > \int_0^m (x, y, t) \phi(t) dt
\]
for all \(x, y \in X\), \(k \in (0, 1)\) where \(\phi : [0, 1] \to [0, 1]\)

for every \(x, y \in X, t > 0, \alpha \in (0, 2)\). Then \(A\), \(S\) and \(T\) have unique common fixed point in \(X\).

Let \((X, d)\) be a metric space. Let \(A, B : X \to X\) and \(S, T : X \to CB(X)\) be single and set valued mappings respectively such that the maps \((A, S)\) and \((B, T)\) are owc and satisfy the inequality
\[
\int_0^H(Sx, Ty, kx) d'(x, y) \phi(t) dt < k \int_0^H d'(x, y) \phi(t) dt
\]
where
\[
d'(x, y) = \max \left\{ \frac{d(Ax, By) + d(Ax, Sx) + d(By, Ty)}{2} \right\}
\]

for every \(x, y \in X, k \in (0, 1)\), and \(\phi : R^+ \to R^+\) is a Lebesgue interable mapping which is summable, non-negative and such that
\[
\int_0^\varepsilon \phi(t) dt > 0 \text{ for each } \varepsilon > 0
\]
Then A, B, S and T have unique common fixed point \( x \) in \( X \).

**Proof.** The proof follows from theorem 1. Considering the induced fuzzy metric space \((X, \mathcal{M}, *)\) where \( a * b = \min \{ a, b \} \) and \( M(x,y,t) = \frac{t}{t + d(x,y)} \)

**Theorem 3.** Let \((X,d)\) be a metric space. Let \( A, B : X \rightarrow X \) and \( S, T : X \rightarrow CB(X) \) be single and set valued mappings respectively such that the maps \((A,S)\) and \((B,T)\) are owc and satisfy the inequality

\[
H_{\{S x, T y\}} \leq \int_{0}^{d'(x, y)} \phi(t)dt < k \int_{0}^{d'(x, y)} \phi(t)dt
\]

where \( d'(x, y) = \max \left\{ d(A x, B y), \frac{d(A x, S x) + d(B y, T y)}{2}, \frac{d(A x, T y) + d(S x, B y)}{2} \right\} \)

for every \( x, y \in X, k \in (0,1) \) and \( \phi: \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) is a Lebesque integrable mapping which is sum able, non-negative and such that

\[
\int_{0}^{\varepsilon} \phi(t)dt > 0 \quad \text{for each } \varepsilon > 0
\]

Then A, B, S and T have unique common fixed point \( x \) in \( X \).

**Proof.** The proof follows from theorem 1. Considering the induced fuzzy metric space \((X, \mathcal{M}, *)\) where \( a * b = \min \{ a, b \} \) and \( M(x,y,t) = \frac{t}{t + d(x,y)} \)

**Remark 1.** In view of theorem 2, it is clear that the some results of [1,2,3,4,5,6,7,8] are special cases of our main results in fuzzy metric space.

**References**

[1] M. Abbas and B. E. Rhoades, Common fixed point theorems for hybrid pairs of occasionally weakly compatible mappings satisfying generalized condition of integral type, Fixed point theory appl., Art. ID 54101, 9pp


Received: October, 2010