\( \vec{P}_7 \) - Factorization of Complete Bipartite Symmetric Digraph

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Abstract

\( \vec{P}_{2p} \) -factorization of a complete bipartite graph for \( p \), an integer was studied by Wang [1]. Further, Beiling [2] extended the work of Wang[1], and studied the \( \vec{P}_{2k} \) -factorization of complete bipartite multigraphs. For even value of \( k \) in \( \vec{P}_k \) -factorization the spectrum problem is completely solved [1, 2, 3]. However for odd value of \( k \) i.e. \( P_3 \), \( P_5 \) and \( P_7 \), the path factorization have been studied by a number of researchers [4, 5, 6]. Again \( \vec{P}_3 \) -factorization of complete bipartite symmetric digraph was studied by Beiling [7]. \( \vec{P}_5 \) -factorization of complete bipartite symmetric digraph was studied by Rajput and Shukla [8]. In the present paper, \( \vec{P}_7 \) -factorization of complete bipartite symmetric digraph has been studied. It is shown that the necessary and sufficient conditions for the existence of \( \vec{P}_7 \) -factorization of complete bipartite symmetric digraph are:

1. \( 4m \geq 3n \),
2. \( 4n \geq 3m \),
3. \( m + n \equiv 0 (mod 7) \),
4. \( 7mn/\lfloor 3(m + n) \rfloor \) is an integer.

Mathematics Subject Classification: 68R10, 05C70

Keywords: Complete bipartite graph, factorization of graph

1 Introduction

A \( \vec{P}_7 \) -factorization of \( K^*_{m,n} \) is the sum of arc disjoint \( \vec{P}_7 \) -factors, where \( K^*_{m,n} \) is the complete bipartite symmetric digraph.

Theorem 1.1: \( K^*_{m,n} \) has \( \vec{P}_7 \) -factorization if and only if
(1) $4m \geq 3n$,
(2) $4n \geq 3m$,
(3) $m + n \equiv 0 \pmod{7}$,
(4) $7mn/\left[3(m + n)\right]$ is an integer.

As $\overrightarrow{P}_7$-factorization of $K^*_{m,n}$ gives rise to a $P_7$-factorization of $2K_{m,n}$ where $m$ and $n$ are positive integers. We get $P_7$ -factorization [6] as a particular case of $\overrightarrow{P}_7$ -factorization.

2 Mathematical Analysis

**Theorem 2.1**: Let $m$ and $n$ be positive integers. Then $K^*_{m,n}$ has $\overrightarrow{P}_7$ -factorization if and only if :
(1) $4m \geq 3n$,
(2) $4n \geq 3m$,
(3) $m + n \equiv 0 \pmod{7}$,
(4) $7mn/\left[3(m + n)\right] \text{ is an integer.}$

To prove this theorem the following well established number theoretic result will also be used.

**Lemma 2.1**: Let $g, h, p$ and $q$ be any positive integers. If gcd $(p, q) = 1$, then gcd $(p.q, p + g.q) = \gcd(p, g)$. Similarly, if gcd $(gp, hq) = 1$ then gcd $(gp + hq, pq) = 1$.

The following theorem will be used in the proof.

**Theorem 2.2**: If $K^*_{m,n}$ has $\overrightarrow{P}_7$ -factorization then $K^*_{sm,sn}$ has also $\overrightarrow{P}_7$ -factorization for every positive integer $s$.

**Proof**: Let $K_{s,s}$ be 1-factorable[9], and $\{H_1, H_2, ..., H_s\}$ be a 1-factorization of it. For each $i$ with $1 \leq i \leq s$, replace every edge of $H_i$ with a $K^*_{m,n}$ to get spanning subgraphs $G_i$ of $K^*_{sm,sn}$ such that the $G_i$'s $\{1 \leq i \leq s\}$ are pair wise edge disjoint and there sum is $K^*_{m,n}$. Since $K^*_{m,n}$ is $\overrightarrow{P}_7$ -factorable, therefore it is obvious that each $G_i$ is also $P_7$ -factorable, and hence, $K^*_{m,n}$ is also $\overrightarrow{P}_7$ -factorable.

Now to prove the theorem (2.1), consider three cases:-

**Case I** $(4m = 3n)$: In this case $K^*_{m,n}$ has a $\overrightarrow{P}_7$ -factorization, since $K^*_{3,4}$ (trivial case) has $\overrightarrow{P}_7$ -factorization.

$y_1x_1y_2x_2y_3x_3y_4, y_4x_3y_3x_2y_2x_1y_1, y_3x_1y_4x_2y_1x_3y_2, y_2x_3y_1x_2y_4x_1y_3$.

**Case II** $(4n = 3m$, swapping the values of $m, n$): Obviously $K^*_{m,n}$ has $\overrightarrow{P}_7$ -factorization.

**Case III** $(4m > 3n$ and $4n > 3m$): In this case, Let $a = \frac{4m - 3n}{7}, b = \frac{4m - 3n}{7}, t = \frac{m + n}{7}$ and $r = \frac{7mn}{3(m + n)}$. 

Where $a, b, t$ and $r$ will be integers (for $m, n$ satisfying theorem 2.1), we have
$m = 3a + 4b$, $n = 4a + 3b$, $r = 4(a + b) + \frac{ab}{3(a+b)}$, where $z = \frac{ab}{3(a+b)}$.

Here,

$t$ = the number of copies of $P_7$ in any factor,

$r$ = the number of $P_7$ - factors in the factorization,

$a$ = the number of copies of $P_7$ with its endpoints in $Y$ in a particular $P_7$ - factor(type M),

$b$ = the number of copies of $P_7$ with its endpoints in $X$ in a particular $P_7$ - factor(type W),

$c$ = the total number of copies of $P_7$ in the whole factorization.

Let $\gcd (3a, 4b) = d$ and therefore $3a = dp$, $4b = dq$ for some $p, q$. where $\gcd (p, q) = 1$, Then $z = \frac{dpq}{3(4p + 3q)}$.

These equalities imply the following equalities;

$$d = \frac{3(4p + 3q)z}{pq}, m = \frac{3(p+q)(4p+3q)z}{4pq},$$

$$n = \frac{(4p + 3q)(16p+9q)z}{4pq}, r = \frac{pq}{4pq},$$

$a = \frac{3p(4p + 3q)z}{4pq}$ and $b = \frac{3q(4p + 3q)z}{4pq}$,

Now to compute the values of $m, n, a, b$ and $r$, we established the following lemma:

**Lemma 2.2:**

Case (1): If $\gcd (p, 9) = 1$ and $\gcd (q, 16) = 1$, then there exist a positive integer $s$ such that,

$m = 12(p+q)(4p + 3q)s, n = (16p + 9q)(4p + 3q)s,$

$a = 12p(4p + 3q)s, b = 3q(4p + 3q)s$ and $r = 4(p + q)(16p + 9q)s$.

Case (2): If $\gcd (p, 9) = 1$ and $\gcd (q, 16) = 2$. Let $q = 2q_1$, then there exist a positive integer $s$ such that

$m = 12(2p + 3q_1)(p + 2q_1)s, n = 2(2p + 3q_1)(8p + 9q_1)s,$

$a = 12p(2p + 3q_1)s, b = 6q_1(2p + 3q_1)s$ and $r = 4(p + 2q_1)(8p + 9q_1)s$.

Case (3): If $\gcd (p, 9) = 1$ and $\gcd (q, 16) = 4$. Let $q = 4q_2$, then there exist a positive integer $s$ such that

$m = 12(p + 3q_2)(p + 4q_2)s, n = 4(p + 3q_2)(4p + 9q_2)s,$

$a = 12p(p + 3q_2)s, b = 12q_2(p + 3q_2)s$ and $r = 4(p + 4q_2)(4p + 9q_2)s$.

Case (4): If $\gcd (p, 9) = 1$ and $\gcd (q, 16) = 8$. Let $q = 8p_3$, then there exist a positive integer $s$ such that

$m = 6(p + 6q_3)(p + 8q_3)s, n = 4(p + 6q_3)(2p + 9q_3)s,$

$a = 6p(p + 6q_3)s, b = 12q_3(p + 6q_3)s$ and $r = 4(p + 8q_3)(2p + 9q_3)s$.

Case (5): If $\gcd (p, 9) = 1$ and $\gcd (q, 16) = 16$. Let $q = 16q_4$, then there exist a positive integer $s$ such that

$m = 3(p + 12q_4)(p + 16q_4)s, n = 4(p + 9q_4)(p + 12q_4)s,$

$a = 3p(p + 12q_4)s, b = 12q_4(p + 12q_4)s$ and $r = 4(p + 16q_4)(p+9q_4)s$.

Case (6): If $\gcd (p, 9) = 3$, and $\gcd(q, 16) = 1$. Let $p = 3p_1$, then there exist a positive integer $s$ such that

$$m = 3(p + 12q_4)(p + 16q_4)s, n = 4(p + 9q_4)(p + 12q_4)s,$$
\[ m = 12(4p_1 + q)(3p_1 + q)s, \quad n = 3(4p_1 + q)(16p_1 + 3q)s, \]
\[ a = 12p_1(4p_1 + q)s, \quad b = 3q(4p_1 + q)s \quad \text{and} \quad r = 4(3p_1 + q)(16p_1 + 3q)s. \]

Case (7): If gcd \((p,9) = 3\) and gcd \((q,16) = 2\). Let \(p = 3p_1\) and \(q = 2q_1\), then there exist a positive integer \(s\) such that
\[ m = 12(2p_1 + q_1)(3p_1 + 2q_1)s, \quad n = 6(2p_1 + q_1)(8p_1 + 3q_1)s, \]
\[ a = 12p_1(2p_1 + q_1)s, \quad b = 6q_1(2p_1 + q_1)s \quad \text{and} \quad r = 4(3p_1 + 2q_1)(8p_1 + 3q_1)s. \]

Case (8): If gcd \((p,9) = 3, \)gcd \((q,16) = 4\). Let \(p = 3p_1, \quad q = 4q_2, \)then there exist a positive integer \(s\) such that
\[ m = 12(p_1 + q_2)(3p_1 + 4q_2)s, \quad n = 12(p_1 + q_2)(4p_1 + 3q_2)s, \]
\[ a = 36p_1(p_1 + q_2)s, \quad b = 12q_2(p_1 + q_2)s \quad \text{and} \quad r = 4(3p_1 + 4q_2)(4p_1 + 3q_2)s. \]

Case (9): If gcd \((p,9) = 3\) gcd \((q,16) = 8\). Let \(p = 3p_1, \quad q = 8q_3, \)then there exist a positive integer \(s\) such that
\[ m = 6(p_1 + 2q_3)(3p_1 + 8q_3)s, \quad n = 12(p_1 + 2q_3)(2p_1 + 3q_3)s, \]
\[ a = 18p_1(p_1 + 2q_3)s, \quad b = 12q_3(p_1 + 2q_3)s \quad \text{and} \quad r = (3p_1 + 8q_3)(2p_1 + 3q_3)s. \]

Case (10): If gcd \((p,9) = 3, \)gcd \((q,16) = 16\). Let \(p = 3p_1, \quad q = 16q_4, \)then there exist a positive integer \(s\) such that
\[ m = 3(p_1 + 4q_4)(3p_1 + 16q_4)s, \quad n = 12(p_1 + 4q_4)(p_1 + 3q_4)s, \]
\[ a = 9p_1(p_1 + 4q_4)s, \quad b = 12q_4(p_1 + 4q_4)s \quad \text{and} \quad r = 4(3p_1 + 16q_4)(p_1 + 3q_4)s. \]

Case (11): If gcd \((p,9) = 9, \)gcd \((q,16) = 1. \) Let \(p = 9p_2, \quad q = 16q_3, \)then there exist a positive integer \(s\) such that
\[ m = 4(12p_2 + q)(9p_2 + q)s, \quad n = 3(12p_2 + q)(16p_2 + q)s, \]
\[ a = 36p_2(12p_2 + q)s, \quad b = q(12p_2 + q)s \quad \text{and} \quad r = 4(9p_2 + q)(16p_2 + q)s. \]

Case (12): If gcd \((p,9) = 9, \)gcd \((q,16) = 2. \) Let \(p = 9p_2, \quad q = 2q_1, \)then there exist a positive integer \(s\) such that
\[ m = 4(6p_2 + q_1)(9p_2 + 2q_1)s, \quad n = 6(6p_2 + q_1)(8p_2 + q_1)s, \]
\[ a = 36p_2(6p_2 + q_1)s, \quad b = 6q_1(6p_2 + q_1)s \quad \text{and} \quad r = 4(9p_2 + 2q_1)(8p_2 + q_1)s. \]

Case (13): If gcd \((p,9) = 9, \)gcd \((q,16) = 4. \) Let \(p = 9p_2, \quad q = 4q_2, \)then there exist a positive integer \(s\) such that
\[ m = 4(3p_2 + q_2)(9p_2 + 4q_2)s, \quad n = 12(3p_2 + q_2)(4p_2 + q_2)s, \]
\[ a = 36p_2(3p_2 + q_2)s, \quad b = 4q_2(3p_2 + q_2)s \quad \text{and} \quad r = 4(9p_2 + 4q_2)(4p_2 + q_2)s. \]

Case (14): If gcd \((p,9) = 9, \)gcd \((q,16) = 8. \) Let \(p = 9p_2, \quad q = 8q_3, \)then there exist a positive integer \(s\) such that
\[ m = 2(3p_2 + 2q_3)(9p_2 + 8q_3)s, \quad n = 12(3p_2 + 2q_3)(2p_2 + 3q_3)s, \]
\[ a = 18p_2(3p_2 + 2q_3)s, \quad b = 4q_3(3p_2 + 2q_3)s \quad \text{and} \quad r = 4(9p_2 + 8q_3)(2p_2 + 3q_3)s. \]

Case (15): If gcd \((p,9) = 9, \)gcd \((q,16) = 16. \) Let \(p = 9p_2, \quad q = 16q_4. \) Then there exist a positive integer \(s\) such that
\[ m = (3p_2 + 4q_4)(9p_2 + 16q_4)s, \quad n = 12(3p_2 + 4q_4)(p_2 + q_4)s, \]
\[ a = 9p_2(3p_2 + 4q_4)s, \quad b = 4q_4(3p_2 + 4q_4)s \quad \text{and} \quad r = 4(9p_2 + 16q_4)(p_2 + q_4)s. \]

**Proof:** Here we are giving the proof of case (1). If gcd \((p,q) = 1, \)gcd \((p,9) = 1\) and gcd \((q,16) = 1, \)then there exist gcd \((16p + 9q,2) = 1 = \text{gcd}(4p + 3q,2) \) and gcd \((16p,9q) = p(4p,3q) = 1\) hold.

It is easy \( n = \frac{(16p + 9q)(4p + 3q)}{4pq} \). Hence gcd \((16p + 9q, pq) = \text{gcd}(4p + 3q, pq) = 1. \)
Therefore, $P_{7}$ must be an integer. Let $s = \frac{3}{4pq}$ then the equality (1) hold. The proofs of other equalities are similar to (1). For main result, we need the direct constructions at $s = 1$.

**Lemma 2.3:** For any positive integer $p$ and $q$ let $m = 12(p + q)(4p + 3q)$, and $n = (4p + 3q)(16p + 9q)$. Then $K_{m,n}^{*}$ has $P_{7}$-factorization.

**Proof:** Let $a = 12p(4p + 3q)$, $b = 3q(4p + 3q)$, $r = 4(p + q)(16p + 9q)$, $r_1 = 4(p + q)$ and $r_2 = (16p + 9q)$, $m_0 = m/r_1 = 3(4p + 3q)$, $n_0 = n/r_2 = (4p + 3q)$. Let $X, Y$ be two partite sets of $K_{m,n}^{*}$ and

$X = \{x_{i,j}; 1 \leq i \leq r_1, 1 \leq j \leq m_0\}$,

$Y = \{y_{i,j}; 1 \leq i \leq r_2, 1 \leq j \leq n_0\}$.

We will construct a $P_{7}$-factorization of $K_{m,n}^{*}$. Where first subscript of $x_{i,j}$'s and $y_{i,j}$'s taken addition modulo $r_1$ and $r_2$ the second subscript of $x_{i,j}$'s and $y_{i,j}$'s taken addition modulo $3(4p + 3q)$ and $(4p + 3q)$ in $\{1, 2, ..., 3(4p + 3q)\}$ and $\{1, 2, ..., (4p + 3q)\}$.

For each $1 \leq i \leq 4p$, let $E_i = \{x_{i,j}+(4p+3q)(u-1)+(4p+3q)v; 1 \leq j \leq (4p + 3q), 1 \leq u \leq 2, 0 \leq v, w \leq 1\}$,

For each $1 \leq i \leq q$, let $E_{4p+4i} = \{x_{4p+4(i-1)+u+v,j}+(4p+3q)v; 1 \leq j \leq (4p + 3q), 0 \leq u \leq 3, 1 \leq w \leq 3, 0 \leq v \leq 1\}$.

Let $F = U_{1 \leq i \leq 4p+3q}E_i$. Then the graph $F$ is a $P_{7}$-factor of $K_{m,n}^{*}$. Define a bijection $\sigma : X \cup Y \text{ onto } X \cup Y$ in such a way that $\sigma(x_{i,j}) = x_{i+1,j}$ and $\sigma(y_{i,j}) = y_{i+1,j}$. For each $i \in (1, 2, ..., r_1)$ and each $j \in (1, 2, ..., r_2)$, let $F_{i,j} = \{\sigma^i(x)\sigma^j(y) : x \in X, y \in Y, xy \in F\}$.

It is easy to show that the graph, $F_{i,j} \{1 \leq i \leq 4(p + q), 1 \leq j \leq (4p + 3q)\}$, are edge disjoint $P_{7}$-factor of $K_{m,n}^{*}$ and its union is also $K_{m,n}^{*}$.

Thus $\{F_{i,j} : 1 \leq i \leq 4(p + q), 1 \leq j \leq (4p + 3q)\}$ is a $P_{7}$-factorization of $K_{m,n}^{*}$.

This is the proof of lemma(2.3), similarly we give the proof of other cases in lemma (2.2).

**Proof:** By using theorem 2.2 with lemma 2.3, it can be seen that when the parameters $m$ and $n$ satisfy condition (1)-(4), the digraph $K_{m,n}^{*}$ has $P_{7}$-factorization. This completes the proof of theorem 2.1.

**References**


Received: February, 2011