Convergence of Quotients of Consecutive Terms of a Generalized Secondary Fibonacci Sequence

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Abstract

In this paper we consider a generalized Secondary Fibonacci sequence. We study the convergence of the sequence of quotients of its consecutive terms and we give an estimation for the speed of its convergence.

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1 Introduction

Some numbers are well known because of their mathematical properties, applications and presence in different areas, e.g., \( \phi = \frac{1+\sqrt{5}}{2} \) and \( \theta = 1 + \sqrt{2} \). These constants are called the golden mean and the silver mean, respectively [12, 7, 5, 9]. \( \phi \) and \( \theta \) have a lot of geometric, algebraic and analytical mathematical properties which are useful in the study of architectural proportion, art, computer science, and growth of some biological systems [6, 3, 10, 17, 15].

Analogously to the way the golden mean is related to the Fibonacci Sequence, the silver mean is related to the Pell Sequence [9]. These sequences are defined recursively by:
\[ F_n = \begin{cases} 0 & \text{if } n = 1 \\ 1 & \text{if } n = 2 \\ F_{n-1} + F_{n-2} & \text{if } n \geq 3 \end{cases} \tag{1} \]

and

\[ P_n = \begin{cases} 0 & \text{if } n = 1 \\ 1 & \text{if } n = 2 \\ 2P_{n-1} + P_{n-2} & \text{if } n \geq 3 \end{cases} \tag{2} \]

respectively. The theory and generalization of Fibonacci Numbers and the golden mean is an important branch of modern mathematics \[20, 15\]. According to \[18\], \( \varphi \) and \( \theta \) are members of a very special group of quadratic positive irrational numbers known as ”the Metallic Means Family (MMF)” and arise as positive solution of the equations

\[ x^2 - px - q = 0; \; p, q \in \mathbb{Z}^+. \tag{3} \]

With appropriate values for \( p \) and \( q \) in (3) a member of the Metallic Means Family can be obtained, i.e., the golden mean, by setting \( p = 1 \) and \( q = 1 \); the silver mean, with \( p = 2 \) and \( q = 1 \); the cooper mean by setting \( p = 1 \) and \( q = 2 \) and so on[17].

The Fibonacci and Pell sequences are particular cases of the generalized secondary Fibonacci sequence” (GSFS). They satisfy relations of the type

\[ c_n = \begin{cases} a & \text{if } n = 1 \\ b & \text{if } n = 2 \\ pc_{n-1} + qc_{n-2} & \text{if } n \geq 3 \end{cases}; \; p, q \in \mathbb{Z}^+. \tag{4} \]

**Theorem 1.** Suppose that \( a, b, p, q > 0 \) are positive numbers such that

\[ \frac{b}{a} = r \neq \mu = \frac{p + \sqrt{p^2 + 4q}}{2}. \tag{5} \]

Then the sequence \( \{a_n\}_{n=1}^{\infty} \) defined by

\[ a_n = \frac{c_{n+1}}{c_n}, \tag{6} \]

where \( \{c_n\}_{n=1}^{\infty} \) is defined by (4) converges and

\[ \lim_{n \to \infty} a_n = \frac{p + \sqrt{p^2 + 4q}}{2}. \tag{7} \]
Proof. Let \( a, b, p, q > 0 \). Suppose that
\[
 r > \mu. 
\] (8)
The case when \( r < \mu \) may be considered analogously.
First at all, \( c_n > 0 \) for all \( n = 1, 2, 3, \ldots \). Indeed, \( c_1 = a > 0 \) and \( c_2 = b > 0 \).
Suppose that \( a_j > 0 \) for \( j = 2, \ldots, k \). Then
\[
c_{k+1} = pc_k + qc_{k-1} > pc_k > 0.
\]
Now, observe that
\[
0 < a_n = \frac{c_{n+1}}{c_n} = \frac{pc_n + qc_{n-1}}{c_n} = p + \frac{qc_{n-1}}{c_n} = p + \frac{q}{a_{n-1}} \quad \text{for } n = 2, 3, \ldots \quad (9)
\]
It follows from (9) that
\[
a_n = p + \frac{q}{p + \frac{q}{a_{n-2}}} \quad \text{for } n = 3, 4, \ldots \quad (10)
\]
In particular,
\[
a_{n+2} - a_n = \frac{q^2}{(q + pa_n)(q + pa_{n-2})} (a_n - a_{n-2}) \quad \text{for } n = 3, 4, 5, \ldots \quad (11)
\]
It is clear that \( \mu \) is the positive root of the equation \( x^2 - px - q = 0 \). In particular,
\[
\mu^2 = p\mu + q \text{ and since } r > \mu, r^2 - pr - q > 0. \quad (12)
\]
We now will show that the subsequence \( \{a_{2n}\}_{n=1}^\infty \) of even terms is increasing and the subsequence \( \{a_{2n-1}\}_{n=1}^\infty \) of odd terms is decreasing and
\[
a_{2m} \leq \mu \leq a_{2n-1} \text{ for any } m, n = 1, 2, 3, \ldots \quad (13)
\]
Indeed, observe that \( a_1 = \frac{a_2}{c_1} = \frac{b}{a} = r \) and from (10) and (12) we obtain
\[
a_3 - a_1 = p + \frac{q}{p + \frac{q}{r}} - r = -p\frac{r^2 - pr - q}{q + pr} < 0. \quad (14)
\]
On the other hand, from (9)
\[
a_4 - a_2 = p + \frac{q}{a_3} - \left(p + \frac{q}{a_1}\right) = \frac{q(a_1 - a_3)}{a_1 a_3} > 0. \quad (15)
\]
Then

$$0 < a_3 < a_1 \text{ and } a_4 > a_2 > 0.$$  \hfill (16)

We now proceed by induction. Suppose that

$$a_{2k+1} < a_{2k-1} \text{ and } a_{2k+2} > a_{2k} \text{ for some } k \geq 1.$$

Making use of (11) gives

$$a_{2k+3} - a_{2k+1} = q^2 \frac{a_{2k+1} - a_{2k-1}}{(q + pa_{2k+1})(q + pa_{2k-1})} < 0 \hfill (17)$$

and

$$a_{2k+4} - a_{2k+2} = q^2 \frac{a_{2k+2} - a_{2k}}{(q + pa_{2k+2})(q + pa_{2k})} > 0 \hfill (18)$$

By the principle of mathematical induction,

$$a_{2n+1} < a_{2n-1} \text{ and } a_{2n+2} > a_{2n} \text{ for all } n = 1, 2, 3, \ldots.$$

We now will show that

$$a_{2n-1} > \mu \text{ and } a_{2n} < \mu \text{ for all } n \geq 1.$$ \hfill (20)

We again proceed inductively. In view of (8) we have $-\mu > -r$ and then

$$a_2 - \mu = p + \frac{q}{r} - \mu > p + \frac{q}{r} - r = -\frac{1}{r}(r^2 - pr - q) < 0$$

and

$$a_1 - \mu = r - \mu > 0.$$

We have proved that

$$a_1 > \mu \text{ and } a_2 < \mu.$$

Suppose that

$$a_{2k-1} > \mu \text{ and } a_{2k} < \mu \text{ for some } k \geq 1.$$

Then, by virtue of (11),

$$a_{2k+1} - \mu = p + \frac{q}{p + \frac{q}{a_{2k-1}}} - \mu > p + \frac{q}{p + \frac{q}{\mu}} - \mu = -p\mu^2 - p\mu - q \frac{q + p\mu}{q + p\mu} = 0$$
and
\[ a_{2k+2} - \mu = p + \frac{q}{p + \frac{q}{a_{2k}}} - \mu < \frac{q}{p + \frac{q}{\mu}} - \mu = -p \frac{2 \mu^2 - p \mu - q}{q + p \mu} = 0. \]

We have proved (19) and (20), that is,
\[ a_2 < a_4 < a_6 < \cdots < \mu < \cdots < a_5 < a_3 < a_1. \]

By the Weierstrass theorem, there exist
\[ 0 < \alpha = \lim_{n \to \infty} a_{2n} \text{ and } \beta = \lim_{n \to \infty} a_{2n-1}. \]

We have
\[ \alpha = \lim_{n \to \infty} a_{2n+2} = \lim_{n \to \infty} \left( p + \frac{q}{p + \frac{q}{a_{2n}}} \right) = p + \frac{q}{p + \frac{q}{\alpha}} \]
from where
\[ p \frac{\alpha^2 - p \alpha - q}{q + p \mu} = 0 \]
and then \( \alpha = \mu. \) In a similar way, \( \beta = \mu. \) This implies that \( \lim_{n \to \infty} a_n = \mu. \)

Indeed, let be \( \varepsilon > 0 \) any positive number. There exist two positive integer numbers \( N_1 \) and \( N_2 \) such that
\[ |a_{2n} - \mu| < \varepsilon \text{ for } n > N_1 \text{ and } |a_{2n-1} - \mu| < \varepsilon \text{ for } n > N_2. \]

Let \( N = \max(2N_1, 2N_2) \) and let \( k > N. \) If \( k \) is even, say \( k = 2n \) then \( 2n > N \geq 2N_1, \) \( n > N_1 \) and then \( |a_k - \mu| < \varepsilon. \) If \( k \) is odd, say \( k = 2n-1 \) then \( 2n-1 > N \geq 2N_2, \) \( n > N_2 + 1/2 > N_2 \) and then \( |a_k - \mu| < \varepsilon. \) This proves (7).

**Remark.** If \( r < \mu \) then we may show that the subsequence of even terms of the sequence \( \{a_n\}_{n=1}^{\infty} \) is decreasing, while the subsequence of odd terms of this sequence is increasing and
\[ a_1 < a_3 < a_5 < \cdots < \mu < \cdots < a_6 < a_4 < a_2. \]

This happens, for example, if \( a = b = p = q = 1 \) (Fibonacci sequence) and \( a = b = 1, p = 2, q = 1 \) (Pell sequence). Thus, for the Fibonacci sequence,
\[ r = 1 < \mu = \frac{1 + \sqrt{5}}{2}. \]
and for the Pell sequence,

\[ r = 1 < \mu = 1 + \sqrt{2}. \]

On the other hand, if \( r = \mu \) we may show that \( \{a_n\}_{n=1}^{\infty} \) is a constant sequence and each of its terms equals \( \mu \). Then \( \{a_n\}_{n=1}^{\infty} \) converges to \( \mu \). Thus, in any case, \( a_n \to \mu \ (n \to \infty) \).

We now proceed to estimate the speed of convergence of the sequence \( \{a_n\}_{n=1}^{\infty} \). We have the following

**Theorem.** Let \( C = \max\{|a_3 - a_1|, |a_4 - a_2|\} \). If \( r \neq \mu \) there exists a constant \( \rho \) such that \( 0 < \rho < 1 \) for which

\[ |a_n - \mu| < C \rho^{n-1} \]

for any \( n = 1, 2, 3, \ldots \) \( (21) \)

**Proof.** Suppose that \( r > \mu \) (the case \( r < \mu \) is similar). Making use of (17), (18) and (20) we obtain

\[ |a_{2k+3} - a_{2k+1}| = \frac{q^2}{(q + pa_{2k+1}) (q + pa_{2k-1})} |a_{2k+1} - a_{2k-1}| \]

\[ < \frac{q^2}{(q + \mu p) (q + \mu p)} |a_{2k+1} - a_{2k-1}| \]

\[ = \frac{q^2}{(q + \mu p)^2} |a_{2k+1} - a_{2k-1}|. \]

We have proved that

\[ |a_{2k+3} - a_{2k+1}| < \rho_1 |a_{2k+1} - a_{2k-1}| \]

for any \( k \geq 1 \), \( (22) \)

where

\[ \rho_1 = \frac{q^2}{(q + \mu p)^2} \in (0, 1). \] \( (23) \)

On the other hand, since \( a_{2k-1} < a_1 = r \) and \( a_{2k-1} > \mu \),

\[ a_{2k} = p + \frac{q}{a_{2k-1}} = \frac{pa_{2k-1} + q}{a_{2k-1}} > \frac{p\mu + q}{r}. \] \( (24) \)

Similarly,

\[ a_{2k+2} = p + \frac{q}{a_{2k+1}} = \frac{pa_{2k+1} + q}{a_{2k+1}} > \frac{p\mu + q}{r}. \] \( (25) \)
We have
\[ |a_{2k+4} - a_{2k+2}| = \frac{q^2 |a_{2k+2} - a_{2k}|}{(q + pa_{2k+2})(q + pa_{2k})} < \frac{q^2 |a_{2k+2} - a_{2k}|}{(q + p(p\mu + q))(q + p(p\mu + q))} \]
\[ = \frac{q^2 r^2}{(q + p\mu^2 + pq)^2} |a_{2k+2} - a_{2k}| \]
\[ = \frac{q^2}{(q + p\mu^2 + pq)^2} |a_{2k+2} - a_{2k}|. \]

We have proved that
\[ |a_{2k+4} - a_{2k+2}| < \rho_2 |a_{2k+2} - a_{2k}| \quad \text{for any } k \geq 1, \quad (26) \]
where
\[ \rho_2 = \frac{q^2}{(q + p\mu^2 + pq)^2} \in (0, 1). \quad (27) \]

Let \( A_n = a_{2n-1} \) and \( B_n = a_{2n} \) (\( n = 1, 2, 3, \ldots \)). Taking into account (22) we obtain for \( k \geq 1 \)
\[ |a_5 - a_3| < \rho_1 |a_3 - a_1| \]
\[ |a_7 - a_5| < \rho_1 |a_5 - a_3| \]
\[ \vdots \]
\[ |a_{2k-1} - a_{2k-3}| < \rho_1 |a_{2k-3} - a_{2k-5}| \]
\[ |a_{2k+1} - a_{2k-1}| < \rho_1 |a_{2k-1} - a_{2k-3}| \]

Multiplying term by term these inequalities yields
\[ |A_{k+1} - A_k| = |a_{2k+1} - a_{2k-1}| < \rho_1^{k-1} |a_3 - a_1| = \rho_1^{k-1} C_1, \quad \text{where } C_1 = |a_3 - a_1|. \quad (28) \]

In a similar way,
\[ |B_{k+1} - B_k| = |a_{2k+2} - a_{2k}| < \rho_2^{k-1} |a_4 - a_2| = \rho_2^{k-1} C_2, \quad \text{where } C_2 = |a_4 - a_2|. \quad (29) \]
Inequalities (28) and (29) hold for any \( k \geq 1. \)

Let \( m > n. \) We have

\[
|A_m - A_n| < \sum_{k=n}^{m-1} |A_{k+1} - A_k| < \sum_{k=n}^{m-1} \rho_1^{k-1} C_1 < \sum_{k=n}^{\infty} \rho_1^{k-1} C_1 = \frac{C_1 \rho_1^{n-1}}{1 - \rho_1}.
\]

We thus have

\[
|A_n - A_m| < \frac{C_1 \rho_1^{n-1}}{1 - \rho_1} \text{ for any } m > n.
\]

Letting \( m \to \infty \) in this last inequality gives

\[
|a_{2n-1} - \mu| \leq \frac{C_1 \rho_1^{n-1}}{1 - \rho_1} \text{ for any } n \geq 1. \tag{30}
\]

Similarly,

\[
|a_{2n} - \mu| \leq \frac{C_2 \rho_2^{n-1}}{1 - \rho_2} \text{ for any } n \geq 1 \tag{31}
\]

Finally, observe that

\[
\rho_2 = \frac{q^2}{(q + \frac{p\nu^2}{\mu})^2} > \frac{q^2}{(q + \frac{p\mu^2}{\mu})^2} = \frac{q^2}{(q + p\mu)^2} = \rho_1 \tag{32}
\]

and then

\[
\frac{C_2 \rho_2^{n-1}}{1 - \rho_2} > \frac{C_2 \rho_1^{n-1}}{1 - \rho_1} \tag{33}
\]

It is clear from (30), (31) and (33) that

\[
|a_n - \mu| < \frac{C \rho^{n-1}}{1 - \rho}, \tag{34}
\]

where \( C = \max\{C_1, C_2\} \) and \( \rho = \rho_2 = \max\{\rho_1, \rho_2\}. \) This proves the theorem.
2 Conclusions

We generalized some results concerning the Fibonacci and Pell numbers. We did not make use of the theorem about the existence and uniqueness of the solution of a general linear recurrence sequence of second order. The main result in this work, i.e. Theorem 1, can also be obtained directly by using Binnet’s Formula [20],[3].

References


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