On Fuzzy Topology on KS-semigroups

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Abstract

In this paper, we introduce and investigate fuzzy topology on a KS-semigroup and fuzzy topological KS-semigroup and see how it affects its image and preimage under a homomorphism.

1 Introduction

Two classes of abstract algebras; the BCK-algebra and BCI-algebra were introduced by Y. Imai and K. Iseki in [2]. It is known that the class of BCK-algebra is a proper subclass of BCI-algebra. Since then, a great deal of studies has been produced on the theory of BCK-algebra and BCI-algebra and recently, Walendziak in [9] introduced the notion of BF-algebras. In [3], K.H. Kim introduced a class of sets called KS-semigroups and in [11], Zadeh introduced the concepts of fuzzy sets after which Rosenfeld in [4] introduced the notion of fuzzy groups. Following this idea, fuzzy BCK-algebras in [10] were studied. In [8], Prince Williams and Husain studied fuzzy KS-semigroups. In [11], fuzzy topology on a set was introduced and in [5], Saeid and Resvani introduced fuzzy topology on BF-algebras. In this paper, we define fuzzy topology on a KS-semigroup and introduce the fuzzy normal topology and discuss its properties specifically on the concepts of fuzzy continuity and fuzzy openness. We also define a fuzzy topological KS-semigroup and study some of its properties.
2 Preliminary

We now review some definitions and results that will be used in this paper.

\textbf{Definition 2.1} \cite{8} An algebraic system \((X, \ast, 0)\) is called a \textit{BCK-algebra} if it satisfies the following conditions: For all \(x, y, z \in X\),

1. \(((x \ast y) \ast (x \ast z)) \ast (z \ast y) = 0\),
2. \((x \ast (x \ast y)) \ast y = 0\),
3. \(x \ast x = 0\),
4. \(0 \ast x = 0\),
5. \(x \ast y = 0\) and \(y \ast x = 0\) implies \(x = y\).

It is easy to show that in a BCK-algebra, \(x \ast 0 = x\).

\textbf{Definition 2.2} \cite{1} Let \(X\) be a nonempty set. The system \((X, \cdot)\) is called a \textit{semigroup} if " \(\cdot"\) is an associative binary operation.

\textbf{Definition 2.3} \cite{8} An algebraic system \((X, \ast, \cdot, 0)\) is called a \textit{KS-semigroup} if it satisfies the following conditions:

1. \((X, \ast, 0)\) is a BCK algebra,
2. \((X, \cdot)\) is a semigroup,
3. The operation \(\cdot\) is distributive (on both sides) over \(\ast\), that is, for all \(x, y, z \in X\)
   
   \begin{enumerate}
   \item \(x \cdot (y \ast z) = (x \cdot y) \ast (x \cdot z)\)
   \item \((x \ast y) \cdot z = (x \cdot z) \ast (y \cdot z)\)
   \end{enumerate}

For convenience, we write \(x \cdot y\) by \(xy\).

\textbf{Definition 2.4} \cite{8} Let \(X\) be any set. A mapping \(A : X \rightarrow [0, 1]\) is called a \textit{fuzzy set} of \(X\).

\textbf{Definition 2.5} \cite{8} A nonempty subset \(Y\) of a KS-semigroup \(X\) with binary operations " \(\ast\)" and " \(\cdot\)" is called a \textit{sub KS-semigroup} if \(Y\) is closed under " \(\ast\)" and " \(\cdot\)" , that is,

1. \(x \ast y \in Y\) for all \(x, y \in Y\), and
(2) \( xy \in Y \) for all \( x, y \in Y \).

**Definition 2.6** [8] Let \( X \) be a KS-semigroup and \( A \) a fuzzy set in \( X \) such that for all \( x, y \in X \)

(1) \( A(xy) \geq \min \{ A(x), A(y) \} \), and

(2) \( A(x*y) \geq \min \{ A(x), A(y) \} \).

Then \( A \) is called a *fuzzy sub KS-semigroup* of \( X \).

**Definition 2.7** [8] Let \( X \) and \( X' \) be two nonempty sets, \( f : X \rightarrow X' \) a mapping, and \( A \) a fuzzy set in \( X' \). The *preimage* of \( A \) under \( f \), denoted by \( f^{-1}(A) \) (or \( A_f \)), is a fuzzy set of \( X \) defined by \( f^{-1}(A)(x) = A_f(x) = A(f(x)) \) for all \( x \in X \).

**Definition 2.8** [5] Let \( A \) be a fuzzy set in \( X \) and \( f : X \rightarrow Y \) a mapping. The mapping \( f(A) : Y \rightarrow [0, 1] \) defined by \( f(A)(y) = \sup \{ A(x) \}_{x \in f^{-1}(y)} \) if \( f^{-1}(y) \neq \emptyset \) and \( f(A)(y) = 0 \) if \( f^{-1}(y) = \emptyset \), is called the *image of \( A \) under \( f \)*, where \( f^{-1}(y) = \{ x \in X : f(x) = y \} \).

**Definition 2.9** [5] Let \( X \) be a nonempty set. A *fuzzy topology on \( X \)* is a family \( \tau \) of fuzzy sets in \( X \) which satisfies the following conditions:

(1) For any \( c \in [0, 1] \), the fuzzy set \( \mu_c \) defined by \( \mu_c(x) = c \), \( \forall x \in X \), is in \( \tau \),

(2) If \( A, B \in \tau \), then \( A \cap B \in \tau \), where \( (A \cap B)(x) = \min \{ A(x), B(x) \} \), and

(3) If \( A_j \in \tau \) for \( j \in J \), then \( \bigcup_{j \in J} A_j \in \tau \) where \( \bigcup_{j \in J} A_j(x) = \sup \{ A_j(x) \}_{j \in J} \).

The pair \( (X, \tau) \) is called a *fuzzy topological space* and the members of \( \tau \) are called *fuzzy open sets*.

**Definition 2.10** [5] Let \( (X, \tau) \) be a fuzzy topological space and \( A \) a fuzzy set in \( X \). Then the *induced fuzzy topology on \( A \)*, denoted by \( \tau_A \) is defined as the set \( \tau_A = \{ A \cap G : G \in \tau \} \). The pair \( (A, \tau_A) \) is called a *fuzzy subspace* of \( (X, \tau) \).

**Definition 2.11** [5] Let \( f : (X, \tau) \rightarrow (Y, \upsilon) \) be a mapping of fuzzy topological spaces. Then \( f \) is *fuzzy continuous* if for each \( V \in \upsilon \), \( f^{-1}(V) \in \tau \). \( f \) is said to be *fuzzy open* if for each \( U \in \tau \), \( f(U) \in \upsilon \).
Definition 2.12 [5] Let \((A, \tau_A)\) and \((B, \upsilon_B)\) be fuzzy subspaces of \((X, \tau)\) and \((Y, \upsilon)\) respectively and let \(f : (X, \tau) \rightarrow (Y, \upsilon)\). Then \(f\) is a mapping of \((A, \tau_A)\) into \((B, \upsilon_B)\) if \(f(A) \subseteq B\). That is, \(f(A)(y) \leq B(y), \forall y \in Y\). In addition, \(f\) is said to be relatively fuzzy continuous if for each \(V \in \upsilon_B, f^{-1}(V) \cap A \in \tau_A\). \(f\) is said to be relatively fuzzy open if for each \(U \in \tau, f(U) \in \upsilon_B\).

Definition 2.13 [5] Let \(X\) and \(Y\) be two nonempty sets, \(f : X \rightarrow Y\) be a mapping, and \(A\) be a fuzzy set in \(X\). Then \(A\) is said to be \(f\)-invariant if for each \(x, y \in X\) such that \(f(x) = f(y)\), we have \(A(x) = A(y)\).

Definition 2.14 [3] Let \(X\) and \(Y\) be KS-semigroups and \(f : X \rightarrow Y\) be a function. Then \(f\) is a KS-semigroup homomorphism or simply homomorphism if \(f(xy) = f(x)f(y)\) and \(f(x \ast y) = f(x) \ast f(y)\) for all \(x, y \in X\).

3 On Fuzzy Topology on a KS-semigroup

Let \(X\) be a KS-semigroup and \(\tau\) is a family of fuzzy sets in \(X\) satisfying the conditions in Definition 2.9, then \(\tau\) is a fuzzy topology on a KS-semigroup \(X\) and \((X, \tau)\) is a fuzzy topological space.

Example 3.1 Consider any KS-semigroup \(X\) and let

\[
\tau = \{A : X \rightarrow [0, 1] : A(x \ast y) = A(y \ast x), A(xy) = A(yx)\}.
\]

Let us verify if the family \(\tau\) satisfies the conditions of a fuzzy topology. First, we show that the fuzzy set \(\mu_c \in \tau\) where \(\mu_c(x) = c\) for \(c \in [0, 1]\). Now, let \(x, y \in X\). Then \(\mu_c(x \ast y) = c = \mu_c(y \ast x)\) and \(\mu_c(xy) = c = \mu_c(yx)\), hence \(\mu_c \in \tau\). Next, let \(A, B \in \tau\). Then

\[
(A \cap B)(x \ast y) = \min \{A(x \ast y), B(x \ast y)\} = \min \{A(y \ast x), B(y \ast x)\} = (A \cap B)(y \ast x).
\]

and

\[
(A \cap B)(xy) = \min \{A(xy), B(xy)\} = \min \{A(yx), B(yx)\} = (A \cap B)(yx).
\]

Thus \(A \cap B \in \tau\). Lastly, let \(A_i \in \tau, i \in I\). Then

\[
\left(\bigcup_{i \in I}(A_i)\right)(x \ast y) = \sup \{A_i(x \ast y)\}_{i \in I} = \sup \{A_i(y \ast x)\}_{i \in I} = \left(\bigcup_{i \in I}(A_i)\right)(y \ast x).
\]
Thus \( \bigcup_{i \in I} (A_i) \in \tau \) and so \( \tau \) is a fuzzy topology on \( X \).

**Definition 3.2** The family \( \tau \) in Example 3.1 is called the *fuzzy normal topology* in \( X \).

The next definition appeared first in [6].

**Definition 3.3** Let \( X \) be a KS-semigroup and \( A \) a fuzzy set on \( X \). Then \( A \) is called a *fuzzy normal sub KS-semigroup* of \( X \) if it satisfies the following:

1. \( A \) is a fuzzy sub KS-semigroup of \( X \)
2. \( A(xy) = A(yx) \), and
3. \( A(x * y) = A(y * x) \).

**Theorem 3.4** Let \( (X, \tau) \) and \( (Y, \upsilon) \) be two fuzzy topological spaces, \( \tau \) is the fuzzy normal topology in \( X \) and \( f : (X, \tau) \rightarrow (Y, \upsilon) \) be an epimorphism of KS-semigroups. If \( f \) is fuzzy continuous, then any fuzzy sub KS-semigroup \( V \) in \( Y \) such that \( V \in \upsilon \) is a fuzzy normal sub KS-semigroup of \( Y \).

**Proof:** Since \( V \) is a fuzzy sub KS-semigroup of \( Y \), we are left to show normality. Now, let \( y_1, y_2 \in Y \). Since \( f \) is onto, there exist \( x_1, x_2 \in X \) such that \( f(x_1) = y_1 \) and \( f(x_2) = y_2 \) so that

\[
V(y_1 * y_2) = V(f(x_1) * f(x_2)) \\
= V(f(x_1 * x_2)) \\
= V'(x_1 * x_2) \\
= f^{-1}(V)(x_1 * x_2).
\]

Since \( f \) is fuzzy continuous and \( V \in \upsilon \), \( f^{-1}(V) \in \tau \). Also, since \( \tau \) is the fuzzy normal topology in \( X \), we have

\[
V(y_1 * y_2) = f^{-1}(V)(x_2 * x_1) \\
= V'(x_2 * x_1) \\
= V(f(x_2 * x_1)) \\
= V(f(x_2) * f(x_1)) \\
= V(y_2 * y_1).
\]
Similarly,

\[
V(y_1y_2) = V(f(x_1)f(x_2)) \\
= V(f(x_1x_2)) \\
= V^I(x_1x_2) \\
= f^{-1}(V)(x_1x_2) \\
= f^{-1}(V)(x_2x_1) \\
= V^I(x_2x_1) \\
= V(f(x_2x_1)) \\
= V(f(x_2)f(x_1)) \\
= V(y_2y_1).
\]

\[\square\]

**Lemma 3.5** [11] Let \(X\) and \(Y\) be two nonempty sets, \(f : X \rightarrow Y\) be a mapping, and \(A\) be an \(f\)-invariant fuzzy set in \(X\). Then \(f^{-1}(f(A)) = A\).

**Theorem 3.6** Let \(\tau\) and \(\nu\) be the fuzzy normal topology on \(X\) and \(Y\) respectively. Then any homomorphism \(f : (X, \tau) \rightarrow (Y, \nu)\) is fuzzy continuous. Moreover, if \(f\) is onto and every fuzzy set in \(X\) is \(f\)-invariant, then \(f\) is fuzzy open.

**Proof:** Let \(V \in \nu\). We want to show that \(f^{-1}(V) \in \tau\). Now, let \(x_1, x_2 \in X\). Then

\[
f^{-1}(V)(x_1 * x_2) = V(f(x_1 * x_2)) \\
= V(f(x_1) * f(x_2)) \\
= V(f(x_2) * f(x_1)) \\
= V(f(x_2 * x_1)) \\
= f^{-1}(V)(x_2 * x_1).
\]

Also,

\[
f^{-1}(V)(x_1x_2) = V(f(x_1x_2)) \\
= V(f(x_1)f(x_2)) \\
= V(f(x_2)f(x_1)) \\
= V(f(x_2x_1)) \\
= f^{-1}(V)(x_2x_1).
\]
Thus \( f^{-1}(V) \in \tau \). Moreover, suppose that \( f \) is onto and every fuzzy set in \( X \) is \( f \)-invariant and let \( U \in \tau \) and \( y_1, y_2 \in Y \). Then there exist \( x_1, x_2 \in X \) such that \( f(x_1) = y_1 \) and \( f(x_2) = y_2 \) so that

\[
\begin{align*}
    f(U)(y_1 \ast y_2) &= f(U)(f(x_1) \ast f(x_2)) \\
    &= f(U)(f(x_1 \ast x_2)) \\
    &= f^{-1}(f(U))(x_1 \ast x_2).
\end{align*}
\]

Since \( U \) is \( f \)-invariant, by Lemma 3.5, \( f^{-1}(f(U)) = U \). Hence

\[
\begin{align*}
    f(U)(y_1 \ast y_2) &= U(x_1 \ast x_2) \\
    &= U(x_2 \ast x_1) \\
    &= f^{-1}(f(U))(x_2 \ast x_1) \\
    &= f(U)(f(x_2) \ast f(x_1)) \\
    &= f(U)(y_2 \ast y_1).
\end{align*}
\]

Similarly,

\[
\begin{align*}
    f(U)(y_1 y_2) &= f(U)(f(x_1) f(x_2)) \\
    &= f(U)(f(x_1 x_2)) \\
    &= f^{-1}(f(U))(x_1 x_2) \\
    &= U(x_1 x_2) \\
    &= U(x_2 x_1) \\
    &= f^{-1}(f(U))(x_2 x_1) \\
    &= f(U)(f(x_2) f(x_1)) \\
    &= f(U)(y_2 y_1).
\end{align*}
\]

\[\square\]

**Definition 3.7** Let \( X \) be a KS-semigroup and \( a \in X \). We define the right translation of \( X \) by \( T_a(x) = x \ast a \) and the right dilation of \( X \) by \( D_a(x) = xa \).

**Definition 3.8** Let \( \tau \) be a fuzzy topology on a KS-semigroup \( X \) and \( H \) a fuzzy set in \( X \) with induced fuzzy topology \( \tau_H \). Then \( H \) is called fuzzy topological KS-semigroup of \( X \) if for each \( a \in X \), we have

1. \( T_a : (H, \tau_H) \longrightarrow (H, \tau_H) \) is relatively fuzzy continuous, and
(2) $D_a : (H, \tau_H) \to (H, \tau_H)$ is relatively fuzzy continuous.

Let $f : (X, \tau) \to (Y, v)$ a mapping of fuzzy topological spaces. By $f^{-1}(v) = \tau$, we mean that every fuzzy set in $\tau$ can be written as $f^{-1}(V)$ for some $V \in v$, and conversely, the preimage of every $V \in v$ is in $\tau$. By $f(\tau) = v$, we mean that every fuzzy set in $v$ can be written as $f(U)$ for some $U \in \tau$, and conversely, the image of every $U \in \tau$ is in $v$.

**Lemma 3.9** [11] Let $f : (X, \tau) \to (Y, v)$ be a mapping of fuzzy topological spaces. Then the following are true:

1. If $f(\tau) = v$ then $f$ is fuzzy open. Moreover, if $f$ is onto, $f$ is fuzzy continuous.

2. If $f^{-1}(v) = \tau$ then $f$ is fuzzy continuous.

**Lemma 3.10** [11] Let $(A, \tau_A)$ and $(B, v_B)$ be fuzzy subspaces of $(X, \tau)$ and $(Y, v)$ respectively and let $f : (X, \tau) \to (Y, v)$ be fuzzy continuous such that $f(A) \subseteq B$. Then $f$ is relatively fuzzy continuous.

**Lemma 3.11** [11] Let $X$ and $Y$ be two nonempty sets, $f : X \to Y$ a mapping, and $A$ and $B$ be fuzzy sets in $Y$. Then $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$.

**Theorem 3.12** Let $f : (X, \tau) \to (Y, v)$ be a KS-semigroup homomorphism such that $f^{-1}(v) = \tau$ and let $G$ be a fuzzy topological KS-semigroup of $Y$. Then $f^{-1}(G)$ is a fuzzy topological KS-semigroup of $X$.

**Proof:** We want to show that for each $a \in X$,

$$T_a : (f^{-1}(G), \tau_{f^{-1}(G)}) \to f^{-1}(G), \tau_{f^{-1}(G)})$$

and

$$D_a : (f^{-1}(G), \tau_{f^{-1}(G)}) \to f^{-1}(G), \tau_{f^{-1}(G)})$$

are relatively fuzzy continuous. Now let $U \in \tau_{f^{-1}(G)}$. Then $U = f^{-1}(G) \cap A$ for some $A \in \tau$. Since $f^{-1}(v) = \tau$, there exists $V' \in v$ such that $f^{-1}(V') = A$ so that $U = f^{-1}(G) \cap f^{-1}(V') = f^{-1}(G \cap V')$ by Lemma 3.11. Since $V' \in v$, $G \cap V' \in v_G$. Set $V = G \cap V'$. Then $V \in v_G$ so that $U = f^{-1}(V)$. Now, let $x \in X$. Since $f$ is a homomorphism,

$$T_a^{-1}(U)(x) = U^{T_a}(x) = U(T_a(x))$$

$$= U(x \ast a) = f^{-1}(V)(x \ast a)$$

$$= V_f(x \ast a) = V(f(x \ast a))$$

$$= V(f(x) \ast f(a)) = V(T_{f(a)}(f(x)))$$

$$= V^{T_{f(a)}}(f(x)) = (V^{T_{f(a)}})^f(x)$$

$$= f^{-1}(V^{T_{f(a)}})(x)$$

$$= f^{-1}(T_{f(a)}^{-1}(V))(x).$$
Thus, \( T_a^{-1}(U) = f^{-1}(T_{f(a)}^{-1}(V)) \) so that

\[
T_a^{-1}(U) \cap f^{-1}(G) = f^{-1}(T_{f(a)}^{-1}(V)) \cap f^{-1}(G).
\]

Since \( G \) is a fuzzy topological KS-semigroup of \( Y \), for each \( b \in Y \), \( T_b : (G, v_G) \to (G, v_G) \) is relatively fuzzy continuous. That is, \( T_{f(a)}^{-1}(V) \cap G \in v_G \) since \( V \in v_G \). Hence, we can write \( T_{f(a)}^{-1}(V) \cap G = G \cap O \), where \( O \in v \). Thus \( f^{-1}(T_{f(a)}^{-1}(V) \cap G) = f^{-1}(G \cap O) \) so that \( f^{-1}(T_{f(a)}^{-1}(V)) \cap f^{-1}(G) = f^{-1}(G) \cap f^{-1}(O) \) by Lemma 3.11. Note that by Lemma 3.9, \( f \) is fuzzy continuous so that \( f^{-1}(O) \in \tau \). Thus \( f^{-1}(G) \cap f^{-1}(O) \in \tau_{f^{-1}(G)} \), hence, \( T_a^{-1}(U) \cap f^{-1}(G) \in \tau_{f^{-1}(G)} \). Therefore, \( T_a : (f^{-1}(G), \tau_{f^{-1}(G)}) \to f^{-1}(G), \tau_{f^{-1}(G)}) \) is relatively fuzzy continuous.

Similarly,

\[
D_a^{-1}(U)(x) = U^{D_a}(x) = U(D_a(x)) = U(xa) = f^{-1}(V)(xa) = V^f(xa) = V(f(xa)) = V(f(x)f(a)) = V(D_{f(a)}(f(x))) = V^{D_{f(a)}}(f(x)) = f^{-1}(V^{D_{f(a)}}(x)) = f^{-1}(D_{f(a)}^{-1}(V))(x).
\]

Thus, \( D_a^{-1}(U) = f^{-1}(D_{f(a)}^{-1}(V)) \) so that

\[
D_a^{-1}(U) \cap f^{-1}(G) = f^{-1}(D_{f(a)}^{-1}(V)) \cap f^{-1}(G).
\]

Since \( G \) is a fuzzy topological KS-semigroup of \( Y \), for each \( b \in Y \), \( D_b : (G, v_G) \to (G, v_G) \) is relatively fuzzy continuous. That is, \( D_{f(a)}^{-1}(V) \cap G \in v_G \). Hence, we can write \( D_{f(a)}^{-1}(V) \cap G = G \cap H \), where \( H \in v \). Thus \( f^{-1}(D_{f(a)}^{-1}(V) \cap G) = f^{-1}(G \cap H) \) so that \( f^{-1}(D_{f(a)}^{-1}(V)) \cap f^{-1}(G) = f^{-1}(G) \cap f^{-1}(H) \) by Lemma 3.11. Note that by Lemma 3.9, \( f \) is fuzzy continuous so that \( f^{-1}(H) \in \tau \). Thus \( f^{-1}(G) \cap f^{-1}(H) \in \tau_{f^{-1}(G)} \), hence, \( D_a^{-1}(U) \cap f^{-1}(G) \in \tau_{f^{-1}(G)} \). Therefore, \( D_a : (f^{-1}(G), \tau_{f^{-1}(G)}) \to f^{-1}(G), \tau_{f^{-1}(G)}) \) is relatively fuzzy continuous. \( \square \)

Lemma 3.13 [11] Let \( X \) and \( Y \) be two nonempty sets, \( f : X \to Y \) be a mapping, and \( A \) be an \( f \)-invariant fuzzy set in \( X \). Then for any fuzzy set \( B \) in \( X \), we have \( f(A \cap B) = f(A) \cap f(B) \).

Theorem 3.14 Let \( f : (X, \tau) \to (Y, \nu) \) be an onto KS-semigroup homomorphism such that \( f(\tau) = \nu \) and let \( H \) be a fuzzy topological KS-semigroup of \( X \) and \( H \) is \( f \)-invariant. Then \( f(H) \) is a fuzzy topological KS-semigroup of \( Y \).
Proof: We want to show that for each $b \in Y$
\[ T_b : (f(H), v_{f(H)}) \longrightarrow (f(H), v_{f(H)}) \]
and
\[ D_b : (f(H), v_{f(H)}) \longrightarrow (f(H), v_{f(H)}) \]
are relatively fuzzy continuous. Now, let $V \in v_{f(H)}$ and $b \in Y$. Since $f$ is onto, there exist $a \in X$ such that $f(a) = b$. Since $f$ is a homomorphism,
\[
\begin{align*}
  f^{-1}(T_b^{-1}(V))(x) & = f^{-1}(V^{T_b})(x) = (V^{T_b})(f(x)) \\
                      & = V(T_b(f(x))) = V(f(x) \ast b) \\
                      & = V(f(x) \ast f(a)) = V(f(x \ast a)) \\
                      & = V^{f}(x \ast a) = V^{f}(T_a(x)) \\
                      & = (V^{f})T_a(x) = T_a^{-1}(V^{f})(x) \\
                      & = T_a^{-1}(f^{-1}(V))(x).
\end{align*}
\]
Thus, $f^{-1}(T_b^{-1}(V)) = T_a^{-1}(f^{-1}(V))$ so that
\[ f^{-1}(T_b^{-1}(V)) \cap H = T_a^{-1}(f^{-1}(V)) \cap H. \]
Since $f(\tau) = v$ and $f$ is onto, by Lemma 3.9, $f$ is fuzzy continuous. Now, we claim first that $f^{-1}(V) \cap H = f^{-1}(V)$. Now, since $V \in v_{f(H)}$, $V = f(H) \cap O$, where $O \in v$ so that
\[
\begin{align*}
  f^{-1}(V) & = f^{-1}(f(H) \cap O) \\
             & = f^{-1}(f(H) \cap f^{-1}(O)) \\
             & = H \cap f^{-1}(O).
\end{align*}
\]
Hence, $f^{-1}(V) \cap H = H \cap f^{-1}(O) \cap H = f^{-1}(O) \cap H = f^{-1}(V)$. This proves our claim. Since $T_a : (H, \tau_H) \longrightarrow (H, \tau_H)$ is relatively fuzzy continuous and $f^{-1}(V) \in \tau_H$, $T_a^{-1}(f^{-1}(V)) \cap H \in \tau_H$. Note that $f : (H, \tau_H) \longrightarrow (f(H), v_{f(H)})$ is relatively fuzzy open for if $U \in \tau_H$, $U = H \cap K$ where $K \in \tau$. Thus $f(U) = f(H \cap K) = f(H) \cap f(K)$ by Lemma 3.13. Since $K \in \tau$, and $f$ is fuzzy open, $f(K) \in v$. This means that $f(U) \in v_{f(H)}$. Thus $f$ is relatively fuzzy open. Since $T_a^{-1}(f^{-1}(V)) \cap H \in \tau_H$, $f(T_a^{-1}(f^{-1}(V)) \cap H) \in v_{f(H)}$. Thus $f(f^{-1}(T_b^{-1}(V)) \cap H) = f(f^{-1}(T_b^{-1}(V))) \cap f(H) = T_b^{-1}(V) \cap f(H) \in v_{f(H)}$. Therefore, $T_b : (f(H), v_{f(H)}) \longrightarrow (f(H), v_{f(H)})$ is relatively fuzzy continuous.
Similarly,
\[
\begin{align*}
  f^{-1}(D_b^{-1}(V))(x) & = f^{-1}(V^{D_b})(x) = (V^{D_b})(f(x)) \\
  & = V(D_b(f(x))) = V(f(x)b) \\
  & = V(f(x)f(a)) = V(f(ax)) \\
  & = V^f(xa) = V^f(D_a(x)) \\
  & = (V^f)^{-1}(x) = D^{-1}_a(V^f)(x) \\
  & = D^{-1}_a(f^{-1}(V))(x).
\end{align*}
\]

Thus, \( f^{-1}(D_b^{-1}(V)) = D_a^{-1}(f^{-1}(V)) \) so that
\[
  f^{-1}(D_b^{-1}(V)) \cap H = D_a^{-1}(f^{-1}(V)) \cap H.
\]

Again, since \( V \in v_{f(H)} \), \( f^{-1}(V) \cap H = f^{-1}(V) \in \tau_H \) for \( f : (H, \tau_H) \rightarrow (f(H), v_{f(H)}) \) is relatively fuzzy continuous. Now, since \( D_a : (H, \tau_H) \rightarrow (H, \tau_H) \) is relatively fuzzy continuous and \( f^{-1}(V) \in \tau_H, D_a^{-1}(f^{-1}(V)) \cap H \in \tau_H \).

Since \( f : (H, \tau_H) \rightarrow (f(H), v_{f(H)}) \) is relatively fuzzy open and \( D_a^{-1}(f^{-1}(V)) \cap H \in \tau_H, f(D_a^{-1}(f^{-1}(V)) \cap H) = v_{f(H)} \). Thus \( f(f^{-1}(D_b^{-1}(V)) \cap H) = f(f^{-1}(D_b^{-1}(V))) \cap f(H) = D^{-1}_b(V) \cap f(H) \in v_{f(H)} \). Therefore, \( D_b : (f(H), v_{f(H)}) \rightarrow (f(H), v_{f(H)}) \) is relatively fuzzy continuous.

\[\square\]

References


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