Reg(S) in E-inversive Semigroups and Semidirect Product of Cancellative Semigroups

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Abstract

Properties of Reg(S) in an E-inversive semigroup are studied. A study of the characterizations of semidirect product of cancellative semigroups and weakly cancellative semigroups is also made.

Introduction

A semigroup $S$ is said to be E-inversive if for every $a \in S$ there exists $x \in S$ such that $a.x$ is an idempotent. A semigroup $S$ is said to be E-semigroup if $E(S)$ is a band. Characterization of an E-inversive semigroup in which Reg(S) is a group is obtained. A sufficient condition for Reg(S) is subsemigroup in an E-inversive semigroup is obtained and an example is also obtained to show that it is not necessary. Further it is observed that a regular semigroup satisfying $(*)$ condition is inverse semigroup and an inverse semigroup need not satisfy the $(*)$ condition. More over in an E-inversive semigroup $S$ the set of all regular congruences on $S$ is a filter in the lattice of all congruences on $S$.

Let $S$ and $T$ be two semigroups and let $\alpha : S \to End(T)$ be a homomorphism of $S$ into endomorphisms of $T$. If $s \in S$ and $t \in T$ denote $t(s\alpha)$ by $t^s$. Then the set $U = S \times T$ is a semigroup with respect to the multiplication $(s, t)(r, u) = (sr, t^ru)(s, r \in S$ and $t, u \in T)$ called the semidirect product of $S$ and $T$ with structure map $\alpha$ and denoted by $S \times_\alpha T$. The necessary and sufficient condition for semidirect product of two cancellative semigroups to be cancellative is obtained. On the other hand if $S \times_\alpha T$ is cancellative then $S$ and $T$ are cancellative if and only if $\{s\alpha|s \in S\}$ separates $T$. Further it is observed that if $S \times_\alpha T$ is weakly cancellative then both $S$ and $T$ are weakly cancellative whenever $\{s\alpha|s \in S\}$ separates $T$ and an example is obtained to show that if $S \times_\alpha T$ weakly cancellative and $\{s\alpha|s \in S\}$ does not separate $T$ then $T$ need not be weakly cancellative. More over it is observed that if $S$ and $T$ are weakly
cancellative semigroups and \( \alpha \) is a homomorphism from \( S \) into \( \text{Mono}(T) \) then \( S \times _{\alpha} T \) is weakly cancellative and an example is obtained to show that if \( S \) and \( T \) are weakly cancellative semigroups and \( \alpha \) is a homomorphism from \( S \) into \( \text{End}(T) \) then \( S \times _{\alpha} T \) need not be weakly cancellative even though \( \{ s\alpha | s \in S \} \) separates \( T \).

§1. Reg(S) in E-inversive Semigroups

In an E-inversive semigroup \( S \) a sufficient condition for \( \text{Reg}(S) \) is subsemigroup is that \( w(e) = v(e) \), for all \( e \in E(S) \)

**Theorem-1.1:** If \( S \) is an E-inversive semigroup such that \( w(e) = v(e) \), for all \( e \in E(S) \) then \( \text{Reg}(S) \) is a subsemigroup of \( S \).

**Proof:** Let \( e, f \in E(S) \) and \( x \in w(ef) \) then \( fxe \in w(e) \cap w(f) \) and hence \( fxe \in v(e) \cap v(f) \). Therefore \( efxe = e \) and hence \( ef.x.ef = ef \). Hence \( ef \) is a regular element. Thus product of any two idempotents is regular. Therefore by Theorem-2.3 in [5], \( \text{Reg}(S) \) is a subsemigroup of \( S \).

The condition \( w(e) = v(e) \) for all \( e \in E(S) \) is only sufficient but not necessary.

**Example 1.2** Let \( S \) be a chain with more than one elements. Since \( S \) is a chain, \( S \) is regular. For any \( a, b \in S \) such that \( a \neq b \) either \( a \leq b \) or \( b \leq a \) so that \( ab = a \) or \( ba = b \). If \( ab = a \) then \( aba = a^2 = a \). Hence \( a \in w(b) \) and \( w(b) \neq v(b) \).

The following theorem is the characterization of an E-inversive semigroup in which \( \text{Reg}(S) \) is a group.

**Theorem-1.3:** In an E-inversive semigroup \( S \), \( \text{Reg}(S) \) is a subgroup of \( S \) if and only if \( w(a) \) is singleton for all \( a \in S \).

**Proof:** First we assume that \( \text{Reg}(S) \) is a subgroup of \( S \). Let \( a \in S \) and \( x, y \in w(a) \) then \( xax = x \) and \( yay = y \). Since \( \text{Reg}(S) \) is a group and \( E(S) \subseteq \text{Reg}(S) \), \( S \) has only one idempotent say \( e \). Now \( xa, ya, ax, ay \) are all idempotents and hence \( xa = ya = ax = ay = e \). Therefore \( x = xax = yax = yay = y \). Thus \( w(a) \) is singleton for all \( a \in S \). Conversely assume that \( w(a) \) is singleton for all \( a \in S \). Since \( w(e) \cap w(f) \neq \emptyset \) for all \( e, f \in E(S) \), we have \( E(S) \) is singleton let it be \( e \). Clearly \( \text{Reg}(S) \) is a subsemigroup of \( S \). Let \( x \in \text{Reg}(S) \) then there exists \( x' \in \text{Reg}(S) \) such that \( xx'x = x \). Then \( xx', x'x \in E(S) \) and hence \( xx' = x'x = e \) so that \( xe = ex = x \). Therefore \( \text{Reg}(S) \) is a subgroup of \( S \).

**Def 1.4:** Let \( S \) be an E-inversive semigroup. Then \( S \) is said to satisfy the condition (*) if for any \( a, b \in S \), \( ab, ba \in E(S) \) implies \( ab = ba \).
A regular semigroup satisfying the (*) condition is inverse semigroup.

**Theorem 1.5:** If $S$ is a regular semigroup satisfying the (*) condition then $S$ is an inverse semigroup.

**Proof:** Let $a \in S$ and $a',a'' \in v(a)$ then $a = aa'a = aa''a$ and $a'aa' = a',aa'' = a''$ so that we have $aa'.aa'' = aa''$ and $aa''.aa' = aa'$ and $aa',aa'' \in E(S)$. Therefore by (*) condition, $aa'.aa'' = aa''.aa'$ and hence $aa' = aa''$. Similarly we can prove that $a'a = a''a$. Hence $a' = a''$. Thus $S$ is an inverse semigroup.

The following example shows that an inverse semigroup need not satisfy the (*) condition.

**Example 1.6** Let $X = \{1,2,3,4,5,6,7,8\}$ and $S$ be the set of all partial one-one mappings on $X$. Then $S$ is an inverse semigroup with respect to the mapping composition.

Let $A = \{1,2,3\}, B = \{4,5,6\}, C = \{5,6,7\}$ and $D = \{2,3,8\}$

Define $\alpha : A \to B$ by $1 \to 4, 2 \to 5, 3 \to 6$

$\beta : C \to D$ by $5 \to 2, 6 \to 3, 7 \to 8$

Then $\alpha, \beta$ is an identity map on $\{2,3\}$ and $\beta, \alpha$ is an identity map on $\{5,6\}$.

Hence $\alpha, \beta, \beta, \alpha \in E(S)$ and $\alpha, \beta \neq \beta, \alpha$. Therefore $S$ does not satisfy the (*) condition.

**Proposition 1.7:** Let $S$ be a semigroup such that $E(S) \subseteq C(S)$ then $S$ satisfies (*), where $C(S)$ is the center of $S$. i.e., for any $a \in C(S)$, $ax = xa$ for all $x \in S$.

**Proof:** Let $a, b \in S$ be such that $ab, ba \in E(S)$. Then $ab = ab.ab = a.ba = ba.a.b = ba.a = b.a.a = b.a = ba$.

In an E-inversive semigroup the set of regular congruences is a filter in the lattice of all congruences.

**Theorem 1.8:** If $S$ is an E-inversive semigroup then the set of all regular congruences on $S$ is a filter in the lattice of all congruences on $S$.

**Proof:** Let $\rho$ be a regular congruence $\sigma$ be a congruence on $S$. Let $a \in S$ then there exists $a' \in S$ such that $(a, aa'a) \in \rho \subseteq \rho \vee \sigma$. Therefore $\rho \vee \sigma$ is also regular congruence. Therefore the set of all regular congruences on $S$ is a filter in the lattice of all congruences in $S$. 
§2. Semidirect Product of Cancellative and Weakly cancellative Semigroups

**Def-2.1:** A semigroup $S$ is said to be cancellative if for any $a, x, y \in S$, $ax = ay$ or $xa = ya$ then $x = y$.

**Def-2.2:** A semigroup $S$ is said to be weakly cancellative if for any $a, b, x, y \in S$, $ax = ay$ and $xb = yb$ implies $x = y$.

The following theorem is the characterization of semidirect product of cancellative semigroups to be cancellative.

**Theorem-2.3:** If $S$ and $T$ are cancellative semigroups and $\alpha : S \rightarrow \text{End}(T)$ is a homomorphism then $S \times_\alpha T$ is cancellative if and only if $s\alpha$ is one-one for each $s \in S$.

**proof:** First assume that $S \times_\alpha T$ is cancellative and suppose that there exists $s \in S$ such that $s\alpha$ is not one-one. So there exists $t_1, t_2 \in T$ such that $t_1 \neq t_2$ and $t_1^s = t_2^s$. Then $(s, t_1)(s, t_1) = (s, t_2)(s, t_1)$ and $S \times_\alpha T$ is cancellative. Therefore $(s, t_1) = (s, t_2)$ so that $t_1 = t_2$ which is not true. Therefore each $s\alpha$ is one-one. Conversely assume that $s\alpha$ is one-one for all $s \in S$ and suppose that $(s, t)(s_1, t_1) = (s, t)(s_2, t_2)$. Then $s.s_1 = s.s_2$ and $t^{s_1}t_1 = t^{s_2}t_2$. Since $S$ is cancellative we have $s_1 = s_2$ and hence $t^{s_1}t_1 = t^{s_2}t_2$ and $T$ is cancellative therefore $t_1 = t_2$. Hence $S \times_\alpha T$ is left cancellative. Similarly we can prove that $S \times_\alpha T$ is right cancellative. Thus $S \times_\alpha T$ is cancellative semigroup.

**Def-2.4:** Let $S$ be a semigroup and $\mathcal{H}$ be a non empty subset of $\text{End}(S)$. Then $\mathcal{H}$ is said to separate $S$ if for any $s_1 \neq s_2$ in $S$ there exists $\theta$ in $\mathcal{H}$ such that $s_1\theta \neq s_2\theta$.

If $S \times_\alpha T$ is cancellative then the necessary and sufficient condition for both $S$ and $T$ are cancellative is obtained.

**Theorem-2.5:** Let $\alpha : S \rightarrow \text{End}(T)$ be a homomorphism such that $S \times_\alpha T$ is a cancellative semigroup. Then $\{s\alpha | s \in S\}$ separates $T$ if and only if both $S$ and $T$ are cancellative.

**Proof:** Assume that $\{s\alpha | s \in S\}$ separates $T$. First we observe that $S$ is cancellative. Let $s_1, s_2, s_3 \in S$ be such that $s_1s_2 = s_1s_3$ and let $t \in T$. Then $(s_1, t^{s_1}).(s_2, t) = (s_1, t^{s_1}).(s_3, t)$ and since $S \times_\alpha T$ is cancellative we have $(s_2, t) = (s_3, t)$. Therefore $s_2 = s_3$. Now suppose that $s_2s_1 = s_3s_1$ and let $t \in T$. Then $(s_2, t).(s_1, t) = (s_3, t).(s_1, t)$ and $S \times_\alpha T$ is cancellative therefore $s_2 = s_3$. Now we prove that $T$ is cancellative. Let $t_1, t_2, t_3 \in T$ be such that $t_1t_2 = t_1t_3$. If $t_2 \neq t_3$ then there exists $s \in S$ such that $t_2^s \neq t_3^s$. Also we have $(s, t_1).(s, t_2^s) =
Let \((s, t_1), (s, t_3^s)\). Since \(S \times \alpha T\) is cancellative we have \((s, t_2^s) = (s, t_3^s)\) and hence \(t_2^s = t_3^s\) which is not true. Hence \(t_2 = t_3\). Therefore \(T\) is right cancellative. Similarly we can prove that \(T\) is left cancellative. Thus \(T\) is cancellative semigroup.

**Corollary-2.6:** Let \(S\) and \(T\) be two semigroups and \(\alpha : S \to \text{Mono}(T)\) be a homomorphism. Then \(S \times \alpha T\) is cancellative if and only if \(S\) and \(T\) are cancellative.

If \(S \times \alpha T\) is cancellative then \(\{s \alpha | s \in S\}\) separates \(T\) is an equivalent condition for both \(S\) and \(T\) are cancellative where as in the case of weakly cancellative semigroups it is only sufficient but not necessary.

**Lemma-2.7:** Let \(S\) and \(T\) be two semigroups and \(\alpha : S \to \text{End}(T)\) be a homomorphism such that \(S \times \alpha T\) is cancellative. If \(\{s \alpha | s \in S\}\) separates \(T\) then both \(S\) and \(T\) are weakly cancellative.

**Proof:** First we prove that \(S\) is weakly cancellative. Let \(s_1, s_2, s_3, s_4 \in S\) be such that \(s_1s_2 = s_1s_3\) and \(s_2s_4 = s_3s_4\). Let \(t \in T\). Then we have \((s_1, t^s_1) . (s_2, t) = (s_1, t^s_1)(s_3, t)\) and \((s_2, t) . (s_4, t) = (s_3, t)(s_4, t)\) and \(S \times \alpha T\) is weakly cancellative. Therefore \((s_2, t) = (s_3, t)\) and hence \(s_2 = s_3\). Thus \(S\) is weakly cancellative. Now we prove that \(T\) is weakly cancellative. Let \(t_1, t_2, t_3, t_4 \in T\) be such that \(t_1t_2 = t_1t_3\) and \(t_4t_2 = t_4t_3\). If \(t_2 \neq t_3\) then there exists \(s \in S\) such that \(t_2^s \neq t_3^s\). For this \(s \in S\) we have \((s, t_1) . (s, t_2^s) = (s, t_1) . (s, t_3^s)\) and \((s, t_2^s) . (s, t_4^s) = (s, t_3^s) . (s, t_4^s)\) and \(S \times \alpha T\) is weakly cancellative so that we have \((s, t_2^s) = (s, t_3^s)\) and hence \(t_2^s = t_3^s\) which is not true. Therefore \(T\) is weakly cancellative.

In the following example it is observed that if we drop the condition that \(\{s \alpha | s \in S\}\) separates \(T\) in the above lemma then \(T\) need not be weakly cancellative.

**Example-2.8:** Let \(S\) be any weakly cancellative semigroup and let \(T = \{0, a, b\}\) be a semigroup with the following multiplication table

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<td>a</td>
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Define \(\alpha : S \to \text{End}(T)\) by \(s \alpha = \theta\) for all \(s \in S\) where \(\theta : T \to T\) by \(\theta = a\) for all \(t \in T\). Clearly \(\alpha\) is a homomorphism. Suppose that \((s_1, t_1) . (s_2, t_2) = (s_1, t_1) . (s_3, t_3)\) and \((s_2, t_2) . (s_4, t_4) = (s_3, t_3) . (s_4, t_4)\) in \(S \times \alpha T\). Then we have \(s_1s_2 = s_1s_3, s_2s_4 = s_3s_4\) and \(t_1^s_1t_2 = t_1^s_3t_3, t_2^s_4t_4 = t_3^s_4t_4\). Since \(S\) is weakly cancellative \(s_2 = s_3\) and hence \(t_1^s_1t_2 = t_1^s_3t_3\) and \(t_2^s_4t_4 = t_3^s_4t_4\) which implies \(at_2 = at_3\) and \(at_4 = at_4\). Hence \(t_2 = t_3\) (Since \(a\) is identity in
Therefore $S \times_\alpha T$ is weakly cancellative. But $T$ is not weakly cancellative (since $0.a = 0.b$, $a.0 = b.0$ and $a \neq b$)

If $S$ and $T$ are weakly cancellative and each $s \alpha$ is one-one then $S \times_\alpha T$ is weakly cancellative but not the converse.

**Lemma-2.9:** Let $S$ and $T$ be two weakly cancellative semigroups and $\alpha : S \rightarrow Mono(T)$ is a homomorphism then $S \times_\alpha T$ is weakly cancellative.

**Proof:** Let $(s_1, t_1), (s_2, t_2), (s_3, t_3), (s_4, t_4) \in S \times_\alpha T$ be such that $(s_1, t_1).(s_2, t_2) = (s_1, t_1).(s_3, t_3)$ and $(s_2, t_2).(s_4, t_4) = (s_3, t_3).(s_4, t_4)$. Then $s_1s_2 = s_1s_3$, $s_2s_4 = s_3s_4$ and $t_1^s t_2 = t_1^s t_3$, $t_2^s t_4 = t_3^s t_4$. Since $S$ is weakly cancellative we have $s_2 = s_3$ and hence $t_1^s t_2 = t_1^s t_3$ and $t_2^s t_4 = t_3^s t_4$. Applying $s_4 \alpha$ to the first term we have $t_1^{s_4^s} t_2^{s_4} = t_1^{s_4^s} t_3^{s_4}$ and $t_2^s t_4 = t_3^s t_4$ and we have $T$ is weakly cancellative therefore $t_2^s = t_3^s$ and $s_4 \alpha$ is one-one. Hence $t_2 = t_3$. Thus $S \times_\alpha T$ is weakly cancellative.

The following example shows that if $S$ and $T$ are weakly cancellative semigroups and $\alpha : S \rightarrow End(T)$ is a homomorphism such that not all $s \alpha$ is one-one then $S \times_\alpha T$ need not be weakly cancellative even though $\{s \alpha \mid s \in S\}$ separates $T$.

**Example-2.10:** Let $S = \{s_1, s_2\}$ and $T = \{t_1, t_2, t_3\}$ be two left zero semigroups and $\alpha : S \rightarrow End(T)$ is defined by

$s_1 \alpha$ as

$t_1 \rightarrow t_1$

$t_2 \rightarrow t_2$

$t_3 \rightarrow t_1$

and $s_2 \alpha$ as

$t_1 \rightarrow t_1$

$t_2 \rightarrow t_2$

$t_3 \rightarrow t_2$

Then $s_1 \alpha$ and $s_2 \alpha$ are endomorphisms on $T$, and $\alpha$ is a homomorphism of $S$ into $End(T)$. Clearly $s_1 \alpha, s_2 \alpha$ separate $T$ and both are not one-one. Also we have $(s_1, t_1).(s_1, t_3) = (s_1, t_3).(s_1, t_3)$ and $(s_1, t_1).(s_1, t_1) = (s_1, t_1).(s_1, t_3)$ and $t_1 \neq t_3$. Therefore $S \times_\alpha T$ is not weakly cancellative.

**Corollary-2.11:** Let $S$ and $T$ be two semigroups and $\alpha : S \rightarrow Mono(T)$ is a homomorphism of $S$ into monomorphisms of $T$. Then $S$ and $T$ are weakly cancellative if and only if $S \times_\alpha T$ is weakly cancellative.

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References


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