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Reg(S) in E-inversive Semigroups and Semidirect Product of Cancellative Semigroups

Y. L. Anasuya

Department of Mathematics
Andhra University
Visakhapatnam - 530 003, India
anasuyapamarthi@gmail.com

Abstract

Properties of Reg(S) in an E-inversive semigroup are studied. A study of the characterizations of semidirect product of cancellative semigroups and weakly cancellative semigroups is also made.

Introduction

A semigroup S is said to be E-inversive if for every $a \in S$ there exists $x \in S$ such that $a.x$ is an idempotent. A semigroup S is said to be E-semigroup if $E(S)$ is a band. Characterization of an E-inversive semigroup in which Reg(S) is a group is obtained. A sufficient condition for Reg(S) is subsemigroup in an E-inversive semigroup is obtained and an example is also obtained to show that it is not necessary. Further it is observed that a regular semigroup satisfying $(*)$ condition is inverse semigroup and an inverse semigroup need not satisfy the $(*)$ condition. More over in an E-inversive semigroup S the set of all regular congruences on S is a filter in the lattice of all congruences on S.

Let S and T be two semigroups and let $\alpha : S \to End(T)$ be a homomorphism of S into endomorphisms of T. If $s \in S$ and $t \in T$ denote $t(s\alpha)$ by t^s. Then the set $U = S \times T$ is a semigroup with respect to the multiplication $(s,t)(r,u) = (sr, t^r u)(s, r \in S$ and $t, u \in T)$ called the semidirect product of S and T with structure map α and denoted by $S \times_\alpha T$. The necessary and sufficient condition for semidirect product of two cancellative semigroups to be cancellative is obtained. On the other hand if $S \times_\alpha T$ is cancellative then S and T are cancellative if and only if $\{s\alpha | s \in S\}$ separates T. Further it is observed that if $S \times_\alpha T$ is weakly canellative then both S and T are weakly cancellative whenever $\{s\alpha | s \in S\}$ separates T and an example is obtained to show that if $S \times_\alpha T$ weakly cancellative and $\{s\alpha | s \in S\}$ does not separate T then T need not be weakly canellative. More over it is observed that if S and T are weakly cancellative semigroups and α is a homomorphism from S into $Mono(T)$ then $S \times_\alpha T$ is weakly cancellative and an example is obtained to show that if S and T are weakly cancellative semigroups and α is a homomorphism from S into $End(T)$ then $S \times_\alpha T$ need not be weakly cancellative even though $\{s\alpha | s \in S\}$ separates T.

§1. Reg(S) in E-inversive Semigroups

In an E-inversive semigroup S a sufficient condition for $Reg(S)$ is subsemigroup is that $w(e) = v(e)$, for all $e \in E(S)$

Theorem-1.1: If S is an E-inversive semigroup such that $w(e) = v(e)$, for all $e \in E(S)$ then $Reg(S)$ is a subsemigroup of S.

Proof: Let $e, f \in E(S)$ and $x \in w(ef)$ then $fxe \in w(e) \cap w(f)$ and hence $fxe \in v(e) \cap v(f)$. Therefore $efxe = e$ and hence $ef.x.ef = ef$. Hence ef is a regular element. Thus product of any two idempotents is regular. Therefore by Theorem-2.3 in [5],$Reg(S)$ is a subsemigroup of S.

The condition $w(e) = v(e)$ for all $e \in E(S)$ is only sufficient but not necessary.

Example 1.2 Let S be a chain with more than one elements. Since S is a chain, S is regular. For any $a, b \in S$ such that $a \neq b$ either $a \leq b$ or $b \leq a$ so that $ab = a$ or $ba = b$. If $ab = a$ then $aba = a^2 = a$. Hence $a \in w(b)$ and $w(b) \neq v(b)$.

The following theorem is the characterization of an E-inversive semigroup in which Reg(S) is a group.

Theorem-1.3: In an E-inversive semigroup S, Reg(S) is a subgroup of S if and only if $w(a)$ is singleton for all $a \in S$.

Proof: First we assume that Reg(S) is a subgroup of S. Let $a \in S$ and $x, y \in w(a)$ then $xax = x$ and $yay = y$. Since Reg(S) is a group and $E(S) \subseteq Reg(S)$, S has only one idempotent say e. Now xa, ya, ax, ay are all idempotents and hence $xa = ya = ax = ay = e$. Therefore $x = xax = yax = yay = y$. Thus $w(a)$ is singleton for all $a \in S$. Conversely assume that $w(a)$ is singleton for all $a \in S$. Since $w(e) \cap w(f) \neq \emptyset$ for all $e, f \in E(S)$, we have $E(S)$ is singleton let it be e. Clearly Reg(S) is a subsemigroup of S. Let $x \in Reg(S)$ then there exists $x' \in Reg(S)$ such that $xx'x = x$. Then $xx', x'x \in E(S)$ and hence $xx' = x'x = e$ so that $xe = ex = x$. Therefore $Reg(S)$ is a subgroup of S.

Def 1.4: Let S be an E-inversive semigroup. Then S is said to satisfy the condition $(*)$ if for any $a, b \in S$, $ab, ba \in E(S)$ implies $ab = ba$.

A regular semigroup satisfying the $(*)$ condition is inverse semigroup.

Theorem 1.5: If S is a regular semigroup satisfying the $(*)$ condition then S is an inverse semigroup.

Proof: Let $a \in S$ and $a', a'' \in v(a)$ then $a = aa'a = aa''a$ and $a'aa' = a', a''aa'' = a''$ so that we have $aa'.aa'' = aa''$ and $aa''.aa' = aa'$ and $aa', aa'' \in E(S)$. Therefore by $(*)$ condition, $aa'.aa'' = aa''.aa'$ and hence $aa' = aa''$. Similarly we can prove that $a'a = a''a$. Hence $a' = a''$. Thus S is an inverse semigroup.

The following example shows that an inverse semigroup need not satisfy the $(*)$ condition.

Example 1.6 Let $X = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and S be the set of all partial one-one mappings on X. Then S is an inverse semigroup with respect to the mapping composition.
Let $A = \{1, 2, 3\}, B = \{4, 5, 6\}, C = \{5, 6, 7\}$ and $D = \{2, 3, 8\}$
Define $\alpha : A \to B$ by $1 \to 4, 2 \to 5, 3 \to 6$
$\beta : C \to D$ by $5 \to 2, 6 \to 3, 7 \to 8$
Then $\alpha.\beta$ is an identity map on $\{2, 3\}$ and $\beta.\alpha$ is an identity map on $\{5, 6\}$. Hence $\alpha.\beta, \beta.\alpha \in E(S)$ and $\alpha.\beta \neq \beta.\alpha$. Therefore S does not satisfy the $(*)$ condition.

Proposition 1.7: Let S be a semigroup such that $E(S) \subseteq C(S)$ then S satisfies $(*)$, where $C(S)$ is the center of S .i.e., for any $a \in C(S)$, $ax = xa$ for all $x \in S$.

Proof: Let $a, b \in S$ be such that $ab, ba \in E(S)$. Then $ab = ab.ab = a.ba.b = ba.a.b = b.a.ab = b.ab.a = ba$.

In an E-inversive semigroup the set of regular congruences is a filter in the lattice of all congruences.

Theorem 1.8: If S is an E-inversive semigroup then the set of all regular congruences on S is a filter in the lattice of all congruences on S.

Proof: Let ρ be a regular congruence σ be a congruence on S. Let $a \in S$ then there exists $a' \in S$ such that $(a, aa'a) \in \rho \subseteq \rho \vee \sigma$. Therefore $\rho \vee \sigma$ is also regular congruence. Therefore the set of all regular congruences on S is a filter in the lattice of all congruences in S.

§2. Semidirect Product of Cancellative and Weakly cancellative Semigroups

Def-2.1: A semigroup S is said to be cancellative if for any $a, x, y \in S$, $ax = ay$ or $xa = ya$ then $x = y$.

Def-2.2: A semigroup S is said to be weakly cancellative if for any $a, b, x, y \in S$, $ax = ay$ and $xb = yb$ implies $x = y$.

The following theorem is the characterization of semidirect product of cancellative semigroups to be cancellative.

Theorem-2.3: If S and T are cancellative semigroups and $\alpha : S \to End(T)$ is a homomorphism then $S \times_\alpha T$ is cancellative if and only if $s\alpha$ is one-one for each $s \in S$.

proof: First assume that $S \times_\alpha T$ is cancellative and suppose that there exists $s \in S$ such that $s\alpha$ is not one-one. So there exists $t_1, t_2 \in T$ such that $t_1 \neq t_2$ and $t_1^s = t_2^s$. Then $(s, t_1)(s, t_1) = (s, t_2)(s, t_1)$ and $S \times_\alpha T$ is cancellative. Therefore $(s, t_1) = (s, t_2)$ so that $t_1 = t_2$ which is not true. Therefore each $s\alpha$ is one-one. Conversely assume that $s\alpha$ is one-one for all $s \in S$ and suppose that $(s, t).(s_1, t_1) = (s, t).(s_2, t_2)$. Then $s.s_1 = s.s_2$ and $t^{s_1}.t_1 = t^{s_2}.t_2$. Since S is cancellative we have $s_1 = s_2$ and hence $t^{s_1}.t_1 = t^{s_1}.t_2$ and T is cancellative therefore $t_1 = t_2$. Hence $S \times_\alpha T$ is left cancellative. Similarly we can prove that $S \times_\alpha T$ is right cancellative. Thus $S \times_\alpha T$ is cancellative semigroup.

Def-2.4: Let S be a semigroup and $\mathcal{H}$ be a non empty subset of $End(S)$. Then $\mathcal{H}$ is said to separate S if for any $s_1 \neq s_2$ in S there exists θ in $\mathcal{H}$ such that $s_1\theta \neq s_2\theta$.

If $S \times_\alpha T$ is cancellative then the necessary and sufficient condition for both S and T are cancellative is obtained.

Theorem-2.5: Let $\alpha : S \to End(T)$ be a homomorphism such that $S \times_\alpha T$ is a cancellative semigroup. Then $\{s\alpha | s \in S\}$ separates T if and only if both S and T are cancellative.

Proof: Assume that $\{s\alpha | s \in S\}$ separates T. First we observe that S is cancellative. Let $s_1, s_2, s_3 \in S$ be such that $s_1 s_2 = s_1 s_3$ and let $t \in T$. Then $(s_1, t^{s_1}).(s_2, t) = (s_1, t^{s_1}).(s_3, t)$ and since $S \times_\alpha T$ is cancellative we have $(s_2, t) = (s_3, t)$. Therefore $s_2 = s_3$. Now suppose that $s_2 s_1 = s_3 s_1$ and let $t \in T$. Then $(s_2, t).(s_1, t) = (s_3, t).(s_1, t)$ and $S \times_\alpha T$ is cancellative therefore $s_2 = s_3$. Now we prove that T is cancellative. Let $t_1, t_2, t_3 \in T$ be such that $t_1 t_2 = t_1 t_3$.If $t_2 \neq t_3$ then there exists $s \in S$ such that $t_2^s \neq t_3^s$. Also we have $(s, t_1).(s, t_2^s) =$

$(s,t_1).(s,t_3^s)$. Since $S \times_\alpha T$ is cancellative we have $(s,t_2^s)=(s,t_3^s)$ and hence $t_2^s = t_3^s$ which is not true. Hence $t_2 = t_3$. Therefore T is right cancellative. Similarly we can prove that T is left cancellative. Thus T is cancellative semigroup.

Corollary-2.6: Let S and T be two semigroups and $\alpha : S \to Mono(T)$ is a homomorphism. Then $S \times_\alpha T$ is cancellative if and only if S and T are cancellative.

If $S \times_\alpha T$ is cancellative then $\{s\alpha | s \in S\}$ separates T is an equivalent condition for both S and T are cancellative where as in the case of weakly cancellative semigroups it is only sufficient but not necessary.

Lemma-2.7: Let S and T be two semigroups and α :S→ $End(T)$ be a homomorphism such that $S \times_\alpha T$ is cancellative. If $\{s\alpha | s \in S\}$ separates T then both S and T are weakly cancellative.

Proof: First we prove that S is weakly cancellative. Let $s_1, s_2, s_3, s_4 \in S$ be such that $s_1 s_2 = s_1 s_3$ and $s_2 s_4 = s_3 s_4$. Let $t \in T$. Then we have $(s_1, t^{s_1}).(s_2, t) = (s_1, t^{s_1})(s_3, t)$ and $(s_2, t).(s_4, t) = (s_3, t)(s_4, t)$ and $S \times_\alpha T$ is weakly cancellative. Therefore $(s_2, t) = (s_3, t)$ and hence $s_2 = s_3$. Thus S is weakly cancellative. Now we prove that T is weakly cancellative. Let $t_1, t_2, t_3, t_4 \in T$ be such that $t_1 t_2 = t_1 t_3$ and $t_4 t_2 = t_4 t_3$. If $t_2 \neq t_3$ then there exists $s \in S$ such that $t_2^s \neq t_3^s$. For this $s \in S$ we have $(s,t_1).(s,t_2^s) = (s,t_1).(s,t_3^s)$ and $(s,t_2^s).(s,t_4^s) = (s,t_3^s).(s,t_4^s)$ and $S \times_\alpha T$ is weakly cancellative so that we have $(s,t_2^s) = (s,t_3^s)$ and hence $t_2^s = t_3^s$ which is not true. There fore T is weakly cancellative.

In the following example it is observed that if we drop the condition that $\{s\alpha | s \in S\}$ separates T in the above lemma then T need not be weakly cancellative.

Example-2.8: Let S be any weakly cancellative semigroup and let $T = \{0, a, b\}$ be a semigroup with the following multiplication table

0	0	0
0	a	b
0	b	a

Define $\alpha : S \to End(T)$ by $s\alpha = \theta$ for all $s \in S$ where $\theta : T \to T$ by $t\theta = a$ for all $t \in T$. clearly α is a homomorphism. Suppose that $(s_1,t_1).(s_2,t_2) = (s_1,t_1).(s_3,t_3)$ and $(s_2,t_2).(s_4,t_4) = (s_3,t_3).(s_4,t_4)$ in $S \times_\alpha T$. Then we have $s_1 s_2 = s_1 s_3$, $s_2 s_4 = s_3 s_4$ and $t_1^{s_2} t_2 = t_1^{s_3} t_3$, $t_2^{s_4} t_4 = t_3^{s_4} t_4$. Since S is weakly cancellative $s_2 = s_3$ and hence $t_1^{s_2} t_2 = t_1^{s_2} t_3$ and $t_2^{s_4} t_4 = t_3^{s_4} t_4$ which implies $at_2 = at_3$ and $at_4 = at_4$. Hence $t_2 = t_3$(Since a is identity in

T).Therefore $S \times_\alpha T$ is weakly cancellative. But T is not weakly cancellative (since $0.a = 0.b$, $a.0 = b.0$ and $a \neq b$)

If S and T are weakly cancellative and each $s\alpha$ is one-one then $S \times_\alpha T$ is weakly cancellative but not the converse.

Lemma-2.9: Let S and T be two weakly cancellative semigroups and $\alpha : S \rightarrow Mono(T)$ is a homomorphism then $S \times_\alpha T$ is weakly cancellative.

Proof: Let $(s_1, t_1), (s_2, t_2), (s_3, t_3), (s_4, t_4) \in S \times_\alpha T$ be such that $(s_1, t_1).(s_2, t_2) = (s_1, t_1).(s_3, t_3)$ and $(s_2, t_2).(s_4, t_4) = (s_3, t_3).(s_4, t_4)$. Then $s_1s_2 = s_1s_3$, $s_2s_4 = s_3s_4$ and $t_1^{s_2}t_2 = t_1^{s_3}t_3$, $t_2^{s_4}t_4 = t_3^{s_4}t_4$. Since S is weakly cancellative we have $s_2 = s_3$ and hence $t_1^{s_2}t_2 = t_1^{s_2}t_3$ and $t_2^{s_4}t_4 = t_3^{s_4}t_4$. Applying $s_4\alpha$ to the first term we have $t_1^{s_2s_4}t_2^{s_4} = t_1^{s_2s_4}t_3^{s_4}$ and $t_2^{s_4}t_4 = t_3^{s_4}t_4$ and we have T is weakly cancellative therefore $t_2^{s_4} = t_3^{s_4}$ and $s_4\alpha$ is one-one. Hence $t_2 = t_3$.Thus $S \times_\alpha T$ is weakly cancellative.

The following example shows that if S and T are weakly cancellative semigroups and $\alpha : S \rightarrow End(T)$ is a homomorphism such that not all $s\alpha$ is one-one then $S \times_\alpha T$ need not be weakly cancellative even though $\{s\alpha | s \in S\}$ separates T.

Example-2.10: Let $S = \{s_1, s_2\}$ and $T = \{t_1, t_2, t_3\}$ be two left zero semigroups and $\alpha : S \rightarrow End(T)$ is defined by
$s_1\alpha$ as
$t_1 \longrightarrow t_1$
$t_2 \longrightarrow t_2$
$t_3 \longrightarrow t_1$
and $s_2\alpha$ as
$t_1 \longrightarrow t_1$
$t_2 \longrightarrow t_2$
$t_3 \longrightarrow t_2$
Then $s_1\alpha$ and $s_2\alpha$ are endomorphisms on T. and α is a homomorphism of S into $End(T)$. Clearly $s_1\alpha, s_2\alpha$ separate T and both are not one-one. Also we have $(s_1, t_1).(s_1, t_3) = (s_1, t_3).(s_1, t_3)$ and $(s_1, t_1).(s_1, t_1) = (s_1, t_1).(s_1, t_3)$ and $t_1 \neq t_3$. Therefore $S \times_\alpha T$ is not weakly cancellative.

Corollary-2.11: Let S and T be two semigroups and $\alpha : S \rightarrow Mono(T)$ is a homomorphism of S into monomorphisms of T. Then S and T are weakly cancellative if and only if $S \times_\alpha T$ is weakly cancellative.

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