On Universal Enveloping Locally C*-Algebra for a Locally JB-Algebra

Alexander A. Katz
Department of Mathematics and Computer Science
St. John’s University
300 Howard Avenue, DaSilva Academic Center 314
Staten Island, NY 10301, USA
katza@stjohns.edu

Oleg Friedman
Department of Mathematical Sciences
University of South Africa
POB 392 UNISA
Pretoria 0003, Republic of South Africa
friedman001@yahoo.com

Current Address: Department of Mathematics
Touro College, LCM
75-31 150th Street, Kew Gardens Hills, NY 11367
USA

Abstract
A theorem is presented on existence and uniqueness up to a topological *-isomorphism of the universal locally C*-algebra for arbitrary locally JB-algebra.

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The abstract Banach associative symmetrical *-algebras over C, so called C*-algebras, were introduces first by Gelfand and Naimark in [1]. In the present time the theory of C*-algebras become a vast portion of Functional Analysis
having connections and applications in almost all branches of Modern Mathematics and Theoretical Physics (see for example [2] for the basic theory of C*-algebras).

From the 1940’s and the beginning of 1950’s there were numerous attempts made to extend the theory of C*-algebras to a category wider than Banach algebras. For example, in 1952, while working on the theory of locally-multiplicatively-convex algebras as projective limits of projective families of Banach algebras, Arens in the paper [3] and Michael in the monograph [4] independently for the first time studied projective limits of projective families of functional algebras in the commutative case and projective limits of projective families of operator algebras in the non-commutative case. In 1971 Inoue in the paper [5] explicitly studied topological *-algebras which are topologically *-isomorphic to projective limits of projective families of C*-algebras and obtained their basic properties. He as well suggested a name of locally C*-algebras for that category. Below we will denote these algebras as LC*-algebras. For the present state of the theory of LC*-algebras see recently published monograph of Fragoulopoulou [6].

At the same time there were numerous attempts to extend the theory of C*-algebras to non-associative algebras which are close to associative, in particular to Jordan algebras. In fact, in 1978 Alfsen, Schultz and Størmer published their celebrated paper [7], in which they introduced and studied real Jordan Banach formally real algebras called JB-algebras, which are real non-associative analogues of C*-algebras, and obtained for this category analogues of the results from aforementioned paper [1] by Gelfand and Naïmark. The exposition of elementary theory of JB-algebras can be found in the monograph [8] by Hanche-Olsen and Størmer, published in 1984. In particular, in this monograph there is the following theorem which was for the first time proved in 1980 by Alfsen, Hanche-Olsen and Schultz in the paper [9].

**Theorem 1 (Alfsen, Hanche-Olsen, Schultz [9])** For an arbitrary JB-algebra A there exists a unique up to an isometric *-isomorphism a C*-algebra $C_u^*(A)$ (the universal enveloping C*-algebra for the JB-algebra A), and a Jordan homomorphism $\psi_A : A \to C_u^*(A)_{sa}$ from A to the self-adjoint part of $C_u^*(A)$, such that:

1. $\psi_A(A)$ generates $C_u^*(A)$ as a C*-algebra;

2. for any pair composed of a C*-algebra $\mathfrak{A}$ and a Jordan homomorphism $\varphi : A \to \mathfrak{A}_{sa}$ from A into the self-adjoint part of $\mathfrak{A}$, there exists a *-homomorphism $\tilde{\varphi} : C_u^*(A) \to \mathfrak{A}$ from the C*-algebra $C_u^*(A)$ into C*-algebra $\mathfrak{A}$, such that $\varphi = \tilde{\varphi} \circ \psi_A$;

3. there exists a *-antiautomorphism $\Phi$ of order 2 on the C*-algebra $C_u^*(A)$, such that $\Phi(\psi_A(a)) = \psi_A(a)$, $\forall a \in A$.

From the aforesaid one can see that it is natural and interesting to study
Jordan topological algebras which are projective limits of projective families of JB-algebras. Those algebras under the name of *locally JB-algebras* were introduced and studied by Katz and Friedman in the paper [10] published in 2006. In what follows we will call these algebras *LJB-algebras*.

An important question of the theory of LJB-algebras would have been an analogue of the theorem 1 above. For the further exposition we need the following technical theorem which is a corollary of Theorem 1 and general properties of projective limits of projective families of Banach algebras.

**Theorem 2** Let \( \Lambda \) be a directed set of indices, and an arbitrary LJB-algebra \( A \) be a projective limit \( A = \lim_{\rightarrow} A_\alpha \), where \( \alpha \in \Lambda \), and \( A_\alpha \) be a projective family of JB-algebras. Then the family of C*-algebras \( C_\alpha^*(A) \), where \( \forall \alpha \in \Lambda \), \( C_\alpha^*(A) \) be universal enveloping C*-algebra for the corresponding JB-algebra \( A_\alpha \), is a projective family of C*-algebras.

Using the theorems 1 and 2 we are able to obtain the following main theorem about existence and uniqueness of universal enveloping LC*-algebra for an arbitrary LJB-algebra.

**Theorem 3** For an arbitrary LJB-algebra \( A \) there exists a unique up to a topological *-isomorphism a LC*-algebra \( LC_\alpha^*(A) \) (the universal enveloping locally C*-algebra for the LJB-algebra \( A \) ), and a Jordan homomorphism \( \psi_A : A \to LC_\alpha^*(A)_{sa} \) from \( A \) to the self-adjoint part of \( LC_\alpha^*(A) \), such that:

1). \( \psi_A(A) \) generates \( LC_\alpha^*(A) \) as a LC*-algebra;
2). for any pair composed of a LC*-algebra \( \mathfrak{A} \) and a Jordan homomorphism \( \varphi : A \to \mathfrak{A}_{sa} \) from \( A \) into the self-adjoint part of \( \mathfrak{A} \), there exists a *-homomorphism \( \widehat{\varphi} : LC_\alpha^*(A) \to \mathfrak{A} \) from the LC*-algebra \( LC_\alpha^*(A) \) into LC*-algebra \( \mathfrak{A} \), such that \( \varphi = \widehat{\varphi} \circ \psi_A \);
3). there exists a *-anti-automorphism \( \Phi \) of order 2 on the LC*-algebra \( LC_\alpha^*(A) \), such that \( \Phi(\psi_A(a)) = \psi_A(a) \), \( \forall a \in A \).

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**References**


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