A Note on All Possible Factorizations of a Positive Integer

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Abstract
Let \( n \) be a positive integer different of 1. We define the integer sequence \( q(n) \) where \( q(n) \) is the number of factorizations of \( n \) in positive integers different of 1 (the order of the factors is irrelevant). We establish the conjecture
\[
\lim_{n \to \infty} \frac{q(n)}{n} = 0
\]
and we prove the conjecture for the sequence of quadratfrei numbers and the sequence of prime powers. We also prove that in this conjecture \( n \) can not be replaced by \( n^h \) where \( 0 < h < 1 \).

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1 Introduction

Let us consider a positive integer \( n \geq 2 \).

Let \( q(n) \) be the number of all possible factorizations of \( n \) in positive integer factors different of 1 (the order of the factors is irrelevant).

The sequence of positive integers \( q(n) \) is very variable.

Clearly \( q(n) \) depends of the prime factorization of \( n \) and it does not depend of the primes in the factorization.

For example:

a) If \( n \) is prime then \( q(n) = 1 \) since \( n = n \).

b) If \( n = p_1p_2 \) where \( p_1 \) and \( p_2 \) are different primes then \( q(n) = q(p_1p_2) = 2 \) since in this case there are two possible factorizations, namely \( n = n \) and \( p_1p_2 = n \).
c) If \( n = p_1p_2p_3 \) where \( p_1, p_2 \) and \( p_3 \) are different primes then \( q(n) = q(p_1p_2p_3) = 5 \) since in this case there are five possible factorizations, namely \( n = p_1p_2p_3, (p_1p_2)p_3 = n, (p_1p_3)p_2 = n, (p_2p_3)p_1 = n \) and \( p_1p_2p_3 = n \).

d) In general if \( n \) is a product of \( k \) different primes (that is, a quadratfrei number) then \( q(n) = B_k \) where \( B_k \) is the \( k \)-th Bell number. Since in this case the number \( q(n) \) of factorizations equals the number of partitions of a set of \( k \) elements in disjoint subsets [1, page 214].

e) If \( n = p^4 \) where \( p \) is prime then \( q(n) = q(p^4) = 5 \) since in this case there are five possible factorizations, namely \( n = n, (pp)p = n, (pp)(p) = n, (pp)p = n \) and \( pppp = n \).

f) In general if \( n = p^k \) where \( p \) is prime then \( q(n) = q(p^k) = p(k) \) where \( p(k) \) denotes (as usual) the number of partitions of \( k \).

We now show a short table with the first values of the integer sequence \( q(n) \).

\[
\begin{align*}
q(2) &= 1 & q(3) &= 1 & q(4) &= q(2^2) = 2 & q(5) &= 1 & q(6) &= q(2.3) = 2 \\
q(7) &= 1 & q(8) &= q(2^3) = 3 & q(9) &= q(3^2) = 2 & q(10) &= q(2.5) = 2 & q(11) &= 1 \\
q(12) &= q(2^2.3) = 4 & q(13) &= 1 & q(14) &= q(2.7) = 2 & q(15) &= q(3.5) = 2 \\
q(16) &= q(2^4) = 5 & q(17) &= 1 & q(18) &= q(2.3^2) = 4 & q(19) &= 1 \\
q(20) &= q(2^2.5) = 4 & q(21) &= q(3.7) = 2 & q(22) &= q(2.11) = 2 & q(23) &= 1 \\
q(24) &= q(2^3.3) = 7 & q(25) &= q(5^2) = 2 & q(26) &= q(2.13) = 2 & q(27) &= q(3^3) = 3 \\
q(28) &= q(2^2.7) = 4 & q(29) &= 1 & q(30) &= q(2.3.5) = 5 & q(31) &= 1 \\
q(32) &= q(2^5) = 7 & q(33) &= q(3.11) = 2 & q(34) &= q(2.17) = 2 \\
q(35) &= q(5.7) = 2 & q(36) &= q(2^2.3^2) = 9 & q(37) &= 1 & q(38) &= q(2.19) = 2 \\
q(39) &= q(3.13) = 2 & q(40) &= q(2^3.5) = 7 & q(41) &= 1
\end{align*}
\]

We establish the following conjecture.

**Conjecture 1.1** The following limit holds.

\[
\lim_{n \to \infty} \frac{q(n)}{n} = 0
\]
2 On the limit of \( \frac{q(n)}{n} \)

Let \( c_n \) be the sequence of quadratfrei numbers. This sequence has positive high density \( \frac{6}{\pi^2} \cong \frac{2}{3} \) [6, page 269].

In the following theorem we prove that this sequence of high density satisfies our conjecture.

**Theorem 2.1** If \( c_n \) is the sequence of quadratfrei numbers then

\[
\lim_{n \to \infty} \frac{q(c_n)}{c_n} = 0
\]

(1)

Proof. We have (see either [3, Theorem 24] or [4]).

\[
\log(p_1 p_2 \ldots p_n) = n \log n + n \log \log n - n + o(n)
\]

(2)

Where \( p_1 p_2 \ldots p_n \) is the first quadratfrei with \( n \) prime factors. That is, the product of the first \( n \) primes.

On the other hand, we have [2, pages 102-109]

\[
\log B_n = n \log n - n \log \log n - n + o(n)
\]

(3)

Therefore (2) and (3) give

\[
\log \frac{B_n}{p_1 p_2 \ldots p_n} = -2n \log \log n + o(n)
\]

Hence

\[
\lim_{n \to \infty} \log \frac{B_n}{p_1 p_2 \ldots p_n} = -\infty
\]

and consequently we have

\[
\lim_{n \to \infty} \frac{q(p_1 p_2 \ldots p_n)}{p_1 p_2 \ldots p_n} = \lim_{n \to \infty} \frac{B_n}{p_1 p_2 \ldots p_n} = 0
\]

(4)

Finally, (4) implies (1). The theorem is proved.

In the following theorem we shall show that in conjecture 1.1 we can not replace \( n \) by \( n^h \) where \( 0 < h < 1 \).

**Theorem 2.2** The conjecture

\[
\lim_{n \to \infty} \frac{q(n)}{n^h} = 0 \quad (0 < h < 1)
\]

is false.
Proof. We have (see either [3, Theorem 24] or [4]).

\[
\log(p_1 p_2 \ldots p_n) = n \log n + n \log \log n - n + o(n)
\]

Consequently

\[
\log(p_1 p_2 \ldots p_n)^h = h \log(p_1 p_2 \ldots p_n) = hn \log n + hn \log \log n - hn + o(n) \quad (5)
\]

On the other hand, we have [2, pages 102-109]

\[
\log B_n = n \log n - n \log \log n - hn + o(n) \quad (6)
\]

Therefore (5) and (6) give

\[
\log \frac{B_n}{(p_1 p_2 \ldots p_n)^h} = (1 - h)n \log n - n \log \log n - hn \log \log n - hn + o(n)
\]

Hence

\[
\lim_{n \to \infty} \frac{B_n}{(p_1 p_2 \ldots p_n)^h} = \infty
\]

and consequently we have

\[
\lim_{n \to \infty} \frac{q(p_1 p_2 \ldots p_n)}{(p_1 p_2 \ldots p_n)^h} = \lim_{n \to \infty} \frac{B_n}{(p_1 p_2 \ldots p_n)^h} = \infty
\]

The theorem is proved.

In the following theorem we prove that the sequence of prime powers whose density is zero also satisfies our conjecture.

**Theorem 2.3** If \(d_n\) is the sequence of prime powers then

\[
\lim_{n \to \infty} \frac{q(d_n)}{d_n} = 0 \quad (7)
\]

Proof. The first prime power with \(n\) prime factors is \(2^n\). Let \(p(n)\) be the number of partitions of \(n\). In an elementary way can be proved that [5],

\[
p(n) < \exp(c_1 \log n \sqrt{n} + c_2 \log n) \quad (8)
\]

Where \(c_1\) and \(c_2\) are positive constants.

Consequently (8) gives

\[
\lim_{n \to \infty} \frac{q(2^n)}{2^n} = \lim_{n \to \infty} \frac{p(n)}{2^n} = 0 \quad (9)
\]

Equation (9) implies (7). The theorem is proved.
References


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